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Greedy Drawings of Triangulations

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Abstract *Greedy Routing* is a class of routing algorithms in which the packets are forwarded in a manner that reduces the distance to the destination at every step. In an attempt to provide theoretical guarantees for a class of greedy routing algorithms, Papadimitriou and Ratajczak (Theor. Comput. Sci. 344(1):3–14, 2005) came up with the following conjecture:

> Any 3-connected planar graph can be drawn in the plane such that for every pair of vertices s and t a *distance decreasing* path can be found. A path $s = v_1, v_2, \dots, v_k = t$ in a drawing is said to be *distance decreasing* if $\|v_i - t\| < \|v_{i-1} - t\|, 2 \le i \le k$ where $\|\dots\|$ denotes the Euclidean distance.

We settle this conjecture in the affirmative for the case of triangulations.

A partitioning of the edges of a triangulation G into 3 trees, called the *realizer* of G, was first developed by Schnyder who also gave a drawing algorithm based on this. We generalize Schnyder's algorithm to obtain a whole class of drawings of any given triangulation G. We show, using the Knaster–Kuratowski–Mazurkiewicz Theorem, that some drawing of G belonging to this class is greedy.

Keywords Graph drawing · Routing · Greedy routing · Triangulations · Fixed point theorem · Schnyder realizers · Planar graphs

1 Introduction

With the increasing use of large wireless communication systems comes an increasing need for reliable and scalable routing algorithms. Internet routing is accomplished using Internet Protocol addresses which are hierarchical and encode topological and

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geographic information about the nodes in the network. Such a protocol is not possible in an ad-hoc network, such as sensornets, where little information about geographic proximity or network topology can be gleaned from node identifiers.

One important family of routing algorithms used for such networks is *Geographic* (*or Geometric*) routing. This is a family of algorithms that use the geographic location of the nodes as their addresses. See, for instance [2, 8, 11, 15]. One such algorithm is the *Euclidean Greedy Routing* algorithm which is conceptually quite simple: each node forwards the packet to the *neighbor*, i.e., a node it can communicate directly with, that has the smallest Euclidean distance to the destination. This algorithm has the disadvantage of not being able to deal with *lakes* or *voids* in the network, i.e., nodes which have no neighbor closer to the destination. To deal with this, variants of the algorithm (such as *face* routing, which involves routing around *faces*) have been proposed, see [11, 15].

Geometric routing has the following two drawbacks: (i) it needs the global position of every node in the network, (ii) it relies entirely on the global position and as such cannot account for local obstructions or the topology of the network. Since GPS units are quite expensive in terms of both money and power requirements, it is quite a restrictive limitation to require every node in the network to have one.

Both the above issues were addressed in [22], where a variant of greedy routing which just uses the local connectivity information of the network without needing the global position of any node, was discussed. The algorithm first computes fictitious or *virtual coordinates* for each node, i.e., it *draws* the graph of the network (where each node in the network is represented by a vertex of the graph and two vertices are adjacent iff the pair of nodes they represent can communicate directly) on the Euclidean plane and routes greedily using these locations. The authors obtain experimental evidence showing that this approach makes greedy routing more reliable. However no theoretical guarantees were obtained.

In a bid to place this approach on a more solid theoretical footing, Papadimitriou and Ratajczak [20] investigated classes of graphs on which greedy routing (without having to rely on variants like *face* routing) could be guaranteed to work, i.e., graphs which can be drawn in the plane without *lakes* or *voids*. They came up with the following conjecture.

Let a *distance decreasing* path in a drawing of a graph be a path $s = v_1, v_1, v_2, \ldots, v_k = t$ such that $\|v_i - t\| < \|v_{i-1} - t\|$, $2 \leq i \leq k$ where $\|\ldots\|$ denotes the Euclidean distance.

Conjecture 1 [20] *Any* 3*-connected planar graph can be drawn*[1] *on the Euclidean plane such that there exists a distance decreasing path between every pair of vertices of the graph.*

Such a drawing is called a *Greedy Drawing* of the graph. It is easy to see that using the greedy drawing of a graph (assuming such a drawing exists) as the virtual coordinates of the vertices guarantees that greedy routing will always work.

[1] Note that the conjecture in [20] uses "*embed*" instead of "*draw*". To be consistent with the Graph Drawing literature, we use "*draw*".

1.1 Our Results

We settle Conjecture 1 in the affirmative for the case of planar triangulations and thus obtain the first non-trivial class of graphs for which this class of greedy routing algorithms can be guaranteed to work.

We show, in fact, that a *planar* drawing of any given triangulation can be obtained, i.e., one in which no pair of edges cross.

The result is obtained by applying the *Knaster–Kuratowski–Mazurkiewicz* Theorem, which is known to be equivalent to the *Brouwer Fixed Point Theorem*. We believe that the technique used in obtaining the result might be of independent interest and might prove helpful in showing the existence of plane drawings with other properties.

Note that greedy drawings can be trivially seen to exist for many simple classes of graphs, like graphs with Hamiltonian circuits, all 4-connected planar graphs (since they have a Hamiltonian circuit by a theorem of Tutte [25]) etc. It is not very difficult to show that the *Delaunay triangulation* of any set of points in the plane is also greedy. But thus far no non-trivial class of graphs with this property was known.

2 Preliminaries and Related Work

Given a n-vertex graph $G(V, E)$, a *drawing* of G is a mapping of the vertices of G to points and of the edges of G to curve segments (with the images of the corresponding vertices as end points) in the plane. We consider only those drawings in which the edges are mapped to straight-line segments and so the drawing is fully specified by the images of the vertices.

Recall that a *plane graph* is an abstract planar graph whose embedding has been fixed, using, say the Hopcroft–Tarjan algorithm [9]. In the rest of the paper we assume that G is plane triangulation. We consider only *planar drawings* of graphs, i.e., drawings in which no pair of edges cross, in this paper. So any reference to a drawing of a graph must be taken to mean a planar straight-line drawing.

