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Čech Type Approach to Computing Homology of Maps

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Abstract A new approach to algorithmic computation of the homology of spaces and maps is presented. The key point of the approach is a change in the representation of sets. The proposed representation is based on a combinatorial variant of the Čech homology and the Nerve Theorem. In many situations, this change of the representation of the input may help in bypassing the problems with the complexity of the standard homology algorithms by reducing the size of necessary input. We show that the approach is particularly advantageous in the case of homology map algorithms.

Keywords Homology algorithm · Čech homology · Nerve theorem · Cubical set · Reduction methods

1 Introduction

Effective algorithms for computing homology of spaces and maps are needed in computer assisted proofs in dynamics based on topological tools (see [4, 18, 23, 24] and references therein). Recently, homology algorithms have also been used in robotics [10], material structure analysis [11, 12] and image recognition [3, 37], in particular in medical imaging [31, 38]. The classical approach to computing homology is

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based on the Smith diagonalization algorithm for integer matrices (see [30]). Unfortunately, the complexity of this algorithm, which is supercubical [33, 34], is unsatisfactory for many applications. Therefore, when computing the homology of a space, one first tries to reduce the space by some reduction algorithms to a possibly smaller space with the same homology [17, 19, 27, 29] and only then one applies the Smith algorithm. This approach proved to be useful in many applications. It is particularly strong in the case of cubical homology introduced in [15] (see also [14]). Cubical homology is defined for cubical sets, i.e., subsets X of $\mathbb{R}^d$ which are unions of a finite family $\mathcal{X}$ of cubes of unit size, the so-called elementary cubes. Such sets appear in a natural way in the case of digital imaging. In particular, the book presents algorithms for computing homology of cubical sets as well as continuous maps of cubical sets. However, for the homology of maps, the situation remains unsatisfactory.

In this paper, we present an entirely different approach to the problem of finding fast algorithms for computing homology of spaces and maps. The standard approach to the study of the efficiency of algorithms is to use the theory of computational complexity. One of the goals of the paper is to show that in some situations there is another factor which may crucially influence the efficiency of algorithms: the representation, i.e., the way the objects of interest are encoded in the form acceptable by the algorithms.

To be more precise, observe that algorithms by their very nature deal with finite amount of data. In particular, an object on input of an algorithm must be finite. However, we often are interested in solving problems in which the input data is infinite by its very nature, as in the case of topological spaces and continuous maps. To use an algorithmic approach in such problems, we first select a countable subfamily of all possible input data and some encoding which assigns a finite code to every object in the family. Then, the code is used as the finite input to the algorithm. This is what we mean by choosing a representation.

For example, consider the problem of computing homology of a simplicial complex. The number of all simplices is uncountable but when we restrict the coordinates of a simplex to a countable subset of real numbers, for instance, to the set of rational numbers, then we have only a countable number of simplices and the sequence of coordinates of vertices of a simplex becomes a natural finite encoding of the simplex. Since a simplicial complex consists of a finite family of simplices, we easily obtain a natural encoding of a simplicial complex which may be used as the input for an algorithm computing homology.

Once a representation is fixed, the computational complexity of the algorithm may be addressed by relating the amount of computations needed for some given input to some measure of the size of the encoding of the input. However, in general, there may be many ways of choosing the representation and, as we already mentioned, the cost of running an algorithm may depend on the representation chosen. For instance, when computing the homology of a topological space, among many choices of a representation there are in particular polyhedra, cubical sets and subanalytic sets. The first choice is very natural because it immediately translates the problem of computing the homology of a space to the problem of computing the homology of a simplicial complex. However, in high dimensions cubical sets may turn out to be more efficient. This is because the number of simplices needed to triangulate a d-dimensional cube is at least (see [32])

$$\frac{6^{d/2}d!}{2(d+1)^{(d+1)/2}}.$$

Consequently, the size of the input is reduced at least exponentially with d when cubes are used. Actually, the problem of finding the minimal triangulation of a d-dimensional cube is important in applications where the simplicial representation is necessary simply because of the need to keep the representation small. It has been studied by many authors. Surprisingly, so far a complete solution of this problem is know only up to dimension 4 and up to dimension 7 for the special case of the so-called *vertex triangulations* (see [6] and the references therein for a detailed study of this problem).

The problem of choosing a good representation is even better visible when dealing with continuous maps. The family of piecewise linear maps, which constitutes the most natural choice for the case of polyhedra, is poor and not satisfactory for many applications. The family of cubical maps, i.e., continuous maps which map elementary cubes to elementary cubes is even poorer. In the case of computer assisted proofs in dynamics, the maps studied are neither piecewise linear nor cubical. Even worse, often all what we know about these maps apart from the fact that they are well defined and continuous is an algorithm providing numerical approximations of the values of the map on a finite set of arguments. Therefore, this algorithm must serve as the finite encoding of the map. Since we want to compute the homology of the map, such encoding is sufficient if only the algorithm can produce approximations which are good enough. One way to pass from a continuous map represented by an approximating algorithm to the homology of the map is to use the algorithm to construct a sufficiently good simplicial approximation of the map. The advantage of such an approach lies in the ease of passing from the simplicial map to the chain map needed in the construction of the homology map. Unfortunately, constructing a simplicial approximation is not straightforward and verifying if the approximation is good enough to carry the proper homology leads to the problem of finding lower estimates of inverse images of simplices under the map. Unfortunately, no satisfactory solution to this problem seems to be available. Therefore, the approach in [15] based on [2] uses the so-called multivalued representations. Unlike simplicial approximations, multivalued representations constitute a natural outcome of the so-called enclosure algorithms for dynamical systems and differential equations [20, 28]. Thus, they are much easier to obtain than simplicial approximations. However, the problem is then shifted to constructing the associated chain map, which is not straightforward in this setting. In [2], it is shown that the problem may be reduced to solving systems of linear equations. Unfortunately, the number of the systems as well as the systems themselves is large. Therefore, although the method has been implemented [21], it is difficult to apply in practice. The algorithm in [25], based on ideas of Górniewicz and Granas [13], reduces the problem to finding the homology of projections from the graph of the multivalued representation onto the domain and codomain. This is much easier, especially in the cubical setting, because projections preserve the cubical structure. Nevertheless, the algorithm is still far from being satisfactory.

In this paper, we introduce a representation of a class of topological spaces and continuous maps which is based on the Čech approach to homology theory. By a

Čech structure we mean a finite family of compact convex subsets of $\mathbb{R}^d$ and we define a Čech polyhedron as the union of a Čech structure. Čech structures may serve as an alternative representation of topological spaces in the context of homology computations because of the Nerve Theorem [5, 7, 22, 35, 36]. In the simplified setting of our interest, the theorem states that every Čech polyhedron is homotopy equivalent to the nerve of its Čech structure $\mathcal{X}$, i.e., the abstract simplicial complex whose simplices are the subfamilies of $\mathcal{X}$ with nonempty intersection. A similar approach, but in a different setting, is used in the topological analysis of point cloud data (see [8]), when the space is known only approximately via a finite subset of sampling points.

The representation based on Čech structures is particularly helpful in computing the homology of maps, because the upper estimates of the images of the sets in the covering under the map may be used to obtain the associated chain map directly from the nerves. This allows us to bypass the problem of constructing the chain map from the multivalued approximation by using the fact that the multivalued approximation acts as a simplicial map on the nerve. Consequently, the computation of the homology of a continuous map from its multivalued representation becomes straightforward.

The organization of the paper is as follows. We begin with preliminaries in Sect. 2. We briefly recall the concept of an abstract simplicial complex in Sect. 3. The main concept of the paper, the Čech structures are discussed in Sect. 4. In the following section, we consider the Čech structures in the context of computing homology of continuous maps. The main result of the paper, namely the algorithm for computing the homology of continuous maps, is presented in Sect. 6. Sections 7, 8 and 9 contain auxiliary, technical material. Section 10 presents Mayer–Vietoris Theorem for Čech structures. In Sect. 11, we construct the chain map between the chain complex of a Čech structure and the singular chain complex of the support of the Čech structure and as a byproduct we prove Nerve Theorem. The next section is devoted to showing that the homologies of all Čech structures of a given Čech polyhedron are isomorphic via a system of natural isomorphisms, the co-called connected simple system. In Sect. 13, we present the proofs of the main results of the paper. We finish the paper with some concluding remarks.

2 Preliminaries

Throughout the paper, $\mathbb{N}$ stands for the set of natural numbers and $\mathbb{R}$ for the set of reals. For an integer $q \geq 0$, we let $\mathbb{N}_q := \{1, 2, \dots, q\}$. Note that, in particular, $\mathbb{N}_0 = \emptyset$. Let

$$\mathrm{Perm}_q := \{\sigma : \mathbb{N}_q \to \mathbb{N}_q \mid \sigma \text{ is a bijection}\}$$

denote the set of all permutations of $\mathbb{N}_q$. Observe that $\mathrm{Perm}_q = \{\mathrm{id}_{\mathbb{N}_q}\}$ for $q = 1$ and we make it true for $q = 0$ via considering the empty set as the identity map on $\mathbb{N}_0$.

Recall that there is a unique homomorphism from the group of permutations of $\mathbb{N}_q$ into the multiplicative group $\{-1, 1\}$ which sends each transposition to -1. We denote it by

$$\mathrm{sgn} : \mathrm{Perm}_q \to \{-1, 1\}.$$

In the paper, we are interested in the subsets of the Euclidean space $\mathbb{R}^d$ for some fixed natural number d. We assume that

$$\mathrm{dist}: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$$

is a fixed metric in $\mathbb{R}^d$ induced by any norm in $\mathbb{R}^d$ equivalent to the Euclidean norm. The associated diameter of a compact set $A \subset \mathbb{R}^d$ is denoted by $\operatorname{diam} A$. If $\mathcal{X}$ is a family of compact subsets of $\mathbb{R}^d$ then $\operatorname{diam} \mathcal{X}$ stands for the supremum of the diameters of the elements of $\mathcal{X}$. Given a set $A \subset \mathbb{R}^d$ and $\epsilon > 0$, we denote by

$$A^\epsilon := \left\{x \in \mathbb{R}^d \mid \mathrm{dist}(x, A) \leq \epsilon\right\}$$

the closed ball around A in $\mathbb{R}^d$ of radius ϵ, also called the ϵ-thickening of A.

The family of all nonempty subset of $\mathbb{R}^d$ is denoted by $\mathcal{P}(\mathbb{R}^d)$. The subfamily of all nonempty, compact, convex subsets of $\mathbb{R}^d$ is denoted by $\mathrm{Conv}(\mathbb{R}^d)$. Given $C \subset \mathbb{R}^d$ and $\mathcal{X} \subset \mathcal{P}(\mathbb{R}^d)$, we define $\mathcal{X}(C) := \{A \in \mathcal{X} \mid C \subset A\}$. For $\mathcal{A} \subset \mathcal{P}(\mathbb{R}^d)$, by $\mathcal{A}^*$ we denote the family of all non-empty intersections of finite subfamilies of $\mathcal{A}$.

For a compact set $X \subset \mathbb{R}^d$, we denote the augmented complex of singular chains in X by $C_\#(X)$, and $H_*(X)$ stands for the associated reduced singular homology.

3 Abstract Simplicial Homology

Our basic reference concerning homology theory and homological algebra is [30] and we refer the reader there for the basic concepts in homology theory not defined in this paper. Here we briefly recall the homology theory of abstract simplicial complexes (see [30, Chap. 1.3]) because, as the reader will notice in the sequel, we slightly deviate from the standard approach. We also recall a few standard theorems used frequently in the paper for reference.

An *abstract simplicial complex* is a collection $\mathcal{K}$ of finite sets such that if $\mathcal{S} \in \mathcal{K}$ then every subset of $\mathcal{S}$ belongs to $\mathcal{K}$. The elements of $\mathcal{K}$ are called *simplices*. The *dimension* of a simplex is one less than the number of its elements. The subcollection of q-dimensional simplices in $\mathcal{K}$ will be denoted by $\mathcal{K}_q$. The elements of $V(\mathcal{K}) := \bigcup \mathcal{K}_0$ are called *vertices* of $\mathcal{K}$.

