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Covering the Plane with Translates of a Triangle

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Abstract The minimum density of a covering of the plane with translates of a triangle is $\frac{3}{2}$.

Keywords Covering · Covering density · Triangle

1 Introduction

A collection $\mathcal{C} = \{C_1, C_2, \ldots\}$ of planar convex bodies is called a *covering* of a domain $D \subseteq R^2$ provided $\bigcup_i C_i \supseteq D$. The area of a convex body C is denoted by $|C|$.

For any pair of independent vectors $\mathbf{v}_1$, $\mathbf{v}_2$ in the plain, the *lattice* generated by $\mathbf{v}_1$ and $\mathbf{v}_2$ is the set of vectors $\{k\mathbf{v}_1 + l\mathbf{v}_2 : k,\ l \text{ integers}\}$. A covering of R^2 is a *lattice covering* if it is of the form $(C + \mathbf{v})_{\mathbf{v}\in L}$, where L is a lattice.

The *density* of a collection $\mathcal{C}$ relative to a bounded domain D is defined as

$$d(\mathcal{C}, D) = \frac{1}{|D|}\sum_{C\in\mathcal{C}} |C \cap D|.$$

If the whole plane R^2 is covered with $\mathcal{C}$ then the *lower density* of $\mathcal{C}$ in R^2 is defined as

$$d_-\big(\mathcal{C}, R^2\big) = \liminf_{r\to\infty} d\big(\mathcal{C}, B^2(r)\big),$$

where $B^2(r)$ denotes the circle of radius r centered at the origin.

The *lattice covering density* $\vartheta_L(C)$ is defined as the infimum of $d_-(\mathcal{C}, R^2)$ taken over all lattice coverings with congruent copies of C. The *translative covering density*

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$\vartheta_T(C)$ is defined as the infimum of $d_-(\mathcal{C}, R^2)$ taken over all translative coverings with congruent copies of C. These infima are attained.

By Fáry's theorem [4], the triangles are the least economical convex sets for a lattice covering; we have $\vartheta_L(C) \le \frac{3}{2}$ for every convex body C and equality holds if and only if C is a triangle (see also [1]).

Since $\vartheta_T(C) \le \vartheta_L(C)$, it follows that $\vartheta_T(C) < \frac{3}{2}$ for any convex body C in the plane other than a triangle. The aim of this paper is to show that the triangles are the least economical sets for translative covering, i.e. that $\vartheta_T(\triangle) = \frac{3}{2}$ for any triangle $\triangle$ (see Problem 2, Chap. 1.3 of [3]).

Various results concerning coverings with convex bodies are discussed in [2, 3, 5], and [6].

Theorem *The minimum density of a covering of the plane with translates of a triangle is* $\frac{3}{2}$.

2 Parts Used for the Covering

Let T_0 be the right isosceles triangle whose vertices are $(0,0)$ (called the *right vertex)*, $(-1,0)$, and $(0,1)$ (called the *left* and the *upper vertex*, respectively). For any translate T_i of T_0 each of the right, the left, and the upper vertex of T_i is defined as the vertex corresponding to the identically named vertex of T_0. The coordinates of the right vertex of T_i are denoted by $(x(T_i), y(T_i))$.

Consider a covering $\mathcal{T}$ of the plane with translates of T_0. We assume that in the covering $\mathcal{T}$ no two triangles coincide. Moreover, we assume that the number of triangles that intersect $B^2(r)$ is finite, for each $r > 0$.

Let T_w and T_i be two different triangles of $\mathcal{T}$. We say that T_w *cuts* T_i provided at least one of the following three conditions is fulfilled:

(c_1) the right vertex of T_w belongs to the interior of T_i;
(c_2) the vertical leg of T_w intersects both a leg of T_i and the hypotenuse of T_i;
(c_3) the horizontal leg of T_w without the left vertex intersects both a leg of T_i and the hypotenuse of T_i.

For instance, $T^d(i)$ cuts T_l, but T_l does not cut $T^d(i)$ in the left-hand picture in Fig. 5. If the interiors of T_w and T_i intersect, then either T_w cuts T_i or T_i cuts T_w.

Let r be an arbitrary number greater than 4. All triangles of $\mathcal{T}$ that intersect $B^2(r-4)$ are denoted by $T_1, \dots, T_s$ so that if $i < j$, then either $y(T_i) < y(T_j)$ or $y(T_i) = y(T_j)$ and, at the same time, $x(T_i) > x(T_j)$. Furthermore, denote by $T_{s+1}, \dots, T_z$ the remaining triangles that intersect $B^2(r)$.

Let $i \in \{1, \dots, z\}$ and let S_i be the union of the triangles that cut T_i. The part $U_i \subset T_i$ *used for the covering* is defined as the closure of $T_i \setminus S_i$.