2.1 Drawing Planar Graphs in the Plane

An overview of graph drawing algorithms can be obtained from [19, 24]. We describe some well-known algorithms for obtaining planar straight-line drawings of planar graphs.

1. *Rubber Band Embedding* [26]: this algorithm has a elegant physical interpretation: fix the positions of the vertices of some face of the graph and replace all other edges by springs (or "rubber bands"). It can be shown that if the graph is 3-connected and planar then the equilibrium position of the nodes gives a planar straight-line drawing. Many interesting generalizations of this approach have been obtained, see for instance [16]. The drawback of this method is that the size of the grid required for the drawing may be large (exponential in the number of vertices).

2. *Canonical Ordering* [4]: this result showed for the first time that a planar straight-line drawing of a planar graph could be obtained on grid of polynomial (in fact $O(n) \times O(n)$) size. This approach was used in [10] to obtain drawings satisfying various bounds on the minimum angle, bends, grid size etc.
3. *Schnyder's Realizers* [23]: The author describes an elegant algorithm for partitioning the edges of a triangulation into three trees and obtaining a planar drawing (on a $O(n) \times O(n)$ grid) of the graph based on this. Our result uses the techniques developed here and so this approach is described in detail in Sect. 4. This was generalized to all 3-connected planar graphs in [5]. Also see [1, 6, 7, 21].

On a related note, it was shown recently, [14], that any graph has a greedy drawing in the *Hyperbolic* plane. But this might require an exponential sized grid, i.e., $\Omega(n)$ bits might be required to store the coordinates of a single vertex, see [13]. This has been further explored in [18]. In contrast, examples of graphs with no greedy drawing in the *Euclidean* plane were obtained in [20].

3 Outline

We describe the approach of [23] in Sects. 4 and 5. The details of how the edges of a triangulation can be partitioned into three trees is described in the former section and the latter section describes how a drawing of the triangulation can be obtained from this partitioning and also describes some interesting geometric properties of these drawings.

In Sect. 6, we investigate greedy paths in drawings and show that any drawing in which every face is *good*, Definition 9, is greedy. In Sect. 7, we prove the main result of the paper that there exists a greedy drawing of the triangulation, by showing that there exists a drawing in which every face is *good*.

In Sect. 8, we prove a technical result on the sum of weights of all *bad* faces of a drawing, which is needed for proving the main result.

4 Schnyder Realizers of a Triangulation

We designate a (triangular) face f_0 of G as the *exterior* face. All vertices (edges) not belonging to f_0 are called the *interior* vertices (edges). Let the vertices of f_0 be P_0, P_1 and P_2. We define the order (P_0, P_1, P_2) to be the "counter-clockwise" (CCW) order.

Theorem 1 [23] *Given a plane triangulation $G(V, E)$, there exist three directed* edge-disjoint *trees, T_0, T_1 and T_2, called the* realizer *of G, Fig. 1, such that*

1. *T_i is rooted at P_i, $i \in \{0, 1, 2\}$ and contains all vertices of G except P_{i+1} and P_{i-1} (the indices are* mod 3).
2. *All edges of T_i are directed towards the root and every edge of G except those belonging to the exterior face are contained in exactly one T_i.*

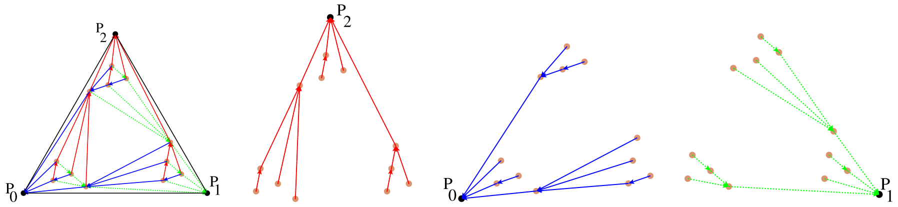

Fig. 1 A triangulation and its realizers. The *leftmost figure* contains all three trees together and the three edges of the exterior face, which do not belong to any tree. The *remaining figures* show each of the three trees separately

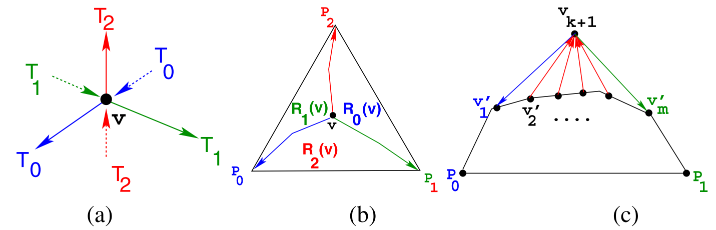

Fig. 2 (**a**) The order of the edges belonging to different trees around an internal vertex v. There are exactly three outgoing edges, one belonging to each tree. There can be any number (including 0) of incoming edges. (**b**) The paths in T_i from v to P_i are vertex disjoint and divide the graph into three regions. (**c**) Obtaining Realizers from the Canonical Order. Note that $m \geq 2$ since G_{k+1} must be biconnected. If $m = 2$ then the edges shown directed towards v_{k+1} will not exist

3. *Each interior vertex*, v, *has exactly* 3 *outgoing edges*, *one for each* T_i. *The edge belonging to* T_0 *is followed by the one belonging to* T_1 *which is followed by the one in* T_2 *in CCW order around* v, *Fig.* 2a.

   Note that there might be any number (*including zero*) *of incoming edges of each* T_i *at any vertex.*

Let $v \in G$ be an interior vertex. Then, it follows from the above that there exist (directed) paths $\mathcal{P}_i(v)$ from v to P_i in T_i, $i = 0, 1, 2$ called the *canonical paths* of v. From the fact the T_i are edge disjoint and the order of the edges around v, it is clear that $\mathcal{P}_i(v)$ and $\mathcal{P}_j(v)$ must be vertex disjoint (except for v itself which appears on all three paths) if $i \neq j$. Hence the $\mathcal{P}_i(v)$, $i = 0, 1, 2$ divide the graph G into three "regions", $R_0(v)$, $R_1(v)$ and $R_2(v)$, see Fig. 2b.