Note that we consider the empty set as a simplex of dimension -1. This is convenient because, for technical reasons, we prefer to use the reduced homology theory. However, this is not a limitation because of the well known one-to-one correspondence between the standard and reduced homology theories.

Let $\mathcal{S}$ be a simplex of dimension q. An *ordering* of $\mathcal{S}$ is a bijection $S: \mathbb{N}_{q+1} \to \mathcal{S}$. Two orderings S, S' of $\mathcal{S}$ are equivalent if there exists an even permutation $\sigma \in \mathrm{Perm}_{q+1}$ such that $S = S'\sigma$.

By a q-chain in $\mathcal{K}$ we mean a function defined on all orderings of q-simplices in $\mathcal{K}$ and satisfying

$$c(S\sigma) = \mathrm{sgn}(\sigma) c(S)$$

for any ordering S of a simplex $\mathcal{S}$ and any permutation $\sigma \in \mathrm{Perm}_{q+1}$.

The set of all such functions with argumentwise addition is an abelian group. We denote it by $C_q(\mathcal{K})$.

For every ordering S of a simplex $\mathcal{S}$ we define a q-chain $\widehat{S} \in C_q(\mathcal{K})$ by

$$\widehat{S}(T) := \begin{cases} \operatorname{sgn}(\sigma) & \exists!\sigma \in \operatorname{Perm}_{q+1} : S = T\sigma, \\ 0 & \text{otherwise.} \end{cases}$$

One easily verifies that the q-chains have the following properties

(i) $\widehat{S\sigma} = \operatorname{sgn}(\sigma)\widehat{S}$.
(ii) $\widehat{S} = \widehat{T}$ iff $S = T\sigma$ for some even permutation σ.
(iii) If $\mathcal{S} \in \mathcal{K}_q$ for $q \geq 1$, then $\widehat{\mathcal{S}} := \{\widehat{S} \mid S$—an ordering of $\mathcal{S}\}$ consists of exactly two mutually inverse elements.

Assume that $\mathcal{K}_q = \{\mathcal{S}_i \mid i = 1, 2, \dots, n\}$ and for every $\mathcal{S}_i$ an ordering S_i is given. It is straightforward to check that $\{\widehat{S_i} \mid i = 1, 2, \dots, n\}$ is a basis of $C_q(\mathcal{K})$.

Let $\kappa_i^q : \mathbb{N}_{q-1} \to \mathbb{N}_q$ be defined by

$$\kappa_i^q(j) := \begin{cases} j, & j < i, \\ j+1, & j \geq i. \end{cases}$$

Define $\partial_q : C_q(\mathcal{K}) \to C_{q-1}(\mathcal{K})$ on the basis by

$$\partial_q(\widehat{S}) := \sum_{i=0}^{q} (-1)^i \widehat{S\kappa_i^q}.$$

A simple, standard calculation (see [30, Lemma 5.3]) shows that

$$\partial_{q-1}\partial_q = 0.$$

Therefore, $(C_\#(\mathcal{K}), \partial) := (C_q(\mathcal{K}), \partial_q)$ is a chain complex. The notation should not be confused with the similar notation used in this paper for the singular chain complex because it is applied to abstract simplicial complexes. Let $Z_q(\mathcal{K}) := \ker \partial_q$ denote the subgroup of *q-cycles* and $B_q(\mathcal{K}) := \operatorname{im} \partial_{q+1}$ the subgroup of *q-boundaries*. The qth *homology group* of $\mathcal{K}$ is defined by

$$H_q(\mathcal{K}) := Z_q(\mathcal{K})/B_q(\mathcal{K}).$$

Let $\mathcal{K}$ and $\mathcal{L}$ be two abstract simplicial complexes and let

$$\mathcal{F} : V(\mathcal{K}) \to V(\mathcal{L})$$

be a map. For $\mathcal{S} \in \mathcal{K}$, define

$$\mathcal{F}(\mathcal{S}) := \{\mathcal{F}(S) \mid S \in \mathcal{S}\}.$$

We call $\mathcal{F}$ *simplicial* if

$$\mathcal{S} \in \mathcal{K} \Rightarrow \mathcal{F}(\mathcal{S}) \in \mathcal{L}.$$

If $\mathcal{F}$ is simplicial then there is an induced chain map

$$C_\#(\mathcal{F}) := \mathcal{F}_\# : C_\#(\mathcal{K}) \to C_\#(\mathcal{L})$$

and a map induced in homology

$$H_*(\mathcal{F}) := \mathcal{F}_* : H_*(\mathcal{K}) \to H_*(\mathcal{L}).$$

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two simplicial complexes and let

$$\iota_i : \mathcal{K}_1 \cap \mathcal{K}_2 \to \mathcal{K}_i$$

and

$$\lambda_i : \mathcal{K}_i \to \mathcal{K}_1 \cup \mathcal{K}_2$$

denote inclusion maps. We have the following theorem.

Theorem 3.1 (Mayer–Vietoris Theorem for simplicial complexes [30, Theorem 25.1]) *The sequence*

$$\begin{aligned} \cdots \to H_q(\mathcal{K}_1 \cap \mathcal{K}_2) &\xrightarrow{(\iota_{1*q}, -\iota_{2*q})} H_q(\mathcal{K}_1) \oplus H_q(\mathcal{K}_2) \\ &\xrightarrow{\lambda_{1*q} + \lambda_{2*q}} H_q(\mathcal{K}_1 \cup \mathcal{K}_2) \to H_{q-1}(\mathcal{K}_1 \cap \mathcal{K}_2) \to \cdots \end{aligned} \tag{1}$$

is exact.

We say that an abstract simplicial complex $\mathcal{K}$ is acyclic if $H_*(\mathcal{K}) = 0$.

A vertex $S_* \in \mathcal{K}_0$ satisfies the *cone condition* if $\mathcal{S} \cup \{S_*\} \in \mathcal{K}$ for every simplex $\mathcal{S} \in \mathcal{K}$. We say that $\mathcal{K}$ is a *cone* if there exists a vertex $S_* \in V(\mathcal{K})$ which satisfies the cone condition.

Theorem 3.2 ([30, Theorem 8.2]) *Every cone is acyclic.*

If $\mathcal{K}$ and $\mathcal{L}$ are abstract simplicial complexes, then an *acyclic carrier* from $\mathcal{K}$ to $\mathcal{L}$ is a function Φ that assigns to each simplex S in $\mathcal{K}$ an abstract, nonempty, acyclic simplicial complex $\Phi(S) \subset \mathcal{L}$ such that if $T \subset S$, then $\Phi(T) \subset \Phi(S)$. A chain map $\varphi : V(\mathcal{K}) \to V(\mathcal{L})$ is *carried* by Φ if $|\varphi(S)| \subset \Phi(S)$ for each $S \in \mathcal{K}$.

Theorem 3.3 (Acyclic Carrier Theorem, [30, Theorem 13.3]) *Let Φ be an acyclic carrier from $\mathcal{K}$ to $\mathcal{L}$. If $\varphi, \psi : V(\mathcal{K}) \to V(\mathcal{L})$ are chain maps carried by Φ, then they are chain homotopic.*

4 Čech Structures

Let $\mathcal{X}$ be a finite family of sets. We define the *support* of $\mathcal{X}$, denoted $|\mathcal{X}|$, as the union of all sets in $\mathcal{X}$, i.e., $|\mathcal{X}| := \bigcup \mathcal{X}$. Note that $\mathcal{X}$ is a covering of its support. Given a finite family of sets $\mathcal{X}$ we build an abstract simplicial complex, called the *nerve* of $\mathcal{X}$ and defined by

$$N(\mathcal{X}) := \left\{ \mathcal{S} \subset \mathcal{X} \mid \bigcap \mathcal{S} \neq \emptyset \right\}.$$

It is straightforward to verify that $N(\mathcal{X})$ satisfies

$$V\bigl(N(\mathcal{X})\bigr) = \mathcal{X}. \tag{2}$$

We denote the subfamily of q-dimensional simplices, the chain complex and the homology groups of the nerve of $\mathcal{X}$, respectively, by $N_q(\mathcal{X})$, $C_\#(\mathcal{X})$ and $H_*(\mathcal{X})$.

If $\mathcal{X}$ and $\mathcal{Y}$ are two families of sets and

$$\mathcal{F} : \mathcal{X} \to \mathcal{Y}$$

is a map, then by (2) the map may be viewed as a map acting on the set of vertices of $K(\mathcal{X})$. We say that $\mathcal{F}$ is *simplicial* if it is simplicial with respect to $N(\mathcal{X})$. If $\mathcal{F}$ is simplicial then there is an induced chain map

$$C_\#(\mathcal{F}) := \mathcal{F}_\# : C_\#(\mathcal{X}) \to C_\#(\mathcal{Y})$$

and a map induced in homology

$$H_*(\mathcal{F}) := \mathcal{F}_* : H_*(\mathcal{X}) \to H_*(\mathcal{Y}).$$

If $\mathcal{X}$ constitutes a family of subsets of a topological space, a natural question arises if the homology of $\mathcal{X}$ is isomorphic to the singular homology of the support of $\mathcal{X}$. In general, the answer is negative. However, under some assumptions about $\mathcal{X}$, this may happen [5, 7, 22, 35, 36]. Results of this type are usually referred to as the Nerve Theorem. Among the simplest settings when the Nerve Theorem holds is the case when all elements of the covering are non-empty, compact, convex subsets of $\mathbb{R}^d$. We will call this type of covering a *Čech structure*. A space which is the support of a Čech structure will be referred to as a *Čech polyhedron*. If $\mathcal{X}$ is a Čech structure such that for some Čech polyhedron X we have $|\mathcal{X}| = X$, then we say that $\mathcal{X}$ is a Čech structure on X.

In Sect. 11, we prove Theorem 11.3 which in particular implies the following form of Nerve Theorem as a straightforward corollary.

Theorem 4.1 *Let $\mathcal{X}$ be a Čech structure. There is a well defined chain map $\varphi^{\mathcal{X}} : C_\#(\mathcal{X}) \to C_\#(|\mathcal{X}|)$ which induces an isomorphism of the homology $H_*(\mathcal{X})$ of the Čech structure $\mathcal{X}$ and the singular homology $H_*(|\mathcal{X}|)$ of the support of $\mathcal{X}$.*

The theorem makes Čech structures a valuable alternative for representing sets whose homology groups have to be computed. Consider the following example. Let

$$\begin{aligned} A &:= [-5,-1] \times [1,3], & B &:= [-1,1] \times [2,5], \\ C &:= [1,5] \times [1,5], & D &:= [2,5] \times [-1,1], \\ E &:= [1,4] \times [-5,-1], & F &:= [-2,1] \times [-3,-2], \\ G &:= [-4,-2] \times [-3,-1], & H &:= [-3,-2] \times [-1,1]. \end{aligned}$$

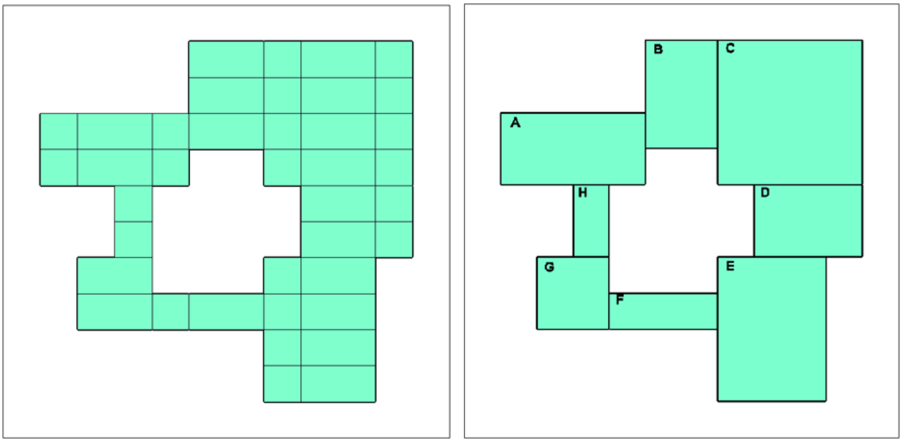

Fig. 1 A planar set with a cubical representation (*left*) and Čech representation (*right*)

Then

$$X_0 := A \cup B \cup C \cup D \cup E \cup F \cup G \cup H \tag{3}$$

is a planar set (see Fig. 1) with an 8-element Čech structure

$$\mathcal{X}_0 := \{A, B, C, D, E, F, G, H\}. \tag{4}$$

Now let us look at the cubical structures of X_0, i.e., families of elementary (unit) cubes whose union is X_0. Recall that such structures may also be used to compute the homology of X_0 (see [15]). It is easy to check that among all possible cubical structures of X_0 with various choices of the unit, the minimal representation requires 47 elementary cubes. When we rescale the sets in $\mathcal{X}_0$ by an integer factor n the situation does not change: we still need 8 elements in the Čech structure and 47 elements in the cubical structure. This is because we can change the unit of cubical structure from 1 to n. However, a simple modification of this example consisting in applying the rescaling to all sets in $\mathcal{X}_0$ except F and rescaling F only in the horizontal direction shows that there exists a sequence of Čech polyhedrons X_n such that the size of the minimal cubical representation of X_n goes to infinity with n, whereas the size of the minimal Čech structure of X_n does not depend on n. Therefore, the Čech structures may be really efficient.