Obviously, U_i is a polygon with two sides contained in the legs of T_i and with the other sides parallel to the legs of T_i. Moreover, sets $U_1, \dots, U_z$ have pairwise disjoint interiors. We show that sets $U_1, \dots, U_s$ cover $B^2(r-4)$. Let $a_0 \in B^2(r-4)$. Denote by $n_1, \dots, n_k$ all integers such that $a_0 \in T_{n_i}$ for $i = 1, \dots, k$. Obviously, $n_1, \dots, n_k \in \{1, \dots, s\}$. Since $a_0 \in T_{n_1} \cap \dots \cap T_{n_k}$, it follows that there exists an integer $j \in \{1, \dots, k\}$ such that $a_0 \in U_{n_j}$. Consequently, $B^2(r-4) \subset U_1 \cup \dots \cup U_s$.

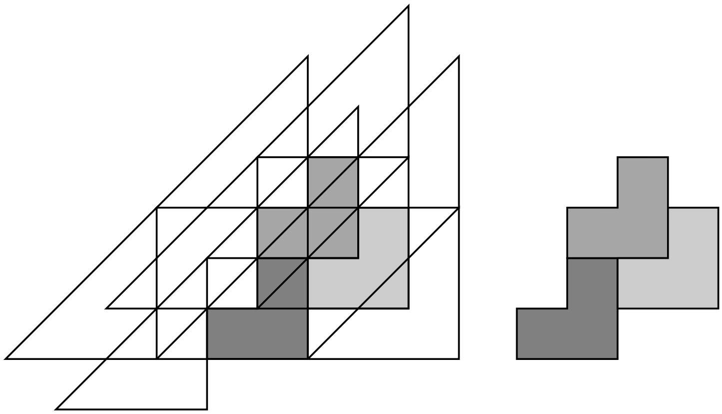


Fig. 1 The optimal lattice covering

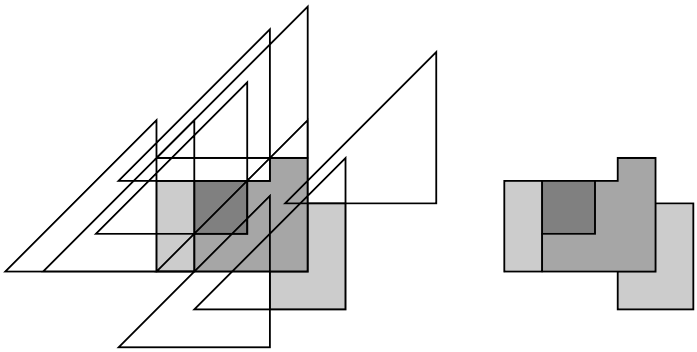


Fig. 2 A translative covering

We say that U_i is *n-tier*, if U_i has $2n+2$ sides. Figure 1 illustrates the optimal lattice covering. In the covering U_i is 2-tier and $|U_i| = \frac{2}{3}|T_i|$ for each triangle T_i. Figure 2 illustrates a translative covering.

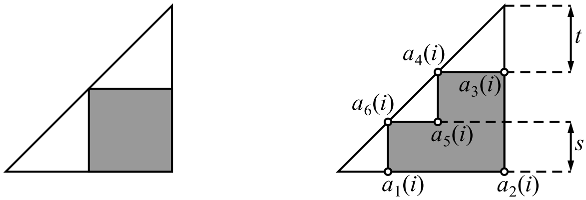
3 Types of Triangles

Note that

$$|U_i| \le \frac{1}{3} = \frac{2}{3}|T_i|$$

provided U_i is either 1- or 2-tier. If U_i is 1-tier, then $|U_i| \le \frac{1}{2}|T_i|$. If U_i is 2-tier, then $|U_i| = t(1-t) + s(1-t-s)$, where the numbers t and s denote the distances shown in Fig. 3. This 2nd degree polynomial in two variables reaches its maximum at $s = t = \frac{1}{3}$. Consequently, $|U_i| \le \frac{1}{3}$.