4.1 Schnyder Realizers from Canonical Ordering

Let $f_0 = (P_0, P_1, P_2)$ be the external face of G. An ordering of the vertices

$$v_1 = P_0, \qquad v_2 = P_1, \quad \ldots, \quad v_i, \ldots, v_n = P_2$$

is called a *Canonical Ordering* [4], if we have the following.

- The graph G_k induced by vertices $v_1, v_2, \ldots, v_k$ is biconnected and the boundary of its exterior face is a cycle C_k containing edge $P_0 P_1$.

- Vertex v_{k+1} lies in the exterior face of G_k and its neighbors form a subinterval (of length at least 2) of the path $C_k - P_0P_1$.

A simple way of using the canonical ordering to find the realizers of G was obtained in [4] and [3]. We describe this below.

We process the vertices in the decreasing order of their rank in the canonical ordering. First, we add all internal edges incident to $v_n(=P_2)$ to tree T_2 and orient them towards v_n. Let the neighbors of v_{k+1} in C_k be $v'_1, v'_2, \dots, v'_m$. We add the edge $v_{k+1}v'_1$ to tree T_0 and orient it towards v'_1. The edge $v_{k+1}v'_m$ is added to tree T_1 and oriented towards v'_m. All other edges (if any) are added to tree T_2 and oriented towards v_{k+1}, Fig. 2c.

5 Schnyder Drawings and Their Properties

Let each internal face f_i be assigned a non-negative weight w_i such that $\sum_{i=1}^{2n-5} w_i = 1$. Let $w_{R_i(v)}$ be the sum of weights of all faces in region $R_i(v)$, Fig. 2b. We can obtain a drawing of G in the following way:

Place vertex v at the point $(w_{R_0(v)}, w_{R_1(v)}, w_{R_2(v)})$.

Recall that we only deal with straight-line drawings and so the drawing is specified by the positions of the vertices. Since the total weight of all faces is 1, every vertex of G is placed on the $x + y + z = 1$ plane. Notice that the external vertices P_0, P_1 and P_2 are always placed at the points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ irrespective of how the weights of the internal faces are assigned and that these points determine an equilateral triangle (on the $x + y + z = 1$ plane). Also notice that all internal vertices are placed inside this equilateral triangle.

The drawing obtained by the above method is defined to be a *Schnyder drawing* of G.

The set of solutions to the equation $\sum_{i=1}^{2n-5} w_i = 1$ such that the w_i are non-negative can be represented by the unit simplex $\mathfrak{S}$ in $2n - 6$ dimensions, with $2n - 5$ vertices. Hence, for each point $p \in \mathfrak{S}$, a Schnyder drawing of G can be obtained.

The following theorem, while a generalization of the result proved in [23], follows directly from the proofs given there.

Theorem 2 [23] *In any Schnyder drawing of a triangulation G, the edges are non-intersecting, i.e., the drawing is planar.*

Definition 3 A *non-degenerate* Schnyder drawing is defined to be one obtained by assigning strictly positive weights to the faces.

In the rest of the paper, we use the same notation for a vertex v of G and the point in the plane it is drawn on. The ray $\overrightarrow{P_0P_1}$ is defined to have a slope of $0°$ and all angles are measured counter-clockwise from this ray. So, the ray $\overrightarrow{P_0P_2}$ has slope $60°$, $\overrightarrow{P_2P_1}$ has slope $300°$ and so on. Recall that all drawings we consider (and the points P_0,P_1 and P_2) lie on the $x + y + z = 1$ plane.

The following is a key property of Schnyder drawings.

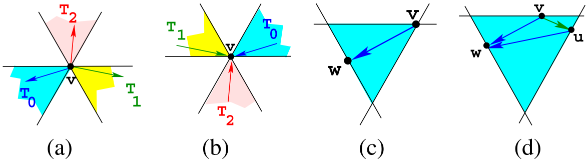

Fig. 3 (**a**) The shaded 60° wedges contain exactly one outgoing edge each. The trees containing the edges are marked. (**b**) All incoming edges (if present) fall in the shaded wedges. (**c**) The equilateral triangle determined by lines through v and w with slopes $0°, 60°$ and $120°$ is free of other vertices. A similar result holds if edge uw were to belong to T_1 or T_2 (with the equilateral triangle changing appropriately). (**d**) The enclosing triangle of a face

Lemma 4 (The Three Wedges Property [21, 23]) *In every Schnyder drawing the three outgoing edges at an internal vertex v have slopes that fall in the intervals* $[60°, 120°]$ (T_2), $[180°, 240°]$ (T_0) *and* $[300°, 360°]$ (T_1), *with exactly one edge in each interval. See Fig.* 3a.

Further, if the drawing is non-degenerate, no edge has slope which is a multiple of $60°$ *and every edge has positive length.*

Lemma 5 *The incoming edges (if present) have slopes in the following ranges*: T_0 : $[0°, 60°]$, $T_1 : [120°, 180°]$ *and* $T_2 : [240°, 300°]$, *Fig.* 3b.

Proof Let $v'v$ be an edge directed from v' towards v. Applying Lemma 4 at v', the result follows. □

Recall that any number of incoming edges might be present at any vertex.

In the rest of the paper, we prove many propositions specifically for non-degenerate Schnyder drawings. Extending them to degenerate Schnyder drawings would make the proof quite messy as degenerate drawings might have zero length edges. Also, non-degenerate drawings are sufficient for our purpose. So we disregard degenerate drawings.