The example may seem to be artificial, but this is what happens when sets exhibit nonuniform or fractal structure, a phenomenon often observed in dynamics. In particular, the nonuniform structure may appear if some parts of the set need some fine-tuning to guarantee some properties. For instance, consider an asymmetric ring R, i.e., the difference of a disk and an internally tangent subdisc (see Fig. 2). If we need a representable covering of R whose Hausdorff distance from R is not greater than a prescribed $\epsilon > 0$ then it is easy to see that the size of the covering consisting of cubes of size ϵ will be proportional to $\frac{1}{\epsilon^2}$, whereas the size will be proportional to $\frac{1}{\epsilon}$ in the case of a non-uniform Čech structure consisting of cubes with the smallest cube size ϵ.

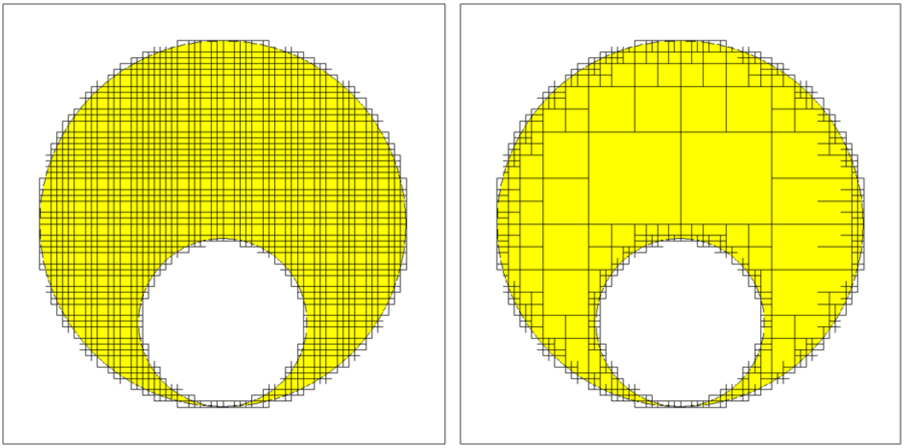

Fig. 2 An asymmetric ring with a cubical representation of 2708 cubes (*left*) and Čech representation of 662 cubes (*right*)

5 Computing Homology of Continuous Maps

We now turn our attention to computing homology of maps. Assume $X \subset \mathbb{R}^d$ and $Y \subset \mathbb{R}^{d'}$ are Čech polyhedrons with some fixed Čech structures $\mathcal{X}$ and $\mathcal{Y}$, and $f : X \to Y$ is a continuous map. Let $\mathcal{A} \subset \mathrm{Conv}(\mathbb{R}^{d'})$ be an arbitrary family, i.e., not necessarily a Čech structure in Y. We say that a map

$$\mathcal{F} : \mathcal{X} \to \mathcal{A} \tag{5}$$

is an *enclosure* of f if

$$f(A) \subset \mathcal{F}(A) \tag{6}$$

for every $A \in \mathcal{X}$. If additionally

$$\mathcal{F}(A) \subset f(A)^\epsilon$$

for some $\epsilon > 0$ and every $A \in \mathcal{X}$, then we say that $\mathcal{F}$ is an ϵ-*enclosure* of f. Note that in (6) the notation $f(A)$ stands for the image of the set A under the map f, whereas $\mathcal{F}(A)$ denotes the value of the map $\mathcal{F}$ at the element $A \in \mathcal{X}$.

Of special interest is an enclosure with values in $\mathcal{Y}$, i.e., a map $\mathcal{F} : \mathcal{X} \to \mathcal{Y}$ which is an enclosure of f. It is straightforward to verify that such an enclosure is a simplicial map, and therefore it induces a homomorphism of the homology groups of the Čech structures $\mathcal{X}$ and $\mathcal{Y}$. In Sect. 7, we will prove that any two such enclosures induce the same homomorphism in homology (see Theorem 7.1). This strongly suggests that the homology of an enclosure of f carries the information about the homology of f. Indeed, in Sect. 11 we prove that the singular homology of $f : X \to Y$ and the homology of an enclosure $\mathcal{F} : \mathcal{X} \to \mathcal{Y}$ coincide up to an isomorphism (see Theorem 11.1).

All this suggests that a way to compute the homology of f is to provide an algorithm constructing an enclosure $\mathcal{F} : \mathcal{X} \to \mathcal{Y}$ of f. However, this simple idea needs some modifications to work in practice. To see why, consider the fundamental question how a continuous map f can be sent to an algorithm. In most applications, in particular in applications to dynamical systems, the map f itself is given in the form of an algorithm. The algorithm takes on input an argument x from the domain of f and an $\epsilon > 0$ and returns a $y \in \mathbb{R}^{d'}$ such that $\operatorname{dist}(y, f(x)) \le \epsilon$. Although such an algorithm may be run only for a finite number of arguments, it is not difficult to extend it (see [20, 28]) to an algorithm, called the *enclosure algorithm* which, given a convex set $A \subset X$, constructs a set $B \in \operatorname{Conv}(\mathbb{R}^{d'})$ such that

$$f(A) \subset B \subset f(A)^\epsilon. \tag{7}$$

Denote the enclosure algorithm by Φ_f^ϵ and let $\Phi_f^\epsilon(A)$ stand for the output of Φ_f^ϵ applied to input A. Running the enclosure algorithm on all elements of $\mathcal{X}$ we obtain an enclosure of f. Unfortunately, even though f maps X into Y, the enclosure algorithm does not have the feature that $\Phi_f^\epsilon(A) \subset Y$ for $A \subset X$, hence the enclosure need not have its values in $\mathcal{Y}$. This is why we need a modification of our approach. The point is that by (7) we have

$$\Phi_f^\epsilon(A) \subset Y^\epsilon. \tag{8}$$

Therefore, Φ_f^ϵ may be viewed as a map

$$\Phi_f^\epsilon : \mathcal{X} \to \Phi_f^\epsilon(\mathcal{X}) \cup \mathcal{Y}^\epsilon,$$

where

$$\mathcal{Y}^\epsilon := \{T^\epsilon \mid T \in \mathcal{Y}\}$$

is a Čech structure on Y^ϵ. Note that Property (8) implies that

$$\left|\Phi_f^\epsilon(\mathcal{X}) \cup \mathcal{Y}^\epsilon\right| = \left|\mathcal{Y}^\epsilon\right|.$$

Therefore, $\Phi_f^\epsilon(\mathcal{X}) \cup \mathcal{Y}^\epsilon$ is another Čech structure on Y^ϵ. As we will show in Sect. 13 (see Theorem 13.1), the map

$$\iota^\epsilon : \mathcal{Y} \ni T \mapsto T^\epsilon \in \Phi_f^\epsilon(\mathcal{X}) \cup \mathcal{Y}^\epsilon$$

is a simplicial map which induces an isomorphism in homology for sufficiently small $\epsilon > 0$. Therefore, the way to get a useful algorithm computing the homology of f is to construct an ϵ-enclosure for $\epsilon > 0$ sufficiently small.

In the case of rational functions, the standard method used to obtain enclosures is via interval arithmetic (see [26]). The idea is very simple. We assume a finite subset $\hat{\mathbb{R}} \subset \mathbb{R}$ of the set of real numbers, the so-called *representable numbers*, is given. We consider the set $\mathbb{I}$ of all closed intervals with endpoints in $\hat{\mathbb{R}}$. Let $\diamond \in \{+, -, *, /\}$ be an arithmetic operation. Given $I, J \in \mathbb{I}$ we define $I \diamond J$ as the minimal element of $\mathbb{I}$ which contains $a \diamond b$ for all $a \in I$, $b \in J$. Now, evaluating the rational function on intervals instead of numbers, we obtain the required enclosure.

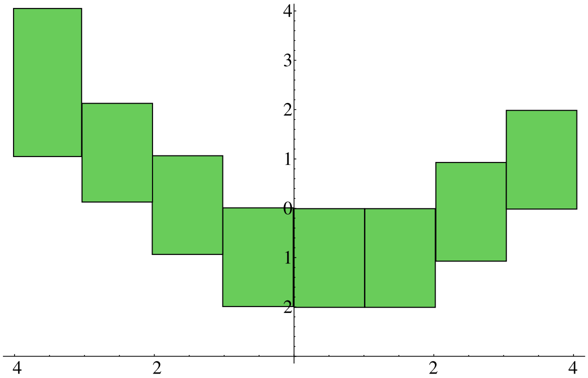

Fig. 3 A polynomial map and its enclosure

Consider the following elementary example. Let $X = [-4, 4]$, $Y = [-\frac{3}{2}, 4]$ and let $f : X \to Y$ be defined by the formula

$$X \ni x \mapsto f(x) := \frac{1}{60}(3x - 8)(5x + 8) \in Y.$$

It is straightforward to verify that f is indeed well defined. Take

$$\mathcal{X} := \{[n, n+1] \mid n = -4, -3, -2, -1, 0, 1, 2, 3\},$$
$$\mathcal{Y} := \left\{ \left[\frac{n}{2}, \frac{n+1}{2} \right] \,\middle|\, n = -3, -2, -1, 0, 1, 2, 3 \right\}.$$

Then $\mathcal{X}$ is a Čech structure on X and $\mathcal{Y}$ is a Čech structure on Y. Now assume for simplicity that the set of representable numbers consists of all integers in the interval $[-1000, 1000]$. We obtain an enclosure $\mathcal{F}$ of f by evaluating our polynomial in our interval arithmetic. The map f and its enclosure $\mathcal{F}$ are presented in Fig. 3. For instance, the computations on interval $[0, 1]$ yield

$$\begin{aligned}(3 * [0, 1] - 8) * (5 * [0, 1] + 8)/60 &= [-8, -5] * [8, 13]/60 \\ &= [-104, -40]/60 = [-2, 0],\end{aligned}$$

therefore $\mathcal{F}([0, 1]) = [-2, 0] \not\subset Y$, which explains the need of considering the Čech structure $\mathcal{Y}^\epsilon$. In our example, taking $\epsilon = \frac{1}{2}$ would suffice.

The discussion and example lead to the following definition. We say that a quintuple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{E}, \mathcal{F})$, where $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are Čech structures and $\mathcal{E} : \mathcal{Y} \to \mathcal{Z}$, $\mathcal{F} : \mathcal{X} \to \mathcal{Z}$, is a *representation* of f if the following four conditions are satisfied

(i) $\mathcal{X}$ is a Čech structure on X and $\mathcal{Y}$ is a Čech structure on Y,
(ii) $Y \subset |\mathcal{Z}|$,
(iii) $\mathcal{E}$ is an enclosure of the inclusion map $i : Y \to |\mathcal{Z}|$ which induces an isomorphism in homology,
(iv) $\mathcal{F}$ is an enclosure of f.