Fig. 3 1- and 2-tier parts



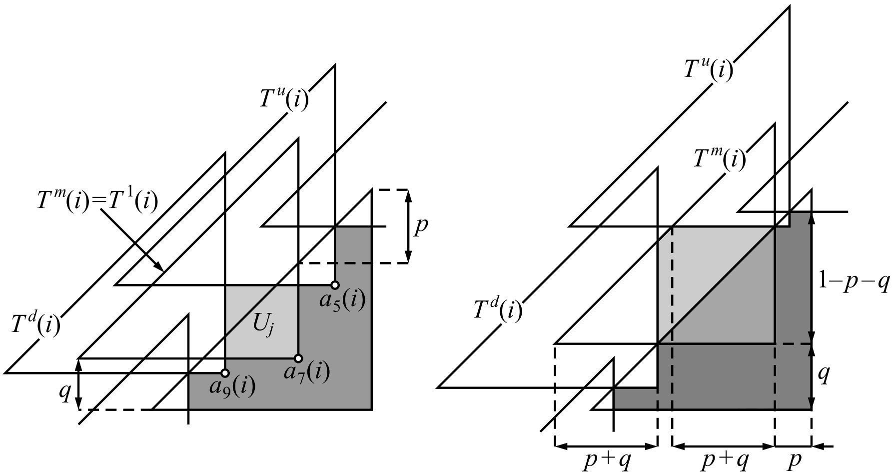


Fig. 4 Triangles of the second type

If $|U_i| \leq \frac{2}{3}|T_i|$ would hold all triangles, it would directly follow that the covering density is at least $\frac{3}{2}$. Unfortunately, if U_i is n-tier and $n \geq 3$, generally it is not the case here, $|U_i|$ might be arbitrarily close to $|T_i|$. Thus, we have to partition the triangles into small groups of 1, 2, 3 or 4 triangles, such that for them

$$\sum |U_i| \leq \sum \frac{2}{3}|T_i|$$

holds.

The vertices of U_i are denoted by $a_1(i), \ldots, a_{2n+2}(i)$ as presented in Fig. 3.

If U_i is n-tier, where $n \geq 3$, then we will define triangle $T^1(i)$ and, if needed, $T^2(i)$ and $T^3(i)$. Denote by $T^u(i)$ the triangle, the right vertex of which is in $a_5(i)$ and by $T^m(i)$ the triangle, the right vertex of which is in $a_7(i)$. Furthermore, denote by $T^d(i)$ the triangle, the right vertex of which is in $a_9(i)$ provided $n \geq 4$ (see Fig. 4).

Assume that $n = 3$. If $T^u(i)$ cuts $T^m(i)$, then $T^1(i) = T^m(i)$. Otherwise, $T^1(i) = T^u(i)$. We do not define $T^2(i)$ (see Fig. 7, where $T^1(i) = T_j$).

Assume that $n \geq 4$. If $T^u(i)$ cuts $T^m(i)$ and if $T^d(i)$ cuts $T^m(i)$, then we take $T^1(i) = T^m(i)$ and we do not define $T^2(i)$ (see Fig. 4). Otherwise, we define $T^1(i)$ and $T^2(i)$ as follows:

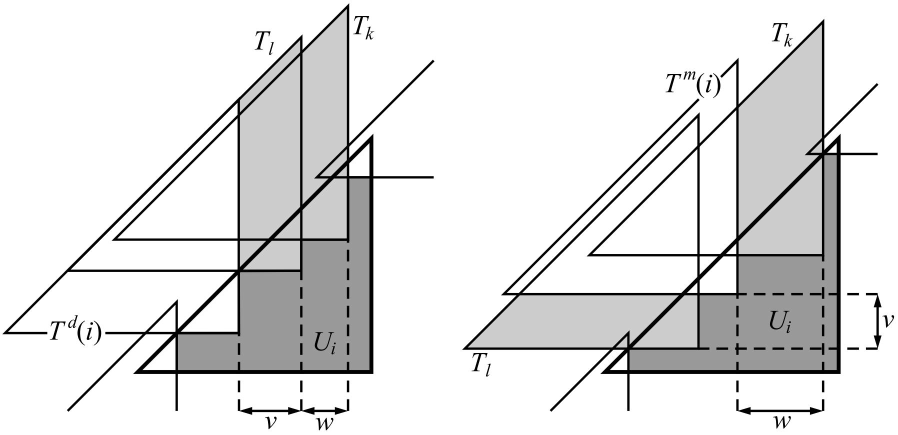


Fig. 5 Triangles of the third type

If $T^u(i)$ cuts $T^m(i)$, then $T^1(i) = T^m(i)$.
If $T^m(i)$ cuts $T^u(i)$, then $T^1(i) = T^u(i)$.
If $T^d(i)$ cuts $T^m(i)$, then $T^2(i) = T^m(i)$.
If $T^m(i)$ cuts $T^d(i)$, then $T^2(i) = T^d(i)$ (see Fig. 5 and the left-hand picture in Fig. 6, where $T^1(i) = T_k$ and $T^2(i) = T_l$).

The triangles $T_1, \dots, T_s$ will be divided into five types. Some of the triangles from among $T_{s+1}, \dots, T_z$ will have defined a type, too. The definition of types is inductive.

First assume that $i = 1$.