Let v be a vertex and (v, w) an outgoing (at v) edge. Let the coordinates of v be $(v_{R_0}, v_{R_1}, v_{R_2})$ and the coordinates of w be $(w_{R_0}, w_{R_1}, w_{R_2})$. If $(v, w) \in T_0$, it follows from Lemma 4 that $w_{R_0} > v_{R_0}$, $w_{R_1} < v_{R_1}$ and $w_{R_2} < v_{R_2}$. Similar conclusions follow if $(v, w) \in T_1$ or if $(v, w) \in T_2$.

Let $\max_0(v, w) = \mathrm{Max}(v_{R_0}, w_{R_0})$ with $\max_1$ and $\max_2$ being defined in a similar manner. The set of all points (x_0, x_1, x_2) in the drawing such that $x_i = c$ is said to be the *line determined by* $x_i = c$, $0 \le i \le 2$.

Lemma 6 (The Enclosing Triangle Property [21, 23])

1. *Let* (v, w) *be an edge of the graph. Consider the* equilateral *triangle determined by the lines* $x_0 = \max_0(v, w)$, $x_1 = \max_1(v, w)$ *and* $x_2 = \max_2(v, w)$, *superscribing the edge* (v, w), *see Fig.* 3c. *This triangle is free of other vertices.*
2. *For any face* $f = (u, v, w)$ *the* equilateral *triangle determined by the lines* $x_i = \max_i(u, v, w)$, $0 \le i \le 2$ *is free of other vertices, see Fig.* 3d. *This triangle is called the* enclosing triangle *of* f.

Proof (1) follows from Lemma 4. (2) follows from (1) and the fact that the drawing is planar. □

6 Greedy Paths in Schnyder Drawings

A face of the triangulation is said to be *cyclic* if its edges form a directed cycle and is said to be *acyclic* otherwise. Any cyclic face of a graph can be *stacked* by adding a vertex adjacent to the three vertices of the face and adding the new edges to each of the trees as shown in Fig. 4a. This breaks the face into three acyclic faces. After a greedy drawing has been found, the new vertex can be deleted without affecting the greedy paths between the other vertices. Hence, we will assume from now on that every face in the triangulation is acyclic.

Notice that any acyclic face must have a vertex (like vertex t in face (u,t,v) in Fig. 4a) with two outgoing face edges which must belong to different trees. The face is said to *belong to tree* T_i if these two outgoing edges belong to trees T_{i-1} and T_{i+1}.

The following lemma will prove useful.

Lemma 7 *Let u be some vertex and (u,v) an edge incident to it. Let (u,x,x') be any equilateral triangle superscribing (u,v) with a vertex at u. Let z be any point in the wedge determined by (x,u,x') not on the same side of the line (x,x') as u, see Fig.* 4b.

Then, $\|v-z\| < \|u-z\|$.

Proof Let l be the perpendicular bisector of uv. It is easy to see that $z \notin l$ and it lies on the same side of l as v. Hence, it follows that $\|v-z\| < \|u-z\|$. □

To show that a drawing of G is greedy, it clearly suffices to show the following.

For every pair of (ordered) distinct vertices $u, v \in V$, there exists some neighbor of u, say u' such that $\|u-v\| > \|u'-v\|$.

In the rest of the paper we will show that a non-degenerate Schnyder drawing of G exists which satisfies the above property.

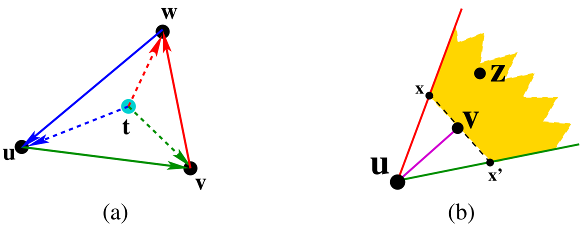

Fig. 4 (**a**) The cyclic face (u,v,w) is *stacked* by adding vertex t and its incident edges. The edge tu is added to the same tree as wu, edge tv the same tree as uv and tw the same tree as vw. (**b**) The triangle (u,x,x') is equilateral. $\|v-z\| < \|u-z\|$ irrespective of where z lies in the shaded region and where v lies on xx'

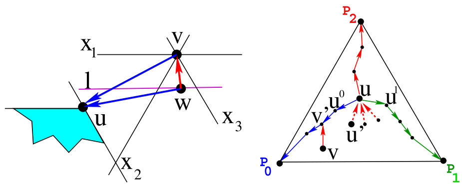

Fig. 5 (**a**) An acyclic face with its active region shaded. The *thin lines* have slopes that are multiples of 60°. The active region at u is bounded by rays with slope 180° and 300°. (**b**) Note that v and v' need not be adjacent. (**c**) Vertices u^0, u and u^1 form a face. Edge u^0u^1 could be directed either way. (**d**) The *greedy region* of face $f = (u, v, w)$ is shown shaded

Let $f = (u, v, w)$ be an acyclic face and let u be the vertex with two incoming edges. Let the coordinates of u be (u_0, u_1, u_2). Without loss of generality, we assume that both these incoming edges belong[2] to T_0. See Fig. 5a. It is easy to see that $u_0 = \max_0(u, v, w)$ and $u_2 = \min_2(u, v, w)$.

Let the *active region* of $\angle uvw$, denoted by $\mathcal{A}_{\angle uvw}$, be the set of points (x_0, x_1, x_2) with $x_0 \geq u_0$ and $x_2 \leq u_2$. It is easy to see that this region is the wedge with sides of slopes 180° and 300° at vertex u, Fig. 5a.

Lemma 8 *Let $f = (u, v, w)$ be an acyclic face of G and in some non-degenerate Schnyder drawing of G, let z be a vertex in the active region of $\angle uvw$. Then,* $\|v - z\| > \min(\|u - z\|, \|w - z\|)$.

Proof From Lemma 4, it follows that u lies below the horizontal line (denoted by l in Fig. 5a) through w.

Since z lies in the active region of vertex u, only two possibilities can arise.