Note that the last condition in particular implies that $\mathcal{F}$ is simplicial. In the sequel, we will often abuse our terminology and refer to the map $\mathcal{F}$ itself as the representation of f, by assuming that the associated Čech structures $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ may be guessed from the context.

In Sect. 13, we will prove the following theorem.

Theorem 5.1 *A representation of $f : X \to Y$ always exists. Moreover, if $\mathcal{F}$ is a representation of f, then the following diagram, in which $\varphi^{\mathcal{X}}$ and $\varphi^{\mathcal{Y}}$ denote the chain maps mentioned in Theorem 4.1, commutes.*

$$\begin{array}{ccc} H_*(X) & \xrightarrow{H_*(f)} & H_*(Y) \\ \uparrow{\scriptstyle H_*(\varphi^{\mathcal{X}})} & & \uparrow{\scriptstyle H_*(\varphi^{\mathcal{Y}})} \\ H_*(\mathcal{X}) & \xrightarrow{H_*(\mathcal{E})^{-1}H_*(\mathcal{F})} & H_*(\mathcal{Y}) \end{array} \tag{9}$$

6 Homology Map Algorithm

As we will see, the proof of Theorem 5.1 is constructive and it may be turned into algorithm `homologyMap` for finding the homology of a continuous map. The algorithm is presented in Table 1. It uses two auxiliary algorithms `inverseIsSimplicial` and `subdivide`. The first verifies if the inverse of the map ι^ϵ is simplicial. The second returns a subdivision of a set in the Čech structure. By a subdivision of a set S in the Čech structure $\mathcal{X}$ we mean a Čech structure $\mathcal{W}$ on S such that $\operatorname{diam} \mathcal{W} \le \operatorname{diam} S/2$. The construction of algorithms `inverseIsSimplicial` and `subdivide` depends on the type of the sets used in the Čech structures. In most cases, the construction is very elementary. We leave the details to the reader.

The algorithm `homologyMap` accepts on input some Čech structures $\mathcal{X}$ in X and $\mathcal{Y}$ in Y, an algorithm Φ approximating a continuous map $f : X \to Y$ and an initial guess of the parameter ϵ.

We say that an algorithm Φ properly encloses a continuous map $f : X \to Y$ on sets in some family $\mathcal{S}$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $S \in \mathcal{S}$ and $\operatorname{diam} S \le \delta$ then the set $\Phi(S)$ returned by Φ satisfies $\operatorname{diam} \Phi(S) \le \epsilon$ and $f(S) \subset \Phi(S)$.

The following theorem will be proved in Sect. 13.

Theorem 6.1 *Assume Algorithm* `homologyMap` *is started with $\mathcal{X}$, $\mathcal{Y}$, Φ, ϵ such that*

(i) *ϵ is positive,*
(ii) *$\mathcal{X}$ and $\mathcal{Y}$ are Čech structures respectively on X and Y,*
(iii) *Φ properly encloses a continuous map $f : X \to Y$ on sets in some family $\mathcal{S}$,*
(iv) *the elements of $\mathcal{X}$ as well as the elements of families produced by applying the iterates of* `subdivide` *to $\mathcal{X}$ belong to $\mathcal{S}$.*

Table 1 Homology map algorithm

```
procedure recurse(𝒳, Φ, ε, var 𝒯, var ℱ)
  foreach S ∈ 𝒳 do
    T := Φ(S);
    if diam T ≤ ε then
      insert T to 𝒯;
      insert (S, T) to ℱ;
    else
      recurse(subdivide(S), Φ, ε, 𝒯, ℱ);
    endif;
  endforeach;
endprocedure;

function homologyMap(𝒳, 𝒴, Φ, ε)
  ι^ε := {(Y, Y^ε) | Y ∈ 𝒴};
  while not inverseIsSimplicial(ι^ε) do ε := ε/2;
  ℱ := ∅;
  𝒯 := ∅;
  recurse(𝒳, Φ, ε, 𝒯, ℱ);
  𝒵 := 𝒴^ε ∪ 𝒯;
  ι' := inclusion map 𝒴^ε → 𝒵;
  ℰ := ι'ι^ε;
  return H_*(ℰ)^{-1} H_*(ℱ);
endfunction;
```

Then the algorithm always stops and returns a homology map which is conjugate to the singular homology of f.

As an example consider the quadratic map on the Euclidean space $\mathbb{R}^2$ treated as the complex plane $\mathbb{C}$, given by

$$\mathbb{C} \ni z \mapsto z^2 \in \mathbb{C}.$$

This map, rewritten in Cartesian coordinates $(x, y) \in \mathbb{R}^2$, where $z = x + iy$, is

$$f_0 : \mathbb{R}^2 \ni (x, y) \mapsto \left(x^2 - y^2, 2xy\right) \in \mathbb{R}^2.$$

One can check that f maps the set X_0 given by (3) into the set

$$Y_0 := K \cup L \cup M \cup N,$$

where

$$K := [-54, 54] \times [2, 52], \qquad L := [-54, 54] \times [-52, -2],$$
$$M := [-32, -2] \times [-54, 54], \qquad N := [2, 32] \times [-54, 54].$$

Therefore, we have a restricted map

$$f_0|_{X_0} : X_0 \to Y_0.$$

Treating the elements of the Čech structure $\mathcal{X}_0$ as pairs of intervals and evaluating the map f_0 in interval arithmetic for the elements of $\mathcal{X}_0$, we obtain an algorithm Φ_0 which properly encloses f_0 on family $\mathcal{S}_0$ of Cartesian products of two intervals. In particular, for the elements of the Čech structure $\mathcal{X}_0$ given by (4) the algorithm respectively returns

$$\begin{aligned} A' &:= [-8, 24] \times [-30, -2], \\ B' &:= [-26, -3] \times [-10, 10], \\ C' &:= [-24, 24] \times [2, 50], \\ D' &:= [3, 26] \times [-10, 10], \\ E' &:= [-24, 15] \times [-40, -2], \\ F' &:= [-11, 0] \times [-6, 12], \\ G' &:= [-5, 15] \times [4, 24], \\ H' &:= [3, 10] \times [-6, 6]. \end{aligned}$$

In particular, we see that $\Phi_0(S) \subset Y_0$ for all $S \in \mathcal{X}_0 \setminus \{F\}$ and there is no $\epsilon > 0$ such that $F' = \Phi_0(F) \subset |Y_0^\epsilon$. Therefore, the procedure `recurse` will call itself recursively for a subdivision of F. Already the simplest subdivision consisting of F_1, F_2 given by

$$\begin{aligned} F_1 &:= [-2, -1] \times [-3, -2], \\ F_2 &:= [-1, 1] \times [-3, -2] \end{aligned}$$

leads to the respective enclosures

$$\begin{aligned} F_1' &:= [-8, 0] \times [4, 12], \\ F_2' &:= [-10, -3] \times [-6, 6], \end{aligned}$$

which are contained entirely in Y. The subdivided Čech structure $\mathcal{X}_0' := \mathcal{X}_0 \setminus \{F\} \cup \{F_1, F_2\}$ together with the enclosures given by Φ_0 are presented in Fig. 4. It is straightforward to verify that the resulting homology map is correct: the homology generator is multiplied by 2. Notice that in this example there is no need to use the set Y_0^ϵ. This is because the interval computations involved are performed on integers and no division is performed, so there is no rounding bound introduced.

Proceeding similarly to the example given in Sect. 4, we may easily extend this example to the case of a sequence of maps f_n such that the cost of finding the homology of f_n by means of the algorithm presented here is constant, whereas the costs of applying the algorithms in [21] and in [25] are supercubical.

7 Comparing Enclosures

We now address the question of comparing two enclosures of the same continuous map.

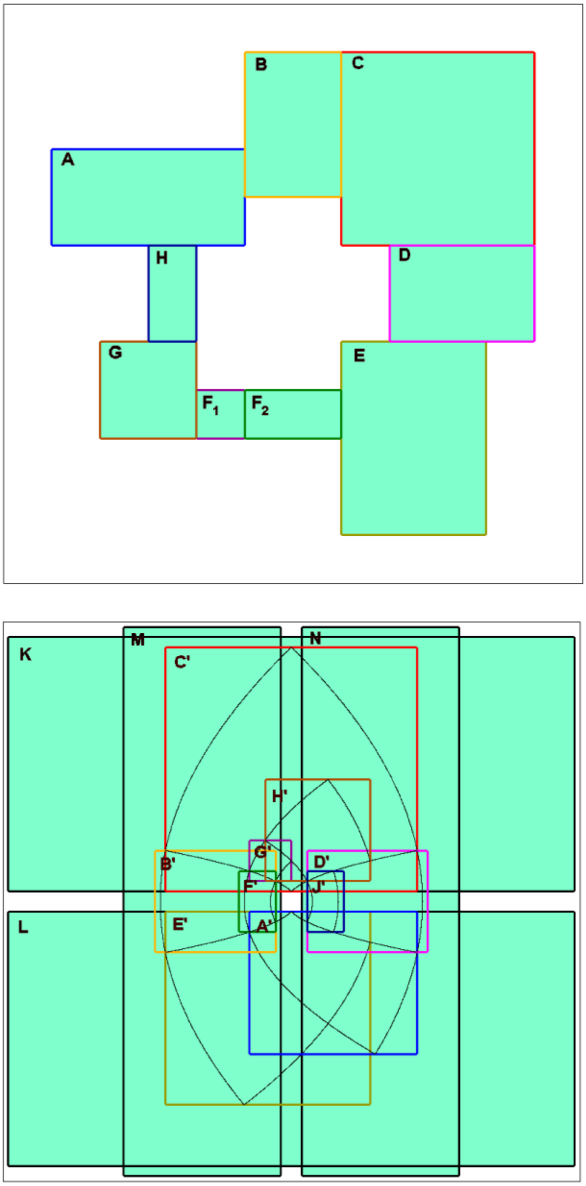

Fig. 4 Set X_0 with Čech structure (*top*) and its image under $z \to z^2$ map (*bottom*). *Curved lines* indicate the approximate images under f_0 of the elements in $\mathcal{X}_0'$. The enclosing boxes obtained by evaluating f_0 in the interval arithmetic are marked with a capital character and a prime

Theorem 7.1 *Assume $\mathcal{X}, \mathcal{Y}$ are Čech structures and $\mathcal{F}, \mathcal{G} : \mathcal{X} \to \mathcal{Y}$ are two enclosures of a continuous map $f : |\mathcal{X}| \to |\mathcal{Y}|$. Then*

$$H_*(\mathcal{F}) = H_*(\mathcal{G}).$$

In order to prove this result, we need the following lemma.

Lemma 7.2 *Assume $\mathcal{F}, \mathcal{G} : \mathcal{X} \to \mathcal{Y}$ are two maps such that*

$$\bigcap \big(\mathcal{F}(\mathcal{S}) \cup \mathcal{G}(\mathcal{S})\big) \neq \emptyset$$

for every $\mathcal{S} \in N(\mathcal{X})$. Then both maps are simplicial and chain homotopic.