(t$_1$) If $|U_i| \le \frac{1}{3}$, then T_i is of the *first type* and T_i is *basic*.
(t$_2$) If $|U_i| > \frac{1}{3}$, if U_i is n-tier, where $n \ge 4$, and if $T^2(i)$ is not defined, then T_i is *basic* and both T_i and $T^1(i)$ are of the *second type*.
(t$_3$) If $|U_i| > \frac{1}{3}$, if U_i is n-tier, where $n \ge 4$, and if $T^2(i)$ has been defined, then T_i is *basic* and T_i, $T^1(i)$ and $T^2(i)$ are of the *third type*.
(t$_4$) If $|U_i| > \frac{1}{3}$, if U_i is 3-tier and if $|U_i| + |U_j| \le \frac{2}{3}$, where $T_j = T^1(i)$, then T_i is *basic* and both T_i and $T^1(i)$ are of the *fourth type*.
(t$_5$) If $|U_i| > \frac{1}{3}$, if U_i is 3-tier and if $|U_i| + |U_j| > \frac{2}{3}$, where $T_j = T^1(i)$, then T_i is *basic* and T_i, $T^1(i)$, $T^1(j)$ and $T^2(j)$ (provided it is defined) are of the *fifth type*. $T^1(j)$ is denoted by $T^2(i)$ and $T^2(j)$ (provided it is defined) is denoted by $T^3(i)$.

Now assume that $i \in \{2, \dots, s\}$ is the smallest integer such that the type of T_i has not been yet defined. The type of T_i is defined by conditions (t$_1$)–(t$_5$).

Assume that T_i and T_l, where $i \ne l$, are basic triangles and that $T^\lambda(i)$ and $T^\mu(l)$ have been defined, where $\lambda, \mu \in \{1, 2, 3\}$. Let j be the integer such that $T_j = T^\lambda(i)$. Since there is only one integer $\eta \ne j$ and there is only one integer κ such that $a_2(j) = a_\kappa(\eta)$, it follows that $T^\lambda(i) \ne T^\mu(l)$.

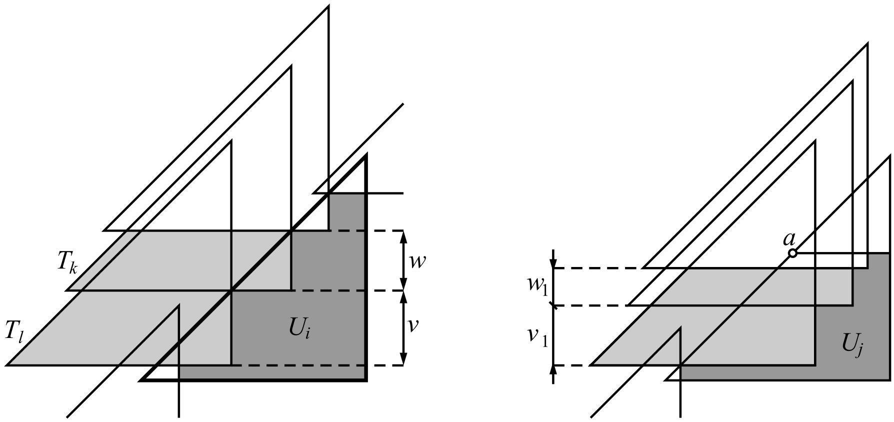


Fig. 6 Triangles of the third and the fifth type

4 Size of Parts Used for the Covering

Obviously, if T_i is a basic triangle of the *first type*, then $|U_i| \le \frac{2}{3}|T_i|$.

Let T_i be a basic triangle of the *second type*. Denote by j the integer such that $T_j = T^1(i)$. Observe that

$$\left|U_j \setminus T_i\right| \le \frac{1}{6}. \tag{1}$$

Denote by p the distance between the vertical leg of T_i and the vertical leg of $T^1(i)$ (see Fig. 4). Furthermore, denote by q the distance between the horizontal leg of $T^1(i)$ and the horizontal leg of T_i. If $p+q \ge \frac{1}{2}$, then $|U_j \setminus T_i| \le \frac{1}{4}|T_j| = \frac{1}{8}$ (see the left-hand picture in Fig. 4). If $p+q < \frac{1}{2}$, then

$$|U_j \setminus T_i| \le \frac{1}{2} - \frac{1}{2}(1-p-q)^2 - (p+q)^2$$

(see the right-hand picture in Fig. 4, where $U_j \setminus T_i$ is contained in the light-grey trapezoid). Since the maximum of this upper bound is reached when $p+q = \frac{1}{3}$, it follows that $|U_j \setminus T_i| \le \frac{1}{6}$. This implies that

$$|U_i| + |U_j| \le \frac{1}{2} + \frac{1}{6} = \frac{2}{3} = \frac{2}{3}\big(|T_i| + |T_j|\big).$$

Let T_i be a basic triangle of the *third type* (see Fig. 5 and the left-hand picture in Fig. 6).