- z lies in the wedge bounded by rays of slope 180° and 240° at vertex v (the wedge x_1vx_2 in Fig. 5a): from Lemma 7, it follows that $\|v - z\| > \|u - z\|$.
- z lies in the wedge bounded by rays of slope 240° and 300° at vertex v (the wedge x_2vx_3 in Fig. 5a): it follows that z must lie below the horizontal line through w since u and so the whole active region lies below this line.

  Now applying Lemma 7 again it follows that $\|v - z\| > \|w - z\|$.

Hence, in every case $\|v - z\| > \min(\|u - z\|, \|w - z\|)$. □

Let u and v be a pair of non-adjacent vertices. It follows that v lies in one of three regions $R_0(u)$, $R_1(u)$ or $R_2(u)$ (or their boundaries), Fig. 2b. Assume wlog, that v lies in region $R_2(u)$, i.e., the region bounded by the edge P_0P_1 of the external face and the paths $\mathcal{P}_i(u)$, $i = 0, 1$ from u to P_0 and P_1, Fig. 5b. The path $\mathcal{P}_2(v)$ from v to P_2 must intersect either $\mathcal{P}_0(u)$ or $\mathcal{P}_1(u)$. Assume wlog, that it intersects $\mathcal{P}_0(u)$ and let $v' = \mathcal{P}_2(v) \cap \mathcal{P}_0(u)$. Let u^0 (u^1) be the neighbor of u on $\mathcal{P}_0(u)$ $(\mathcal{P}_1(u))$.

The following possibilities arise.

[2]Note that both edges must belong to the same tree.

Case I $v' = u$: let $\mathcal{P}_2(v) = (v = v_0, v_1, v_2, \ldots, v_{k-1}, v_k = v' = u, v_{k+1}, \ldots, P_2)$. It follows from Lemmas 7 and 5 that $\|u - v\| > \|v_{k-1} - v\|$ in every non-degenerate Schnyder drawing of G.

Case II *u has one or more edges directed inwards lying between the edges uu^0 and uu^1 in the embedding*: let the edge following uu^0 (in CCW direction) be uu', Fig. 5b. It follows from Lemma 4 that v lies in the active region of $\angle u^0uu'$.

Hence from Lemma 8 it follows that either $\|u^0 - v\| < \|u - v\|$ or $\|u' - v\| < \|u - v\|$ in every non-degenerate Schnyder drawing.

Case III *The vertices u, u^0 and u^1 form a (acyclic) face of G, Fig.* 5c: in this case there might exist some Schnyder drawings in which for every neighbor u^i of u, $\|u^i - v\| > \|u - v\|$. But we will show below that there must exist *some* non-degenerate Schnyder drawing in which $\|u^0 - v\| < \|u - v\|$.

The *greedy region* of a face $f = (u, v, w)$ is the region bounded by the edge vw and the paths $\mathcal{P}_0(v)$ and $\mathcal{P}_1(w)$ as shown in Fig. 5d. Note that even though the greedy region depends on the drawing, the set of vertices falling in this region is fixed by the realizer of G.

Definition 9 Let $f = (u, v, w)$ be a triangular face with edges uv and uw directed away from u, Fig. 5d, and let $\epsilon > 0$ be some small constant depending only on the number of vertices of G. Then, in a Schnyder drawing of G, f is said to be *good* if

I The length of every edge of f is at least $\sqrt{\epsilon}$.
II For every vertex z in the greedy region:

$$\|u - z\|^2 - \|v - z\|^2 \geq \epsilon \quad \text{if } \mathcal{P}_2(z) \cap \mathcal{P}_0(u) \neq \emptyset,$$
$$\|u - z\|^2 - \|w - z\|^2 \geq \epsilon \quad \text{if } \mathcal{P}_2(z) \cap \mathcal{P}_1(u) \neq \emptyset,$$

and is said to be *bad* otherwise.

Note that for every vertex z in the greedy region exactly one of $\mathcal{P}_2(z) \cap \mathcal{P}_0(u)$ and $\mathcal{P}_2(z) \cap \mathcal{P}_1(u)$ is non-empty. *Clearly, a non-degenerate drawing in which every face is good, is greedy.*

The following lemma is not used directly in the paper but is helpful because it provides some intuition as to why the Schnyder drawing framework can lead to greedy drawings of graphs.

Lemma 10 *Given any two vertices $u, v \in G$, then in* any *non-degenerate Schnyder drawing of G, there exists a neighbor of u, say u' and a neighbor of v, say v' such that $\|u - v\| > \min(\|u' - v\|, \|u - v'\|)$.*[3]

Proof Follows from Lemmas 4 and 7. □

[3]This lemma holds more generally for all 3-connected planar graphs and not just triangulations. We will not prove this generalization here as we deal only with triangulations.

7 The Main Result

The following theorem will prove useful.

Theorem 11 (Knaster–Kuratowski–Mazurkiewicz [12]) *Let a d-simplex with vertices $\{v_0, \ldots, v_d\}$, be covered by* closed *sets C_i, $i \in \{0, \ldots, d\}$ such that the following* covering *condition holds.*

For any $Q \subseteq \{0, \ldots, d\}$ the face spanned by the vertices $\{v_i | i \in Q\}$ is covered by $\bigcup_{i \in Q} C_i$.

Then, $\bigcap_{i \in \{0,\ldots,d\}} C_i \neq \emptyset$.

This theorem is known to be equivalent to the Brouwer Fixed Point Theorem [17]. The main result is the following.

Theorem 12 *Given an n-vertex plane triangulation G, there exists a non-degenerate Schnyder drawing of G which is greedy.*

Proof Recall that for each point $p \in \mathfrak{S}$, the unit simplex with $2n - 5$ vertices (in $2n - 6$ dimensions), a Schnyder drawing of G can be obtained.

We define *good sets* $G_{f_1}, \ldots, G_{f_{2n-5}}$ where $G_{f_i} \subseteq \mathfrak{S}$ $\forall i$, in the following way.