Proof Obviously both maps are simplicial. To prove that they are chain homotopic we will show that the maps $\mathcal{F}$ and $\mathcal{G}$ are contiguous in the sense that for every simplex $\mathcal{S} \in N(\mathcal{X})$ the simplices $\mathcal{F}(\mathcal{S})$ and $\mathcal{G}(\mathcal{S})$ are contained in a common simplex of $N(\mathcal{Y})$. Indeed, defining

$$\Phi(\mathcal{S}) := N\big(\mathcal{F}(\mathcal{S}) \cup \mathcal{G}(\mathcal{S})\big)$$

for $\mathcal{S} \in N(\mathcal{X})$, we see that $\Phi(\mathcal{S})$ is such a simplex. Thus it is a cone and by Theorem 3.2 it is acyclic. Therefore, it is straightforward to verify that Φ is an acyclic carrier which carries $C_\#(\mathcal{F})$ and $C_\#(\mathcal{G})$, and the thesis follows from Theorem 3.3. □

Proof of Theorem 7.1 We have

$$\bigcap \big(\mathcal{F}(\mathcal{S}) \cup \mathcal{G}(\mathcal{S})\big) = \bigcap \mathcal{F}(\mathcal{S}) \cap \bigcap \mathcal{G}(\mathcal{S}) \supset f\Big(\bigcap \mathcal{S}\Big) \neq \emptyset.$$

Therefore, the conclusion follows from Lemma 7.2. □

8 Embeddings

In this section, we introduce the technical concept of an embedding needed in particular in the proof of Mayer–Vietoris Theorem in Sect. 10.

Let $\mathcal{F} : \mathcal{X} \to \mathcal{Y}$ be a map of Čech structures. We say that $\mathcal{F}$ is an *embedding* if for every $S \in \mathcal{X}$ the set $\mathcal{F}(S) \in \mathcal{Y}$ contains S. Obviously, every embedding is simplicial. Observe that if $\mathcal{X} \subset \mathcal{Y}$ then the inclusion map

$$\iota : \mathcal{X} \ni S \mapsto S \in \mathcal{Y}$$

is an embedding.

Lemma 8.1 *Assume $\mathcal{X}, \mathcal{Y}$ are Čech structures such that $|\mathcal{X}| \subset |\mathcal{Y}|$. If*

$$\mathcal{F}, \mathcal{G} : \mathcal{X} \to \mathcal{Y}$$

are embeddings then

$$H_*(\mathcal{F}) = H_*(\mathcal{G}).$$

Proof Since both $\mathcal{F}$ and $\mathcal{G}$ are enclosures of the inclusion $\iota : |\mathcal{X}| \to |\mathcal{Y}|$, the conclusion follows immediately from Theorem 7.1. □

Corollary 8.2 *Assume $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ is an embedding. Then*

$$H_*(\mathcal{F}) = \mathrm{id}_{H_*(\mathcal{X})}.$$

We say that $\mathcal{X}$ is *embedded* in $\mathcal{Y}$ and write $\mathcal{X} \sqsubset \mathcal{Y}$ if there exists an embedding $\mathcal{F}: \mathcal{X} \to \mathcal{Y}$. This is equivalent to saying that for every $S \in \mathcal{X}$ there exists a $T \in \mathcal{Y}$ such that $S \subset T$.

Lemma 8.3 *Assume $\mathcal{X} \subset \mathcal{Y}$ and $\mathcal{Y} \sqsubset \mathcal{X}$. Then the inclusion $\iota : \mathcal{X} \subset \mathcal{Y}$ induces an isomorphism in homology.*

Proof Let $\mathcal{F}: \mathcal{Y} \to \mathcal{X}$ be an embedding. Then also $\iota\mathcal{F}: \mathcal{Y} \to \mathcal{Y}$ is an embedding. Redefining $\mathcal{F}$ if necessary, we may assume that $\mathcal{F}\iota = \mathrm{id}_{\mathcal{X}}$. Then $\mathcal{F}_*\iota_* = \mathrm{id}$ and by Corollary 8.2 also $\iota_*\mathcal{F}_* = (\iota\mathcal{F})_* = \mathrm{id}$. The conclusion follows. □

9 Order Complexes

There is another way of associating an abstract simplicial complex with a Čech structure. It has some technical advantages which will become clear in the next section. We define it and study its features in this section.

We say that a family $\mathcal{S} \subset \mathcal{P}(\mathbb{R}^d)$ is *monotone* if $\mathcal{S}$ is linearly ordered by inclusion. Let $\mathcal{X}$ be a Čech structure. Put

$$\bar{N}(\mathcal{X}) := \{\mathcal{M} \subset \mathcal{X} \mid \mathcal{M} \text{ is monotone}\}.$$

Obviously we have

$$\bar{N}(\mathcal{X}) \subset N(\mathcal{X}).$$

It is straightforward to verify that $\bar{N}(\mathcal{X})$ is an abstract simplicial complex. It is called the *order complex* [1, 5] and may be thought of as the first barycentric subdivision of $N(\mathcal{X})$. Obviously, we have

$$V(\bar{N}(\mathcal{X})) = \mathcal{X}. \tag{10}$$

The respective chain complex and homology groups will be denoted by $\bar{C}_\#(\mathcal{X})$ and $\bar{H}_*(\mathcal{X})$.

Recall that for $\mathcal{A} \subset \mathcal{P}(\mathbb{R}^d)$ by $\mathcal{A}^*$ we denote the family of all non-empty intersections of finite subfamilies of $\mathcal{A}$. The following theorem shows the relation between the nerve of a Čech structure $\mathcal{X}$ and the order complex of $\mathcal{X}^*$.

Theorem 9.1 *Let $\mathcal{X}$ be a Čech structure. There is a unique chain equivalence*

$$\zeta : C_\#(\mathcal{X}) \to \bar{C}_\#(\mathcal{X}^*)$$

such that $\zeta(\mathcal{S}) \in \bar{C}_\#(\mathcal{S}^*)$ for $\mathcal{S} \in N(\mathcal{X})$.

Proof Recall that for $C \subset \mathbb{R}^d$ and $\mathcal{X} \subset \mathcal{P}(\mathbb{R}^d)$ we use the notation $\mathcal{X}(C) := \{A \in \mathcal{X} \mid C \subset A\}$. For $\mathcal{S} \in N(\mathcal{X})$, we put

$$\Psi(\mathcal{S}) := N\Big(\mathcal{X}\Big(\bigcap \mathcal{S}\Big)\Big), \tag{11}$$

$$\Lambda(\mathcal{S}) := \bar{N}(\mathcal{S}^*), \tag{12}$$

and for $\mathcal{M} \in \bar{N}(\mathcal{X}^*)$ we put

$$\Phi(\mathcal{M}) := \bar{N}\Big(\mathcal{X}\Big(\bigcap \mathcal{M}\Big)^*\Big), \tag{13}$$

$$\Theta(\mathcal{M}) := N\Big(\mathcal{X}\Big(\bigcap \mathcal{M}\Big)\Big). \tag{14}$$

One easily verifies that $\Psi, \Lambda, \Phi, \Theta$ are acyclic carriers and Ψ and Φ respectively carry $\mathrm{id}_{C_\#(\mathcal{X})}$ and $\mathrm{id}_{\bar{C}_\#(\mathcal{X}^*)}$. Let $\zeta : C_\#(\mathcal{X}) \to \bar{C}_\#(\mathcal{X}^*)$ and $\theta : \bar{C}_\#(\mathcal{X}^*) \to C_\#(\mathcal{X})$ be chain maps respectively carried by Λ and Θ. It is easy to verify that $\zeta\theta$ is carried by Ψ and $\theta\zeta$ is carried by Φ. Therefore, by Theorem 3.3, $\zeta\theta$ is chain homotopic to $\mathrm{id}_{C_\#(\mathcal{X})}$ and $\theta\zeta$ is chain homotopic to $\mathrm{id}_{\bar{C}_\#(\mathcal{X}^*)}$. Thus ζ is a chain equivalence. Its uniqueness follows from the fact that for $\mathcal{S} \in N_q(\mathcal{X})$ there are no $(q+1)$-dimensional simplices in $\bar{N}(\mathcal{S}^*)$. □

Corollary 9.2 *The unique chain equivalence $\zeta : C_\#(\mathcal{X}) \to \bar{C}_\#(\mathcal{X}^*)$ induces an isomorphism $\zeta_* : H_*(\mathcal{X}) \to \bar{H}_*(\mathcal{X}^*)$.*

Let $\mathcal{X}$ and $\mathcal{Y}$ be two Čech structures and let

$$\mathcal{F} : \mathcal{X} \to \mathcal{Y}$$

be a map. By (10), the map $\mathcal{F}$ may be viewed as a map acting on the set of vertices of $\bar{N}(\mathcal{X})$. We say that $\mathcal{F}$ is *monotonically simplicial* if it is simplicial with respect to $\bar{N}(\mathcal{X})$ and $\bar{N}(\mathcal{Y})$. Observe that if $\mathcal{X} \subset \mathcal{Y}$ then the inclusion map

$$\iota : \mathcal{X} \ni S \mapsto S \in \mathcal{Y}$$

is monotonically simplicial.

Proposition 9.3 *If $\mathcal{X}$ contains a unique minimal element then $\bar{N}(\mathcal{X})$ is a cone.*

Assume $\mathcal{F}, \mathcal{G} : \mathcal{X} \to \mathcal{Y}$. We write $\mathcal{F} \subset \mathcal{G}$ if $\mathcal{F}(S) \subset \mathcal{G}(S)$ for each $S \in \mathcal{X}$.

Lemma 9.4 *Assume $\mathcal{F}, \mathcal{G} : \mathcal{X} \to \mathcal{Y}$ are such that $\mathcal{F} \subset \mathcal{G}$. If they are both monotonically simplicial then they are monotonically chain homotopic.*

Proof For $\mathcal{S} \in \bar{N}(\mathcal{X})$ define

$$\bar{\Phi}(\mathcal{S}) := \bar{N}\big(\mathcal{F}(\mathcal{S}) \cup \mathcal{G}(\mathcal{S})\big).$$

By Theorem 3.2 and Proposition 9.3, the abstract simplicial complex $\bar{\Phi}(\mathcal{S})$ is acyclic. It is straightforward to verify that it is an acyclic carrier which carries $\bar{C}_\#(\mathcal{F})$ and $\bar{C}_\#(\mathcal{G})$. Therefore, the thesis follows from Theorem 3.3. □

Theorem 9.5 *If $\mathcal{F}:\mathcal{X}\to\mathcal{Y}$ is simplicial then the diagram*

$$\begin{array}{ccc} H_*(\mathcal{X}) & \xrightarrow{H_*(\mathcal{F})} & H_*(\mathcal{Y}) \\ \downarrow{\scriptstyle \zeta_*} & & \downarrow{\scriptstyle \zeta_*} \\ \bar{H}_*(\mathcal{X}^*) & \xrightarrow{\bar{H}_*(\mathcal{F}^*)} & \bar{H}_*(\mathcal{Y}^*) \end{array}$$

commutes.

Proof One easily verifies that

$$\Theta(\mathcal{S}) := \bar{N}\Big(\mathcal{F}\Big(\mathcal{X}\Big(\bigcap\mathcal{S}\Big)\Big)^*\Big)$$

is an acyclic carrier for both $\zeta_\# C_\#(\mathcal{F})$ and $\bar{C}_\#(\mathcal{F}^*)\zeta_\#$. Therefore, the conclusion follows from Theorem 3.3. □

10 Mayer–Vietoris Sequence

The aim of this section is to prove the Mayer–Vietoris Theorem for Čech structures. For $\mathcal{A},\mathcal{B}\subset\mathcal{P}(\mathbb{R}^d)$, we define

$$\mathcal{A}\,\bar{\cap}\,\mathcal{B} := \{A\cap B \mid A\in\mathcal{A},\ B\in\mathcal{B},\ A\cap B\neq\emptyset\}.$$

Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two Čech structures. Then, obviously, $\mathcal{X}_1\,\bar{\cap}\,\mathcal{X}_2$ is a Čech structure and $\mathcal{X}_1\,\bar{\cap}\,\mathcal{X}_2\sqsubset\mathcal{X}_1$, $\mathcal{X}_1\,\bar{\cap}\,\mathcal{X}_2\sqsubset\mathcal{X}_2$. Let

$$\mu_i : \mathcal{X}_1\,\bar{\cap}\,\mathcal{X}_2\to\mathcal{X}_i$$

be embeddings and let

$$\nu_i : \mathcal{X}_i\to\mathcal{X}_1\cup\mathcal{X}_2$$

denote inclusion maps.