If $T^1(i) = T^u(i)$, then denote by w the distance between the vertical leg of $T^u(i)$ and the vertical leg of $T^m(i)$. If $T^1(i) = T^m(i)$, then denote by w the distance between the horizontal leg of $T^u(i)$ and the horizontal leg of $T^m(i)$. If $T^2(i) = T^m(i)$, then denote by v the distance between the vertical leg of $T^m(i)$ and the vertical leg of $T^d(i)$. If $T^2(i) = T^d(i)$, then denote by v the distance between the horizontal leg

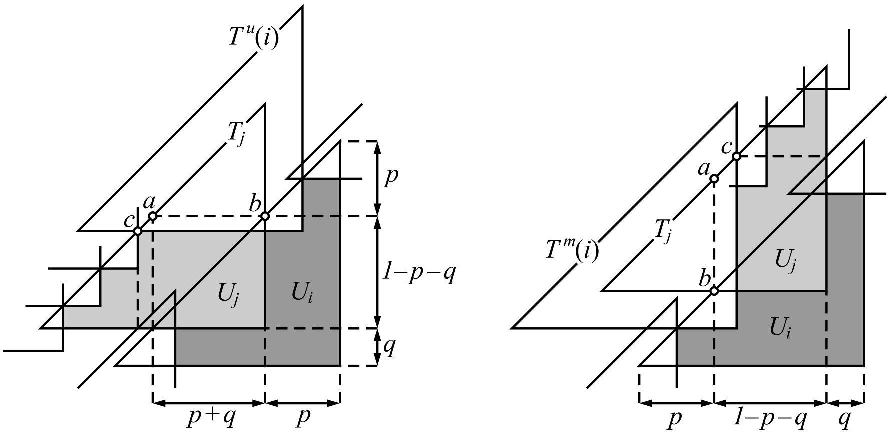


Fig. 7 Triangles of the fourth type

of $T^m(i)$ and the horizontal leg of $T^d(i)$. Furthermore, denote by k and l the integers such that $T_k = T^1(i)$ and $T_l = T^2(i)$. It is easy to see that

$$|U_i| + |U_k| + |U_l| \le \frac{1}{2} + v(1-v) + w(1-w)$$
$$\le \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 = \frac{2}{3}\big(|T_i| + |T_k| + |T_l|\big).$$

If T_i is a basic triangle of the *fourth type* and if $T^1(i) = T_j$, then

$$|U_i| + |U_j| \le \frac{2}{3}(|T_i| + |T_j|).$$

Let T_i be a basic triangle of the *fifth type* and let $T^1(i) = T_j$. Obviously, $|U_i| > \frac{1}{3}$ and $|U_i| + |U_j| > \frac{2}{3}$.

If $T_j = T^m(i)$, then denote by p the distance between the vertical leg of T_j and the vertical leg of T_i and denote by q the distance between the horizontal leg of T_j and the horizontal leg of T_i (see the left-hand picture in Fig. 7). If $T_j = T^u(i)$, then denote by p the distance between the horizontal leg of T_j and the horizontal leg of T_i and denote by q the distance between the vertical leg of T_j and the vertical leg of T_i (see the right-hand picture in Fig. 7).

Note that $\frac{1}{2} < p + q < 0.64$. It is easy to see that

$$|U_i| \le q(1-q) + p(1-q-p) + \frac{1}{4}p^2.$$

If $p + q \le \frac{1}{2}$, then, by a simple calculus argument, this upper bound does not exceed $\frac{1}{3}$. Consequently, T_i is not a basic triangle of the fifth type. Moreover,

$$|U_i| + |U_j| \le q(1-q) + p(1-p-q) + \frac{1}{4}p^2 + \frac{1}{2} - \frac{1}{2}(p+q)^2.$$

If $p+q\geq 0.64$, then, by a simple calculus argument, this upper bound does not exceed $\frac{2}{3}$. Consequently, T_i is not a basic triangle of the fifth type.

We show that U_j is n-tier, where $n\geq 4$. Assume the opposite, that $n\leq 3$.

If $T_j=T^m(i)$, then denote by b the common point of the hypotenuse of T_i and the vertical leg of T_j, by a the common point of the hypotenuse of T_j and the horizontal straight line containing b, and denote by c the common point of the hypotenuse of T_j and the horizontal leg of $T^u(i)$ (see the left-hand picture in Fig. 7). It is easy to see that the area of the part of U_i lying above the horizontal straight line containing b does not exceed $\frac{1}{2}\cdot\frac{1}{2}p^2$. Since U_j is 1-, 2- or 3-tier, the area of the part of U_j lying on the left side of the vertical straight line containing c does not exceed $\frac{2}{3}\cdot\frac{1}{2}(1-p-q)^2$.