Let $w = (w_1, w_2, \ldots, w_{2n-5}) \in \mathfrak{S}$. Then $w \in G_{f_i}$ iff in the Schnyder drawing of G corresponding to w, the face f_i is good. Note that the definition of these good sets depends on the value of ϵ (Definition 9).

In Sect. 8.1, it is shown that in any Schnyder drawing of G the sum of the weights of all the bad faces is always strictly less than 1, if ϵ is small enough (Theorem 15). Let $p = (p_0, \ldots, p_{2n-5}) \in \mathfrak{S}$ lie in the interior of some k-face of $\mathfrak{S}$. Wlog, we can assume that $p_0, p_1, \ldots, p_k > 0$ and $p_{k+1} = \cdots = p_{2n-5} = 0$. Since the sum of weights of bad faces is always less than 1, it follows that some face f_i where $i \in [0, k]$ must be good in the drawing corresponding to point p. Hence $p \in G_{f_i}$ and so the KKM covering condition is satisfied.

It is easy to see that the sets G_{f_i} are closed. The condition that the length of the edges of f_i are at least $\sqrt{\epsilon}$ can be expressed in the form $P \geq \epsilon$ where P is a quadratic polynomial, see (1). It is not very difficult to see that Condition II in Definition 9 can also be expressed as a polynomial (in fact quadratic) inequality. Hence, the set G_{f_i} can be expressed as the set of all points satisfying some weak polynomial inequalities. Hence G_{f_i} is closed.

From this it follows that the G_{f_i} satisfy the conditions of Theorem 11.

Hence, $\bigcap_{i \in \{1,\ldots,2n-5\}} G_{f_i} \neq \emptyset$. Let $g \in \bigcap_{i \in \{1,\ldots,2n-5\}} G_{f_i}$. It follows that every face is good in the Schnyder drawing corresponding to g, which implies that this drawing is greedy.

It is possible that the drawing corresponding to g is degenerate. But since $\epsilon > 0$ and the drawing varies continuously with the set of face weights, we can always pick another point g' close enough to g such that the drawing corresponding to g' is non-degenerate and greedy. □

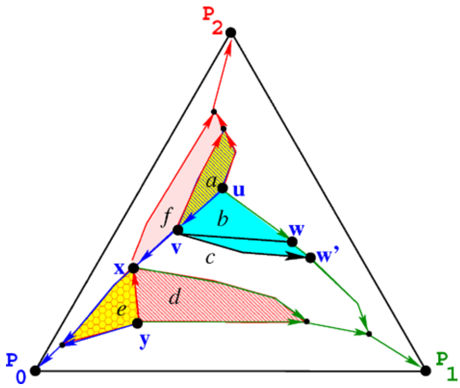

Fig. 6 The sum of weights of faces in different regions are denoted by a, b, c, d, e and f. Note that w and w' could possibly be the same vertex, depending on how the edge wv is directed. The analysis below remains the same in either case

8 Schnyder Drawings and the Weights of Faces

In this section we show that sum of weights of the bad faces in a drawing of the triangulation is always *strictly* less than 1 for ϵ small enough.

Consider the face $F = (u, v, w)$ in Fig. 6. The sum of weights of faces in various regions are marked. All paths shown in the figure are canonical paths ($\mathcal{P}_i(\cdot)$) starting from some vertex. Note that b is the weight of the region demarcated by $uvw'w$ where w and w' could possibly be the same vertex.

The coordinates of the points the vertices are mapped to are given below. Recall that the graph is being drawn on the $x + y + z = 1$ plane, so the points lie on this plane. The vectors corresponding to various edges are also given below. Note that u_0 represents the first coordinate of vertex u, x_1 represents the second coordinate of vertex x and similar is the case with y_2.

$$
\begin{aligned}
u &= (u_0,\ x_1 + a + f,\ y_2 + b + c + d + e),\\
v &= (u_0 + a + b,\ x_1 + f,\ y_2 + c + d + e),\\
y &= (u_0 + a + b + c + d + f,\ x_1 + e,\ y_2),\\
\overrightarrow{u - y} &= (-a - b - c - d - f,\ a + f - e,\ b + c + d + e),\\
\overrightarrow{v - y} &= (-c - d - f,\ f - e,\ c + d + e),\\
\overrightarrow{u - v} &= (-a - b,\ a,\ b).
\end{aligned}
$$

It follows that the length of the edge uv is given by

$$\|u - v\|^2 = 2\big(a^2 + b^2 + ab\big). \tag{1}$$

Lemma 13 *Let $\mathcal{W}_{uvw}$ be the weight of face (u, v, w). If $\mathcal{W}_{uvw} \geq \sqrt{\frac{\epsilon}{2}}$, every edge of face F has length at least $\sqrt{\epsilon}$.*

Proof

$$\|u - v\|^2 = 2\big(a^2 + b^2 + ab\big) \geq 2\mathcal{W}_{uvw}^2 \geq \epsilon. \tag{2}$$

An identical argument applies to edge uw. For edge vw, notice that from Lemma 4, $\angle vuw \geq 60°$. Hence the edge vw is longer than at least one of the other two edges. □

Note that $b \geq \mathcal{W}_{uvw}$ since b is the weight of all faces in region $uvw'w$.

Theorem 14 *Assuming that $b \geq \sqrt{\frac{\epsilon}{2}}$, the following conditions are necessary (but not sufficient) for $\|u-y\|^2 - \|v-y\|^2 < \epsilon$.*

$$a > b, \tag{3}$$

$$e > b, \tag{4}$$

$$a < \sqrt{a-b} \quad \text{and} \quad b < \sqrt{a-b}. \tag{5}$$

Proof

$$\begin{aligned} &\|u-y\|^2 - \|v-y\|^2 < \epsilon \\ &\implies \quad a(a+b+c+d+2f-e) + b(b+2c+2d+f+e) < \frac{\epsilon}{2}. \end{aligned} \tag{6}$$

Since $b \geq \sqrt{\frac{\epsilon}{2}}$ and all variables are non-negative, we must have

$$\begin{aligned} &a+b+c+d+2f-e < 0 \\ &\implies \quad b < e. \end{aligned}$$

Rearranging the terms of (6), we obtain

$$\begin{aligned} &a(a+b+c+d+2f) + b(b+2c+2d+f) + e(b-a) < \frac{\epsilon}{2} \\ &\implies \quad b-a < 0 \quad \implies \quad a > b \end{aligned}$$

for the same reason as before.