Theorem 10.1 *The sequence*

$$\begin{aligned} \cdots\to H_q(\mathcal{X}_1\,\bar{\cap}\,\mathcal{X}_2) &\xrightarrow{(\mu_{1*q},-\mu_{2*q})} H_q(\mathcal{X}_1)\oplus H_q(\mathcal{X}_2) \\ &\xrightarrow{\nu_{1*q}+\nu_{2*q}} H_q(\mathcal{X}_1\cup\mathcal{X}_2)\to H_{q-1}(\mathcal{X}_1\,\bar{\cap}\,\mathcal{X}_2)\to\cdots \end{aligned} \tag{15}$$

is exact.

To present the proof of this theorem, we need some auxiliary definitions and results. For $\mathcal{A},\mathcal{B}\subset\mathcal{P}(\mathbb{R}^d)$, we define

$$\mathcal{A}_\mathcal{B} := \big(\mathcal{A}\cup(\mathcal{A}\,\bar{\cap}\,\mathcal{B})\big)^*.$$

As we will see in the following proofs, the Čech structures $\mathcal{A}_{\mathcal{B}}$ and $\mathcal{B}_{\mathcal{A}}$ may be considered as refinements of the Čech structures $\mathcal{A}$ and $\mathcal{B}$ with the same homology, but behaving nicer with respect to the union and intersection operations.

Proposition 10.2 *We have the following properties*:

(i) *If* $\mathcal{A} \subset \mathcal{B}$ *then* $\mathcal{A}^* \subset \mathcal{B}^*$,
(ii) $\mathcal{A}_{\mathcal{B}} = \{C_0 \cap C_1 \cap C_2 \cap \cdots \cap C_k \mid C_0 \in \mathcal{A}, C_i \in \mathcal{A} \cup \mathcal{B}\}$,
(iii) $\mathcal{A}_{\mathcal{B}} \cap \mathcal{B}_{\mathcal{A}} = (\mathcal{A} \bar{\cap} \mathcal{B})^*$,
(iv) $\mathcal{A}_{\mathcal{B}} \cup \mathcal{B}_{\mathcal{A}} = (\mathcal{A} \cup \mathcal{B})^*$.

Proof Property (i) is straightforward. To prove (ii), assume that $C = C_0 \cap C_1 \cap C_2 \cap \cdots \cap C_k$ is such that $C_0 \in \mathcal{A}$, $C_i \in \mathcal{A} \cup \mathcal{B}$. Without loss of generality, we may assume that for some $l \in \mathbb{N}_k$ we have $C_i \in \mathcal{A}$ for $i = 0, 1, \ldots, l$ and $C_i \in \mathcal{B}$ for $i = l+1, l+2, \ldots, k$. Since

$$C = C_0 \cap C_1 \cap \cdots \cap C_l \cap (C_0 \cap C_{l+1}) \cap \cdots \cap (C_0 \cap C_k),$$

we conclude that $C \in \mathcal{A}_{\mathcal{B}}$. The opposite inclusion is straightforward. Properties (iii) and (iv) follow easily from (i) and (ii). □

Proposition 10.3 *Assume* $\mathcal{X}, \mathcal{Y}$ *are Čech structures. We have the following properties*:

(i) *If* $\mathcal{X} \subset \mathcal{Y}$ *then* $N(\mathcal{X}) \subset N(\mathcal{Y})$ *and* $\bar{N}(\mathcal{X}) \subset \bar{N}(\mathcal{Y})$,
(ii) $\bar{N}(\mathcal{X}_{\mathcal{Y}}) \cup \bar{N}(\mathcal{Y}_{\mathcal{X}}) = \bar{N}(\mathcal{X}_{\mathcal{Y}} \cup \mathcal{Y}_{\mathcal{X}})$,
(iii) $\bar{N}(\mathcal{X}_{\mathcal{Y}}) \cap \bar{N}(\mathcal{Y}_{\mathcal{X}}) = \bar{N}((\mathcal{X} \bar{\cap} \mathcal{Y})^*)$.

Assume $\mathcal{F} : \mathcal{X} \to \mathcal{Y}$ is simplicial and define $\mathcal{F}^* : \mathcal{X}^* \to \mathcal{Y}^*$ by

$$\mathcal{F}^*(C) := \bigcap \mathcal{F}\big(\mathcal{X}(C)\big) \quad \text{for } C \in \mathcal{X}^*.$$

Proposition 10.4 *If* $\mathcal{F}$ *is an embedding then so is* $\mathcal{F}^*$.

Proof Let $C \in \mathcal{X}^*$. Then

$$\mathcal{F}^*(C) = \bigcap \mathcal{F}\big(\mathcal{X}(C)\big) \supset \bigcap \mathcal{X}(C) \supset C.$$ □

Finally, we are ready to present the proof of Theorem 10.1.

Proof of Theorem 10.1 For $i = 1, 2$, put

$$\mathcal{Y}_i := (\mathcal{X}_i)_{\mathcal{X}_{3-i}} = (\mathcal{X}_i \cup \mathcal{X}_1 \bar{\cap} \mathcal{X}_2)^*.$$

Let

$$\iota_i : \bar{N}(\mathcal{X}_1) \cap \bar{N}(\mathcal{X}_2) \to \bar{N}(\mathcal{X}_i)$$

and

$$\lambda_i : \bar{N}(\mathcal{X}_i) \to \bar{N}(\mathcal{X}_1) \cup \bar{N}(\mathcal{X}_2)$$

denote inclusion maps.

By Theorem 3.1, we have the long exact sequence

$$\cdots \to H_q\big(\bar{N}(\mathcal{Y}_1) \cap \bar{N}(\mathcal{Y}_2)\big) \xrightarrow{(\iota_{1*q}, -\iota_{2*q})} H_q\big(\bar{N}(\mathcal{Y}_1)\big) \oplus H_q\big(\bar{N}(\mathcal{Y}_2)\big)$$
$$\xrightarrow{\lambda_{1*q}+\lambda_{2*q}} H_q\big(\bar{N}(\mathcal{Y}_1) \cup \bar{N}(\mathcal{Y}_2)\big) \to H_q\big(\bar{N}(\mathcal{Y}_1) \cap \bar{N}(\mathcal{Y}_2)\big) \to \cdots .$$

By Proposition 10.3, the sequence may be rewritten as

$$\cdots \to H_q\big(\bar{N}\big((\mathcal{X}_1 \bar{\cap} \mathcal{X}_2)^*\big)\big) \xrightarrow{(\iota_{1*q}, -\iota_{2*q})} H_q\big(\bar{N}(\mathcal{Y}_1)\big) \oplus H_q\big(\bar{N}(\mathcal{Y}_2)\big)$$
$$\xrightarrow{\lambda_{1*q}+\lambda_{2*q}} H_q\big(\bar{N}\big((\mathcal{X}_1 \cup \mathcal{X}_2)^*\big)\big) \to H_q\big(\bar{N}\big((\mathcal{X}_1 \bar{\cap} \mathcal{X}_2)^*\big)\big) \to \cdots ,$$

or, using our shorthand notation, as

$$\cdots \to \bar{H}_q\big((\mathcal{X}_1 \bar{\cap} \mathcal{X}_2)^*\big) \xrightarrow{(\iota_{1*q}, -\iota_{2*q})} \bar{H}_q(\mathcal{Y}_1) \oplus \bar{H}_q(\mathcal{Y}_2)$$
$$\xrightarrow{\lambda_{1*q}+\lambda_{2*q}} \bar{H}_q\big((\mathcal{X}_1 \cup \mathcal{X}_2)^*\big) \to H_{q-1}(\mathcal{X}_1 \cap \mathcal{X}_2) \to \cdots .$$

Let

$$\rho_i : \mathcal{X}_1 \bar{\cap} \mathcal{X}_2 \to \mathcal{X}_i \cup \mathcal{X}_1 \bar{\cap} \mathcal{X}_2$$

denote inclusions and let

$$\chi_i : \mathcal{X}_i \cup \mathcal{X}_1 \bar{\cap} \mathcal{X}_2 \to \mathcal{X}_1 \cup \mathcal{X}_2$$

be any embeddings such that $\chi_{i|\mathcal{X}_i} = \mathrm{id}_{\mathcal{X}_i}$. It follows from Proposition 10.4 and Lemma 9.4 that $(\rho_i^*)_* = \iota_*$ and $(\theta_i^*)_* = \lambda_*$. Therefore, we get from Theorem 9.5 that

$$\cdots \to H_q(\mathcal{X}_1 \bar{\cap} \mathcal{X}_2) \xrightarrow{(\rho_{1*q}, -\rho_{2*q})} H_q(\mathcal{X}_1 \cup \mathcal{X}_1 \bar{\cap} \mathcal{X}_2) \oplus H_q(\mathcal{X}_2 \cup \mathcal{X}_1 \bar{\cap} \mathcal{X}_2)$$
$$\xrightarrow{\theta_{1*q}+\theta_{2*q}} H_q(\mathcal{X}_1 \cup \mathcal{X}_2) \to H_{q-1}(\mathcal{X}_1 \bar{\cap} \mathcal{X}_2) \to \cdots . \tag{16}$$

Finally, let

$$\gamma_i : \mathcal{X}_i \to \mathcal{X}_i \cup \mathcal{X}_1 \bar{\cap} \mathcal{X}_2$$

denote the inclusions. It is straightforward to verify that $\rho_i \subset \gamma_i \mu_i$ and $\nu_i = \chi_i \gamma_i$; therefore, we obtain from Lemma 7.2 and Lemma 8.3 the exact sequence (15). □

11 Homology of Čech Polyhedrons

In this section, we define a chain map from the chain complex of the nerve of a Čech structure to the singular chain complex of the associated Čech polyhedron.

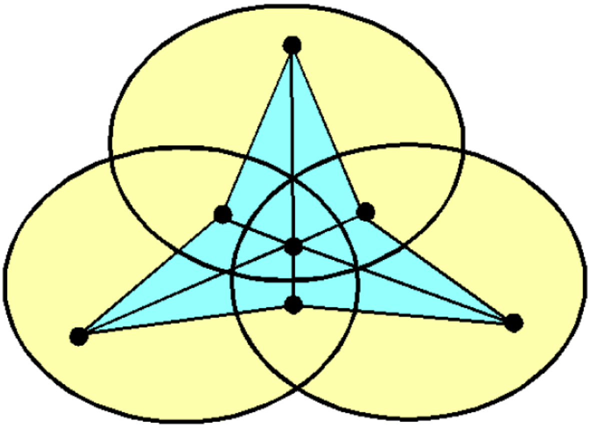

Fig. 5 A simplex in the nerve and the associated singular chain

Recall that given a collection of $n+1$ points $a_0, a_1, \dots, a_n$ in $\mathbb{R}^d$, there is a unique affine map which sends the ith vertex of the standard n-simplex to a_i (see [30, Sect. 29]). We denote this map by $[a_0, a_1, \dots, a_n]$ and call it the *linear singular n-simplex* determined by $a_0, a_1, \dots, a_n$. For $a \in \mathbb{R}^d$, we put

$$a \cdot [a_0, a_1, \dots, a_n] := [a, a_0, a_1, \dots, a_n].$$

If $c = \sum_{i=1}^{m} k_i \sigma_i$ is a linear combination of linear singular simplices, we put

$$a \cdot c := \sum_{i=1}^{m} k_i \, a \cdot \sigma_i.$$

We say that $x = \{x_{\mathcal{S}} \mid \mathcal{S} \in N(\mathcal{X})\}$ is a *selector* of $\mathcal{X}$ if

$$\mathcal{S} \in N(\mathcal{X}) \;\Rightarrow\; x_{\mathcal{S}} \in \bigcap \mathcal{S}.$$

Obviously, by the very definition of $N(\mathcal{X})$, every Čech structure admits at least one selector.