If $T_j=T^u(i)$, then denote by b the common point of the hypotenuse of T_i and the horizontal leg of T_j, by a the common point of the hypotenuse of T_j and the vertical straight line containing b, and denote by c the common point of the hypotenuse of T_j and the vertical leg of $T^m(i)$ (see the right-hand picture in Fig. 7). The area of the part of U_i lying on the left side of the vertical straight line containing b does not exceed $\frac{1}{2}\cdot\frac{1}{2}p^2$. The area of the part of U_j lying above the horizontal straight line containing c does not exceed $\frac{2}{3}\cdot\frac{1}{2}(1-p-q)^2$.

Since $p+q>\frac{1}{2}$, it follows that

$$\begin{aligned}|U_i|+|U_j|\leq{}&\frac{1}{3}(1-p-q)^2+\frac{1}{4}p^2+q(1-q)\\&+p(1-p-q)+(p+q)(1-p-q).\end{aligned}$$

This 2nd degree polynomial in two variables reaches its maximum at $p=\frac{1}{3}$ and $q=\frac{1}{6}$, and therefore $|U_i|+|U_j|\leq\frac{2}{3}$, which is a contradiction.

Since U_j is n-tier, where $n\geq 4$, there are two possibilities: either $T^2(j)$ is defined or $T^2(j)$ is not defined.

Assume that $T^2(j)$ is not defined (see Fig. 8; $T_j=T^m(i)$ in the left-hand picture and $T_j=T^u(i)$ in the right-hand picture). Denote by k the integer such that $T_k=T^1(j)$. The area of $U_k\setminus T_j$ is not greater than $\frac{1}{6}$ (see (1)). Consequently,

$$|U_i|+|U_j|+|U_k|\leq q(1-q)+p(1-p-q)+\frac{1}{4}p^2+\frac{1}{2}-\frac{1}{2}(p+q)^2+\frac{1}{6}.$$

This 2nd degree polynomial in two variables reaches its maximum at $p=\frac{2}{7}$ and $q=\frac{1}{7}$. Consequently,

$$|U_i|+|U_j|+|U_k|\leq\frac{37}{42}<1=\frac{2}{3}\big(|T_i|+|T_j|+|T_k|\big).$$

Now assume that $T^2(j)$ is defined. Denote by k and l the integers such that $T^1(j)=T_k$ and $T^2(j)=T_l$.

Assume that $T_j=T^m(i)$. Denote by R_j the part of T_j lying under the horizontal straight line containing a.

If $T_l=T^d(j)$, then take $v_2=0$ and denote by v_1 the distance between the horizontal leg of $T_m(i)$ and the horizontal leg of T_l (see Fig. 10 and the right-hand picture in Fig. 6).