Rearranging the terms of (6) again we obtain

$$\begin{aligned} &a(a+b+c+d+2f) + b(b+2c+2d+f) + e(b-a) < \frac{\epsilon}{2} \\ &\implies \quad a^2 + e(b-a) < 0 \\ &\implies \quad a^2 < e(a-b) \\ &\implies \quad a < \sqrt{a-b} \quad (\text{since } e < 1). \end{aligned}$$ □

8.1 The Maximum Weight of Bad Faces

Let point $w = (w_1, \ldots, w_{2n-5}) \in \mathfrak{S}$ be such that in the Schnyder drawing, every face f_i with weight $w_i > 0$ is bad. Then, the faces can be divided into three types.

Type A the face has weight 0 and can be either good or bad.

Type B the face has weight strictly less than $\sqrt{\frac{\epsilon}{2}}$ and is bad because either one of its edges is shorter than ϵ (and so violating Condition I in Definition 9) or because it violates Condition II in Definition 9.

Type C the face has weight at least $\sqrt{\frac{\epsilon}{2}}$ and is bad because it violates Condition II in Definition 9.

If ϵ is small enough, then "most" of the weight must be present in faces of Type C.

Theorem 15 *In any Schnyder drawing of G, the sum of weights of all faces of type B and C is strictly less than* 1.

We first give a brief description of the main idea behind the proof.

We try to find a point in $\mathfrak{S}$ such that, in the Schnyder drawing corresponding to it, every face with positive weight is bad. But we run into a contradiction, thus showing that such a point cannot exist.

Let $v_1, v_2, v_3, \ldots, v_n$ be the canonical order of G where v_1 and v_2 are the vertices of the bottom edge of the external face and v_n is the topmost node. We start with the edge v_1v_2 and construct the triangulation by adding vertices one by one according to the canonical order. This also gives us an ordering on the faces. As faces are added, we try to assign weights to them in such a way that no face with positive weight is good. This condition places an upper bound on the weight each face can be assigned. Once we are done with all faces, we show that the sum of weight of all faces (good or bad) is forced to be less than 1, which is a contradiction.

Proof Proof by contradiction. We try to find an assignment of weights to the faces such that every face with positive weight is bad and show that this would require the sum of weights of all faces to be less than 1, which is impossible.

Let $v_1, v_2, v_3, \ldots, v_n$ be the canonical order of G where v_1 and v_2 are the edges of the bottom edge of the external face and v_n is the topmost node.

We start with the edge v_1v_2 and build the graph by adding vertices one by one according to the canonical order. The vertex v_3 and the face, f_1, it forms with v_1 and v_2 are shown in Fig. 7a. Since the greedy region of v_3 contains no vertices, it is clear that f_1 cannot be a type C face. Hence $w_{f_1} < \sqrt{\frac{\epsilon}{2}}$.

Let G_k be the graph induced by the vertices $v_1, v_2, \ldots, v_k$ and let weight of all faces of G_k be W_k. Assume that $W_k \to 0$ as $\epsilon \to 0$. This is clearly satisfied by $W_3 = w_{f_1} < \sqrt{\frac{\epsilon}{2}}$. Let $W = \max(W_k, \sqrt{\frac{\epsilon}{2}})$. We will show that W_{k+1} also satisfies this property.

We now add v_{k+1} to G_k and try to assign weights to the new faces formed. Let W_{new} be the maximum weight that can be assigned to the new faces while ensuring that every face with positive weight is bad. We have the two following possibilities.

Case I v_{k+1} has only two neighbors in G_k. Let them be vertices u and w as shown in Fig. 7b. In this case, the only new face is (u, v_{k+1}, w) which belongs to tree 2. Then it follows from (4) that $W_{\text{new}} < W$, as otherwise the face is good.

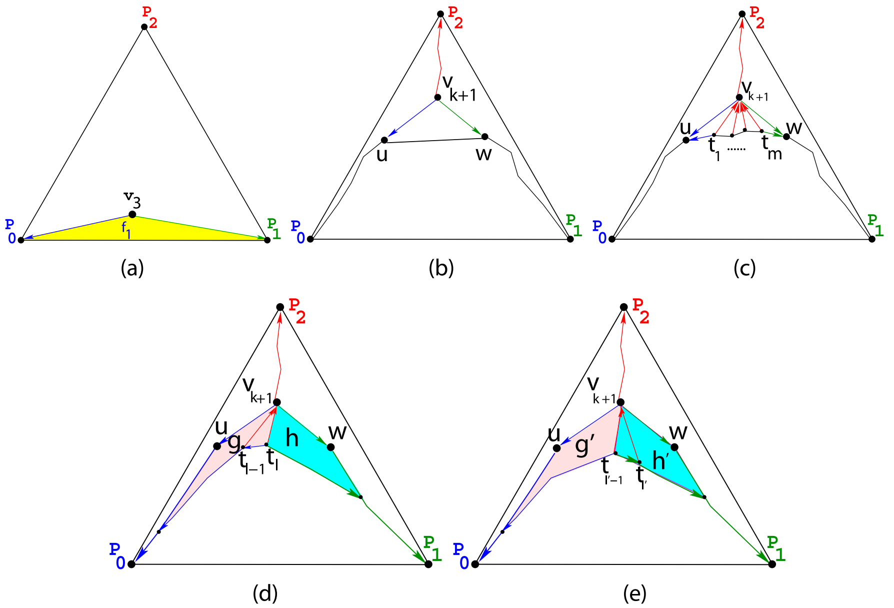

Fig. 7 (**a**) The first faced added. Note that the vertex P_2 and the edges P_2P_0 and P_2P_1 have not yet been added to the graph and are shown only for clarity. (**b**) and (**c**) Faces obtained when the vertex v_{k+1} is added to G_k. Note that the path from v_{k+1} to P_2 is not present in G_k and is shown only for clarity. (**d**) Note that only face $f_l = (v_{k+1}, t_l, t_{l-1})$ is shown to avoid clutter. (**e**) Only face $f_l' = (v_{k+1}, t_{l'}, t_{l'-1})$ is shown for the same reason. Note that $f_{l'}$ lies to the *right* of f_l