For a selector x of $\mathcal{X}$, we define a chain homomorphism

$$\varphi^x : C_\#(\mathcal{X}) \to C_\#(|\mathcal{X}|)$$

from the chain complex of the Čech structure $\mathcal{X}$ to the singular chain complex of the support of $\mathcal{X}$. The geometric idea behind the construction is presented in Fig. 5. The formal definition proceeds recursively. If $\mathcal{S} \in N_0(\mathcal{X})$ we put

$$\varphi^x(\widehat{S}) := x_S.$$

Assuming φ^x is defined for chains of dimensions less than q and given $\mathcal{S} \in N_q(\mathcal{X})$, we put

$$\varphi^x_q(\widehat{S}) := x_S \cdot \varphi^x_{q-1} \partial(\widehat{S}).$$

We need to verify that φ^x is indeed a chain map, i.e.,

$$\partial_q \varphi_q^x = \varphi_{q-1}^x \partial_q.$$

For $q = 0$, the equality is obvious. So assume $q > 0$. We have

$$\partial_q \varphi_q^x(\widehat{S}) = \partial_q x_S \cdot \varphi_{q-1}^x \partial_q(\widehat{S}) = \varphi_{q-1}^x \partial_q(\widehat{S}) - x_S \cdot \partial_{q-1}\varphi_{q-1}^x \partial_q(\widehat{S}) = \varphi_{q-1}^x \partial_q(\widehat{S})$$

because, by the induction assumption, $\partial_{q-1}\varphi_{q-1}^x \partial_q = \varphi_{q-2}^x \partial_{q-1}\partial_q = 0$.

Theorem 11.1 *Assume $\mathcal{F} : \mathcal{X} \to \mathcal{Y}$ is an enclosure of $f : |\mathcal{X}| \to |\mathcal{Y}|$. If x and y are selectors respectively in $\mathcal{X}$ and $\mathcal{Y}$ then the diagram*

$$\begin{array}{ccc} C_\#(|\mathcal{X}|) & \xrightarrow{f_\#} & C_\#(|\mathcal{Y}|) \\ \uparrow \varphi_\#^x & & \uparrow \varphi_\#^y \\ C_\#(\mathcal{X}) & \xrightarrow{\mathcal{F}_\#} & C_\#(\mathcal{Y}) \end{array}$$

is commutative up to a chain homotopy.

Proof The requested chain homotopy $D_q : C_q(\mathcal{X}) \to C_{q+1}(|\mathcal{Y}|)$ will be constructed by induction with respect to q in such a way that for any $q \in \mathbb{Z}$

$$\partial_{q+1} D_q + D_{q-1}\partial_q = \varphi_q^y \mathcal{F}_{\#q} - f_{\#q}\varphi_q^x \tag{17}$$

and for any $\widehat{S} \in N_q(\mathcal{X})$

$$\left|D_q\left((\widehat{S})\right)\right| \subset \left|\mathcal{F}_{\#q}\left((\widehat{S})\right)\right|. \tag{18}$$

We put $D_q = 0$ for $q < 0$. Let $S \in V(\mathcal{X})$. Since $y_{\mathcal{F}(S)} \in \mathcal{F}(S)$, $f(x_S) \in f(S) \subset \mathcal{F}(S)$ and S as a convex set is acyclic, we have

$$y_{\mathcal{F}(S)} - f(x_S) = \partial c_S$$

for some $c \in C_\#(\mathcal{F}(S))$. We put $D_0([S]) := c_S$. It is straightforward to verify that Conditions (17) and (18) are satisfied for $q \leq 0$.

Now let $i > 0$ and assume now that D_q is defined for all $q < i$ in such a way that Conditions (17) and (18) are satisfied. Let $\widehat{S} \in N_q(\mathcal{X})$. Observe that

$$\begin{aligned} &\partial_i\left(D_{i-1}\partial_i - \varphi_i^y \mathcal{F}_{\#i} + f_{\#i}\varphi_i^x\right)(\widehat{S}) \\ &\quad = \left(-D_{i-2}\partial_{i-1} + \varphi_{i-1}^y \mathcal{F}_{\#(i-1)} - f_{\#(i-1)\varphi_{i-1}^x}\right)\partial_i(\widehat{S}) \\ &\qquad + \left(-\varphi_i^y \mathcal{F}_{\#i} + f_{\#i}\varphi_i^x\right)\partial_i(\widehat{S}) = 0. \end{aligned}$$

Therefore, $(D_{i-1}\partial_i - \varphi_i^y \mathcal{F}_{\#i} + f_{\#i}\varphi_i^x)(\widehat{S})$ is a cycle. Moreover, we have

$$\begin{aligned} &\left|\left(D_{i-1}\partial_i - \varphi_i^y \mathcal{F}_{\#i} + f_{\#i}\varphi_i^x\right)(\widehat{S})\right| \\ &\quad \subset \left|D_{i-1}\partial_i(\widehat{S})\right| \cup \left|\varphi_i^y \mathcal{F}_{\#i}(\widehat{S})\right| \cup \left|f_{\#i}\varphi_i^x)(\widehat{S})\right| \subset \bigcup \mathcal{F}(S). \end{aligned}$$

Hence $(D_{i-1}\partial_i - \varphi_i^y \mathcal{F}_{\#i} + f_{\#i}\varphi_i^x)(\widehat{S}) \in Z_i(\bigcup \mathcal{F}(\widehat{S}))$. But $\bigcup \mathcal{F}(\mathcal{S})$ is acyclic. Therefore, there exists a $c \in C_{i+1}(\bigcup \mathcal{F}(\mathcal{S}))$ such that

$$\partial_{i+1} c = \big(D_{i-1}\partial_i - \varphi_i^y \mathcal{F}_{\#i} + f_{\#i}\varphi_i^x\big)(\widehat{S}).$$

Putting $D_i(\widehat{S}) := c$ one easily verifies that (17) and (18) are satisfied for $q = i$. □

From Theorem 11.1 we obtain the following corollary.

Corollary 11.2 *The chain homotopy class of the map $\varphi^x : C_\#(\mathcal{X}) \to C_\#(|\mathcal{X}|)$ is independent of the choice of a selector x. In particular, any two such maps induce the same map in homology.*

In the sequel, we will write $\varphi^{\mathcal{X}}$ to denote the chain homomorphism $\varphi^x : C_\#(\mathcal{X}) \to C_\#(|\mathcal{X}|)$ for some selector x of $\mathcal{X}$. Since these maps serve only as an intermediate step to obtain a map in homology, this will cause no ambiguity.

We can now prove the following version of the Nerve Theorem.

Theorem 11.3 *Let $\mathcal{X}$ be a Čech structure. Then the map $\varphi^{\mathcal{X}} : C_\#(\mathcal{X}) \to C_\#(|\mathcal{X}|)$ induces an isomorphism in homology.*

Proof The proof will proceed by induction in the number of elements in $\mathcal{X}$. If $\mathcal{X}$ consists of just one set, then $N(\mathcal{X})$ is a cone, so it is acyclic. Similarly, $|\mathcal{X}|$ as a convex set is acyclic.

Hence assume that the theorem is proved for Čech structures of no more then k elements and assume $\mathcal{X} = \{S_0, S_1, \dots, S_k\}$. Put $\mathcal{X}_1 := \{\mathcal{S}_0\}$ and $\mathcal{X}_2 := \{S_1, S_2, \dots, S_k\}$. Then $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$ and $\mathcal{X}_1 \bar{\cap} \mathcal{X}_2$ have no more than k elements. We have the following commutative diagram

$$\begin{array}{ccccccccc}
\longrightarrow & H_*(|\mathcal{X}_1| \cap |\mathcal{X}_2|) & \longrightarrow & H_*(|\mathcal{X}_1|) \oplus H_*(|\mathcal{X}_2|) & \longrightarrow & H_*(|\mathcal{X}|) & \longrightarrow \\
 & \uparrow & & \uparrow & & \uparrow & \\
\longrightarrow & H_*(\mathcal{X}_1 \bar{\cap} \mathcal{X}_2) & \longrightarrow & H_*(\mathcal{X}_1) \oplus H_*(\mathcal{X}_2) & \longrightarrow & H_*(\mathcal{X}) & \longrightarrow
\end{array}$$

of Mayer–Vietoris sequences. The top one is the standard Mayer–Vietoris sequence for singular homology and the bottom one is the Mayer–Vietoris sequence established in Theorem 10.1. The conclusion follows now from the induction assumption and the Five Lemma. □

Corollary 11.4 *The homology of a Čech structure $\mathcal{X}$ depends only on the underlying Čech polyhedron $|\mathcal{X}|$ and is isomorphic to the reduced singular homology of $|\mathcal{X}|$.*

Here is another corollary of Theorem 11.3. It will be crucial in the proof of Theorem 5.1 presented in Sect. 13.

Corollary 11.5 *Assume $\mathcal{X} \subset \mathcal{Y}$ are Čech structures such that $|\mathcal{X}| = |\mathcal{Y}|$. Then the inclusion map $\iota : \mathcal{X} \to \mathcal{Y}$ induces an isomorphism $H_*(\iota) : H_*(\mathcal{X}) \to H_*(\mathcal{Y})$.*

Proof Observe that ι is an enclosure of the identity map $\mathrm{id} : |\mathcal{X}| \to |\mathcal{Y}|$ and it obviously induces an identity in homology. Therefore, by Theorem 11.1, we have $H_*(\varphi^{\mathcal{Y}})H_*(\iota) = H_*(\varphi^{\mathcal{X}})$. The conclusion follows now from Theorem 11.3. □

12 Connected Simple Systems

The contents of Corollary 11.4 may be strengthened by showing that for a Čech polyhedron X the isomorphisms between the Čech structures on X may be chosen in a canonical way. In particular, they form a category of isomorphisms. The contents of this section is of interest in itself and is not needed in the sequel, so an uninterested reader may skip it.

Let $\mathcal{C}$ be a category. We identify $\mathcal{C}$ with the collection of objects of $\mathcal{C}$ and we denote by $\mathrm{Mor}(\mathcal{C})$ the collection of all morphisms in $\mathcal{C}$ and by $\mathrm{Iso}(\mathcal{C})$ the collection of all isomorphisms in $\mathcal{C}$. Given two objects $C_1, C_2 \in \mathcal{C}$, we write $\mathcal{C}(C_1, C_2)$ for the collection of morphisms from C_1 to C_2 in $\mathcal{C}$. Let $\mathcal{D}$ be a small subcategory of $\mathcal{C}$. We say that $\mathcal{D}$ is a *pre-connected simple system* (pre-CSS) in $\mathcal{C}$ if the following three conditions are satisfied:

$$\mathrm{Mor}(\mathcal{D}) \subset \mathrm{Iso}(\mathcal{C}), \tag{19}$$

$$\forall E_1, E_2 \in \mathcal{D} \quad \operatorname{card} \mathcal{D}(E_1, E_2) \le 1, \tag{20}$$

$$\forall E_1, E_2 \in \mathcal{D}\, \exists E_3 \in \mathcal{D} :\ \mathcal{D}(E_3, E_1) \ne \emptyset \ne \mathcal{D}(E_3, E_2). \tag{21}$$

If $E_1, E_2 \in \mathcal{D}$ and $\mathcal{D}(E_1, E_2) \ne \emptyset$ then the unique element of $\mathcal{D}(E_1, E_2)$ will be denoted by $\mathcal{D}_{E_2 E_1}$.

We say that $\mathcal{D}$ is a *connected simple system* (CSS) in $\mathcal{C}$ if for any two objects $E_1, E_2 \in \mathcal{D}$ there exists exactly one morphism in $\mathcal{D}(E_1, E_2)$. Obviously, every CSS is also a pre-CSS. The concept of a CSS is due to Conley [9].

Theorem 12.1 (see [16, Theorem 3.1]) *For any $\mathcal{D}$, a pre-CSS in $\mathcal{C}$, there exists a unique CSS $\bar{\mathcal{D}}$ in $\mathcal{C}$ such that*

$$\mathrm{Obj}(\mathcal{D}) = \mathrm{Obj}(\bar{\mathcal{D}}), \tag{22}$$

$$\mathrm{Mor}(\mathcal{D}) \subset \mathrm{Mor}(\bar{\mathcal{D}}). \tag{23}$$

Let X be a Čech polyhedron. Denote by $\mathrm{CS}(X)$ the family of all Čech structures of X.