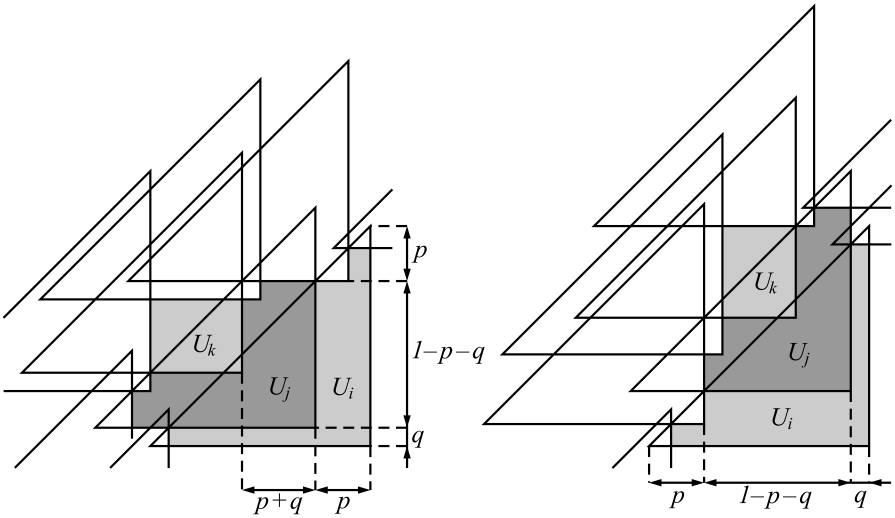


Fig. 8 Triangles of the fifth type, $T^2(j)$ is not defined

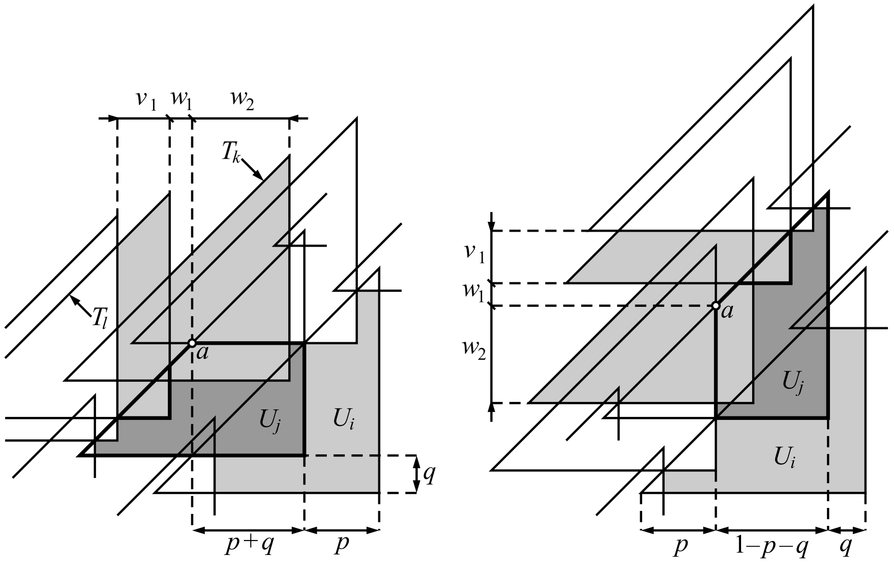


Fig. 9 Triangles of the fifth type, $T^2(j)$ is defined

If $T_l = T^m(j)$ and if $a \in T_k \setminus T_l$, then take $v_2 = 0$ and denote by v_1 the distance between the vertical leg of T_l and the vertical leg of $T^d(j)$ (see the left-hand picture in Fig. 9).

If $T_l = T^m(j)$ and if $a \in T^d(j)$, then take $v_1 = 0$ and denote by v_2 the distance between the vertical leg of T_l and the vertical leg of $T^d(j)$ (see the right-hand picture in Fig. 11).

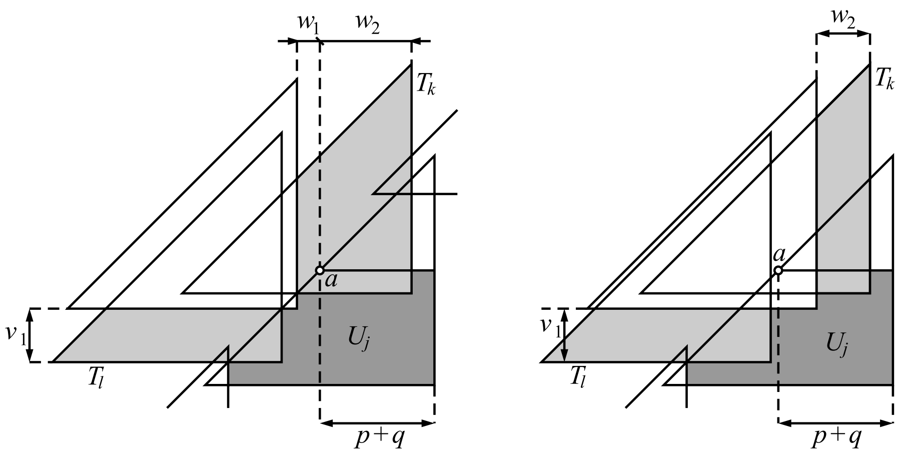


Fig. 10 Triangles of the fifth type, $T^2(j) = T^d(j)$

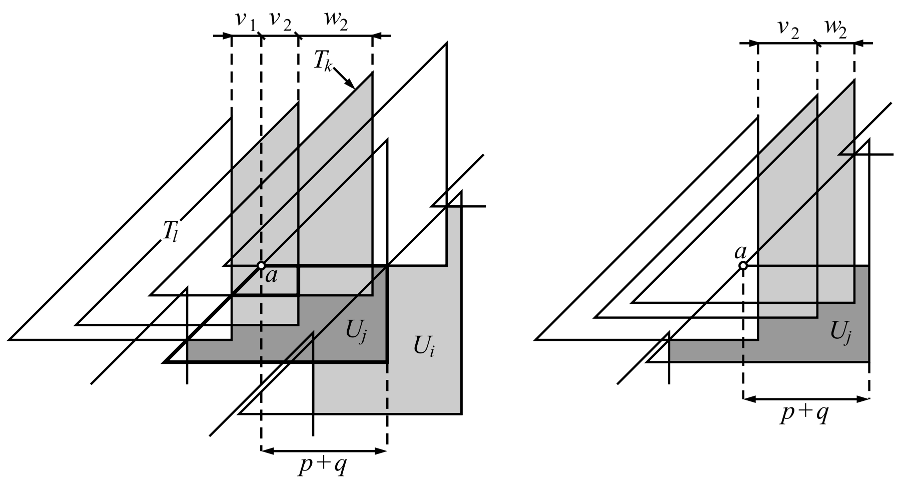


Fig. 11 Triangles of the fifth type, $T^1(j) = T^u(j)$

If $T_l = T^m(j)$ and if $a \in T_l \setminus T^d(j)$, then denote by v_1 the distance between a and the vertical leg of $T^d(j)$ and denote by v_2 the distance between a and the vertical leg of T_l (see the left-hand picture in Fig. 11).