Case II v_{k+1} has more than two neighbors, say $u, t_1, \ldots, t_m$ and w as shown in Fig. 7c. The new faces are $f_1 = (u, t_1, v_{k+1})$ which belongs to tree 1, $f_m = (v_{k+1}, t_m, w)$ which belongs to tree 0 and $m-1$ faces of the form $f_i = (t_i, v_{k+1}, t_{i+1})$ each of which may belong to either tree 0 or 1. Let w_{f_i} be the weight of face f_i.

Case i *none of the new faces have weight more than* W. Hence, $W_{\text{new}} < nW$ as $m + 1 < n$.

Case ii *at least one face, say belonging to tree* 1, *has weight more than* W, *Fig.* 7d.

Let $l \in [1..m]$ be the maximum value such that (a) $w_{f_l} > W$ and (b) f_l belongs to tree 1. Let $l' \in [l + 1..m]$ be the minimum value such that (a) $w_{f_{l'}} > W$ and (b) $f_{l'}$ belongs to tree 0.

Of course such an l' need not exist. If it does not, then, in Fig. 7d every face in the region h has weight at most W. By (3) applied[4] to face f_l, $g < h$ if f_l is to be bad. Every new face must fall in one of

[4]Note that the face shown in Fig. 6 in the derivation of (3) belongs to tree 2 while face f_l and f_l' belong to trees 0 and 1. Of course this does not really change anything as the same argument applies. To see how (3) (or (5)) applies to face f_l, compare Figs. 6 and 7d where vertices v_{k+1}, t_l, t_{l-1} map to v, u, w in that order.

the regions g or h. Since $h < nW$, we have $W_{\text{new}} < g + h < 2nW$ (since $m + 1 < n$).

If l' does exist, then by (3), applied to face $f_{l'}$, $g' > h'$ in Fig. 7e. Note that is possible to have $l = 1$ and/or $l' = m$.

Let $S(g)$, $S(g')$, $S(h)$ and $S(h')$ denote the set of faces in the regions so marked in Figs. 7d and 7e. Since $f_{l'}$ lies to the right of f_l, it is clear that $S(g) \subset S(g')$ and $S(h') \subset S(h)$ and by (3) applied to faces f_l and $f_{l'}$, $h > g$ and $g' > h'$.

Let $D_{gg'} = S(g')\backslash S(g)$ and $D_{hh'} = S(h)\backslash S(h')$ and $W_{gg'}$ ($W_{hh'}$) be the weight of the faces in $D_{gg'}$ ($D_{hh'}$). We have

$$\begin{aligned} g + W_{gg'} = g' \quad &\text{and} \quad h' + W_{hh'} = h \\ \implies \quad &W_{gg'} + W_{hh'} > g' - h' \quad \text{since } h > g \\ \text{and} \quad &W_{gg'} + W_{hh'} > h - g \quad \text{since } g' > h'. \end{aligned}$$

The only new faces in the sets $D_{gg'}$ and $D_{hh'}$ are $f_i, i \in [l + 1, l' - 1]$. Each of these faces have weight at most W (by definition of l and l'). Since the sum of all the old faces is at most W, we have $W_{gg'} + W_{hh'} < 2(W + nW) = 2(n + 1)W$.

From (5) applied to faces f_l and $f_{l'}$, it follows that $g, g', h, h' < \sqrt{2(n + 1)W}$.

Hence $W_{\text{new}} < \max(2nW, c\sqrt{nW})$ where c is some small constant. Since $W \to 0$ as $\epsilon \to 0$, we can assume that ϵ is small enough that $2nW < 1$ and so $W_{\text{new}} < c\sqrt{nW}$. Recall that $W = \max(W_k, \sqrt{\frac{\epsilon}{2}})$. It follows that $W_{k+1} = W_k + W_{\text{new}} < W + c\,\sqrt{nW} < c'\sqrt{nW}$. Hence $W_{k+1} < c'\sqrt{n\ \max(W_k, \sqrt{\frac{\epsilon}{2}})}$. Notice that $W_{k+1} \to 0$ as $\epsilon \to 0$.

Hence it easy to see that by picking ϵ small enough, we can make $W_n < 1$ (in fact, we can make $W_n \to 0$). But the total weight of all faces must be exactly 1 and so this gives us a contradiction and the result follows. □

9 Conclusions

We have been able to show that every triangulation has a planar greedy drawing in the Euclidean plane. As for algorithmic questions, the following iterative approach works quite well in practice.

- Let $\mathcal{W}^i = (w_0, w_1, \ldots, w_{2n-5}) \in \mathfrak{S}$ be the weights of the faces in iteration i.
- Let $\mathcal{W}^{i+1} = \frac{1}{W}(w'_0, w'_1, \ldots, w'_{2n-5})$ where $w'_j = w_j$ if f_j is good in the drawing corresponding to $\mathcal{W}^i$ and $w'_j = 2w_j$ otherwise and W is the normalizing factor such that $\mathcal{W}^{i+1} \in \mathfrak{S}$.
- For $i = 0$, let $w_0 = w_1 = \cdots = w_{2n-5} = \frac{1}{2n-5}$.

This algorithm converges quite fast, but so far no theoretical bounds are known. We are confident that good bounds can be obtained.

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