Proposition 12.2 *Assume $\mathcal{X}, \mathcal{Y}$ are two Čech structures such that $|\mathcal{X}| = |\mathcal{Y}|$. If $\mathcal{F} : \mathcal{X} \to \mathcal{Y}$ is an embedding then $H_*(\mathcal{F}) : H_*(\mathcal{X}) \to H_*(\mathcal{Y})$ is an isomorphism, which does not depend on the particular choice of the embedding $\mathcal{F}$.*

Proof The assumptions imply that $\mathcal{F}$ is an enclosure of $\mathrm{id}_{|\mathcal{X}|}$. The conclusion follows from Theorem 11.1, Theorem 11.3 and Lemma 8.1. □

For any two $\mathcal{X}, \mathcal{Y} \in \mathrm{CS}(X)$ such that $\mathcal{X} \sqsubset \mathcal{Y}$ let

$$\iota_{\mathcal{X},\mathcal{Y}} : H_*(\mathcal{X}) \to H_*(\mathcal{Y})$$

denote the isomorphism given by Proposition 12.2.

Proposition 12.3 *The collection* $(\{H_*(\mathcal{X}) \mid \mathcal{X} \in \mathrm{CS}(X)\}, \{\iota_{\mathcal{X},\mathcal{Y}} \mid \mathcal{X} \sqsubset \mathcal{Y}\})$ *is a pre-CSS.*

Proof Property (20) is obvious. To prove (21) observe that if $\mathcal{X}, \mathcal{Y} \in \mathrm{CS}(X)$ are two Čech structures then $\mathcal{X} \bar{\cap} \mathcal{Y}$ is also a Čech structure and $\mathcal{X} \bar{\cap} \mathcal{Y} \sqsubset \mathcal{X}$ as well as $\mathcal{X} \bar{\cap} \mathcal{Y} \sqsubset \mathcal{Y}$. □

From Theorem 12.1 we obtain the following corollary.

Corollary 12.4 *The collection* $(\{H_*(\mathcal{X}) \mid \mathcal{X} \in \mathrm{CS}(X)\}, \{\iota_{\mathcal{X},\mathcal{Y}} \mid \mathcal{X} \sqsubset \mathcal{Y}\})$ *extends to a unique CSS.*

The connected simple systems in $\mathcal{C}$ form a category (see [16]). This allows us to consider $\tilde{H}(X) := \{H_*(\mathcal{X}) \mid \mathcal{X} \in \mathrm{CS}(X)\}$ as a functor. Details are left to the reader.

13 Proof of Theorems 5.1 and 6.1

We say that an embedding

$$\mathcal{E} : \mathcal{Y} \to \mathcal{Z}$$

of Čech structures is an ϵ-*embedding* if

$$\mathcal{E}(Y) \subset Y^\epsilon \tag{24}$$

for every $Y \in \mathcal{Y}$. We say that $\mathcal{Z}$ is an ϵ-*extension* of $\mathcal{Y}$ if there exists a bijective ϵ-embedding $\mathcal{E} : \mathcal{Y} \to \mathcal{Z}$.

Recall that for a Čech structure $\mathcal{Y}$ we put

$$\mathcal{Y}^\epsilon := \left\{Y^\epsilon \mid Y \in \mathcal{Y}\right\}$$

and we consider the map

$$\iota^\epsilon : \mathcal{Y} \ni Y \mapsto Y^\epsilon \in \mathcal{Y}^\epsilon.$$

This map is obviously surjective and since for every compact convex $A \subset \mathbb{R}^d$ we have

$$\mathbb{R}^d \setminus \left(\mathbb{R}^d \setminus A^\epsilon\right)^\epsilon = A,$$

the map is also injective. Therefore, it is bijective, and consequently it is an ϵ-embedding. Thus, $\mathcal{Y}^\epsilon$ is an ϵ-extension of $\mathcal{Y}$. We have the following theorem.

Theorem 13.1 *If $\mathcal{Y}$ is a Čech structure and $\mathcal{Z}$ is an ϵ-extension of $\mathcal{Y}$ with $\epsilon > 0$ sufficiently small, then every bijective ϵ-embedding $\mathcal{E} : \mathcal{Y} \to \mathcal{Z}$ has an inverse which is simplicial. In particular, it induces an isomorphism in homology.*

Proof To prove this, assume the contrary. Then there exist sequences $\epsilon_n \searrow 0$, $\mathcal{S}_n \subset \mathcal{Y}$ and $\mathcal{E}_n : \mathcal{Y} \to \mathcal{Z}_n$ such that $\mathcal{E}_n$ is a bijective ϵ_n-embedding, $\mathcal{E}_n(\mathcal{S}_n) \in N(\mathcal{Z}_n)$ and $\mathcal{S}_n \notin N(\mathcal{Y})$. Since $\mathcal{Y}$ is finite, without loss of generality, we may assume that $\mathcal{S}_n = \mathcal{S}$ for some $\mathcal{S} \subset \mathcal{Y}$. Let $x_n \in \bigcap \mathcal{E}_n(\mathcal{S}) \subset \bigcap \mathcal{S}^{\epsilon_1}$. Compactness argument lets us replace the sequence x_n by a subsequence convergent to an $x \in \bigcap \mathcal{S}^{\epsilon_1}$. Since, by (24), $\operatorname{dist}(x_n, S) \le \epsilon_n$ for any $S \in \mathcal{S}$, we conclude that $x \in S$ for every $S \in \mathcal{S}$. Therefore, $\bigcap \mathcal{S} \neq \emptyset$, a contradiction. □

Proof of Theorem 5.1 We will first prove that a representation always exists. To this end, fix a Čech structure $\mathcal{Y}$ on Y. By Theorem 13.1, we can choose an $\epsilon > 0$ such that $\iota^\epsilon : \mathcal{Y} \to \mathcal{Y}^\epsilon$ induces an isomorphism in homology. Since X is compact and f is continuous, we can select a $\delta > 0$ such that for every $A \subset X$

$$\operatorname{diam} A \le \delta \ \Rightarrow\ \operatorname{diam} f(A) \le \epsilon.$$

Let $\mathcal{X}$ be a Čech structure on X such that $\operatorname{diam} \mathcal{X} \le \delta$. For $S \in \mathcal{X}$, define

$$\mathcal{F}(S) := \operatorname{conv} f(S).$$

Then $\operatorname{diam} \mathcal{F}(\mathcal{X}) \le \epsilon$. Therefore, $\mathcal{F}(S) \subset \mathcal{Y}^\epsilon$ for every $S \in \mathcal{X}$. In particular, $\mathcal{Z} := \mathcal{Y}^\epsilon \cup \mathcal{F}(\mathcal{X})$ satisfies $|\mathcal{Z}| = |\mathcal{Y}^\epsilon|$. It follows that $\mathcal{Z}$ is a Čech structure on Y^ϵ. Consider the map $\mathcal{E} : \mathcal{Y} \to \mathcal{Z}$ defined as the composition $\iota'\iota^\epsilon$, where $\iota' : \mathcal{Y}^\epsilon \to \mathcal{Z}$ is an inclusion. Obviously, $\mathcal{E}$ is an enclosure of the inclusion map $Y \subset |\mathcal{Z}|$. Since $H_*(\iota^\epsilon)$ is an isomorphism by Theorem 13.1 and $H_*(\iota')$ is an isomorphism by Corollary 11.5, we see that $H_*(\mathcal{E})$ is also an isomorphism. Therefore, $\mathcal{F}$ is a representation of f.

Now, let $\mathcal{F}$ be an arbitrary representation of f. Observe that to prove the commutativity of (9) it is enough to prove the commutativity of the following diagram

$$\begin{tikzcd}
H_*(\mathcal{X}) \arrow[r, "H_*(\varphi^{\mathcal{X}})"] \arrow[d, "H_*(\mathcal{F})"] & H_*(X) \arrow[r, "H_*(f)"] \arrow[d, "H_*(f)"'] & H_*(Y) \arrow[d, "H_*(i)"] & H_*(\mathcal{Y}) \arrow[l, "H_*(\varphi^{\mathcal{Y}})"'] \arrow[d, "H_*(\mathcal{E})"'] \\
H_*(\mathcal{Z}) \arrow[r, "H_*(\varphi^{\mathcal{Z}})"] & H_*(|\mathcal{Z}|) \arrow[r, "\mathrm{id}"] & H_*(|\mathcal{Z}|) & H_*(\mathcal{Z}) \arrow[l, "H_*(\varphi^{\mathcal{Z}})"']
\end{tikzcd},$$

in which the commutativity of the middle square is obvious and the commutativity of the left and right squares follows from Theorem 11.1. □

Proof of Theorem 6.1 To show that the algorithm always stops, assume the contrary. This is possible only if one of the two loops is never exited or the recursive calls of the procedure `recurse` never stop. The **while** loop must be exited by Theorem 13.1. The **foreach** loop always iterates through a finite set, so it also must be exited as long as all calls to `recurse` terminate. This leaves an infinite recursion as the only possibility. Let ϵ_0 denote the value of the ϵ variable immediately after completing the **while** loop. Obviously $\epsilon_0 > 0$. Let $\mathcal{X}_n$ denote the value of the first argument of the procedure `recurse` on its nth call. Then, there is an $S_n \in \mathcal{X}_n$ such that $\operatorname{diam} \Phi(S_n) > \epsilon_0$. By Assumption (iii) of the theorem, we can find a $\delta > 0$ such that if $S \in \mathcal{S}$ and $\operatorname{diam} S \leq \delta$, then $\operatorname{diam} \Phi(S) \leq \epsilon_0$. Since $\mathcal{X}_{n+1}$ is a subdivision of an element of $\mathcal{X}_n$, we see that $\operatorname{diam} S_n \to 0$. Therefore, on some call to `recurse` $\operatorname{diam} S_n \leq \delta$ and consequently $\operatorname{diam} \Phi(S_n) \leq \epsilon_0$, a contradiction.

It follows that the algorithm stops. Observe that when the first call to `recurse` is completed, the variable $\mathcal{F}$ represents a map with all values in $|\mathcal{Y}^\epsilon|$. It follows that $|\mathcal{Z}| = |\mathcal{Y}^\epsilon|$ and by Corollary 11.5 the map $H_*(\iota')$ is an isomorphism. Since $H_*(\iota^\epsilon)$ is an isomorphism by Theorem 13.1, we see that $\mathcal{E}$ induces an isomorphism in homology. Finally, $\mathcal{F}$ is an enclosure of f by Assumption (iii) of the theorem. Therefore, $(\mathcal{X}_k, \mathcal{Y}, \mathcal{Z}, \mathcal{E}, \mathcal{F}_k)$ is a representation of f, and the conclusion follows from Theorem 5.1. □

14 Final Remarks

In order to implement the homology map algorithm presented in Sect. 6, a particular class of convex sets needs to be selected. Among many possible choices, the simplest, but important case is the class of orthotopes, i.e., the Cartesian products of intervals. Orthotopes appear in a natural way in rigorous numerics based on interval arithmetic. Also, the use of orthotopes may significantly reduce the size of the representation of some cubical sets in $\mathbb{R}^d$ with non-uniform structure as in Fig. 2. Another advantage of orthotopes is that it is straightforward to verify if a collection of orthotopes has non-empty intersection, which is needed in the construction of the associated abstract simplicial complex. All this makes orthotopes a good choice for an implementation, and an implementation of the homology map algorithm presented in Sect. 6 and based on orthotopes is in progress. However, the class of orthotopes is not the only possible choice. For instance, general parallelotopes may be better in the context of rigorous computations in dynamics based on Lohner method [20].

In the general Nerve Theorem, instead of assuming the convexity of the elements of the covering one only requires that the intersection of any subfamily of the covering is either empty or acyclic. Thus, a generalization of this paper is possible based on such a family. However, this poses several questions which need to be addressed. The most important is how difficult it is to construct such a covering, a question related to the subject of [29]. This is left for future research.

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