If $T_k = T^m(j)$, then take $w_2 = 0$ and denote by w_1 the distance between the horizontal leg of $T^u(i)$ and the horizontal leg of T_k (see the right-hand picture in Fig. 6).

If $T_k = T^u(j)$ and if $a \notin T_k \setminus T^m(j)$, then take $w_1 = 0$ and denote by w_2 the distance between the vertical leg of $T_m(i)$ and the vertical leg of T_k (see Fig. 11).

If $T_k = T^u(j)$ and if $a \in T_k \setminus T^m(j)$, then denote by w_1 the distance between a and the vertical leg of $T^m(i)$ and by w_2 the distance between a and the vertical leg of T_k (see the left-hand pictures in Figs. 9 and 10).

Consider the case when $v_2 = 0$ (e.g. see the left-hand picture in Fig. 9).

Observe that $\frac{1}{2}v_1^2 \le 0.05$. If $\frac{1}{2}v_1^2 > 0,05$, then

$$|U_i| + |U_j| < q(1-q) + p(1-p-q) + \frac{1}{4}p^2 + \frac{1}{2} - \frac{1}{2}(p+q)^2 - 0.05. \qquad (2)$$

This 2nd degree polynomial in two variables reaches its maximum at $p = \frac{2}{7}$ and $q = \frac{1}{7}$. Consequently, $|U_i| + |U_j| < \frac{2}{3}$, which is a contradiction. This implies that $v \le \sqrt{0.1} < 0.32$ and $|U_l \setminus R_j| \le v_1(1-v_1) < 0.32(1-0.32) < 0.22$.

Obviously,

$$|U_k \setminus R_j| \le (w_1 + w_2)(1 - w_1 - w_2) + \frac{1}{2}w_2^2 = w_1 - w_1^2 + w_2 - \frac{1}{2}w_2^2 - 2w_1w_2.$$

By a simple calculus argument, this upper bound does not exceed $p + q - \frac{1}{2}(p+q)^2$ provided $0 \le w_1 \le 1 - p - q$ and $0 \le w_2 \le p + q$.

Hence, the sum $|U_i| + |U_j| + |U_k| + |U_l|$ is smaller than

$$q(1-q) + p(1-p-q) + \frac{1}{4}p^2 + \frac{1}{2} - \frac{1}{2}(p+q)^2 + p + q - \frac{1}{2}(p+q)^2 + 0.22.$$

This 2nd degree polynomial in two variables reaches its maximum at $p = \frac{2}{5}$ and $q = \frac{1}{5}$. Consequently,

$$|U_i| + |U_j| + |U_k| + |U_l| < \frac{4}{3} = \frac{2}{3}\big(|T_i| + |T_j| + |T_k| + |T_l|\big).$$

Consider the case when $v_2 \ne 0$ (see Fig. 11).

Observe that $\frac{1}{2}v_1^2 + v_1v_2 \le 0.05$, otherwise, by (2), $|U_i| + |U_j| < \frac{2}{3}$, which is a contradiction.

It is easy to see that

$$\big|(U_k \cup U_l) \setminus R_j\big| \le v_1(1 - v_1 - v_2) + v_2 + w_2 - \frac{1}{2}(v_2 + w_2)^2 + v_2(w_2 - v_1).$$

Since $v_2 + w_2 \le p + q < 0.64$, it follows that

$$\big|(U_k \cup U_l) \setminus R_j\big| < v_1(1 - v_1 - v_2) + p + q - \frac{1}{2}(p+q)^2 + v_2(0.64 - v_2 - v_1).$$

If $v_1 \ge 0$, $0 \le v_2 \le 1$ and if $\frac{1}{2}v_1^2 + v_1v_2 \le 0.05$, then, by a calculus argument,

$$v_1(1 - v_1 - v_2) + v_2(0.64 - v_2 - v_1) < 0.22.$$

Consequently, the sum $|U_i| + |U_j| + |U_k| + |U_l|$ does not exceed

$$q(1-q) + p(1-p-q) + \frac{1}{4}p^2 + \frac{1}{2} - \frac{1}{2}(p+q)^2 + p + q - \frac{1}{2}(p+q)^2 + 0.22 < \frac{4}{3}.$$

If $T_j = T^u(i)$, then, arguing in a similar way (e.g. compare the left-hand and the right-hand picture in Fig. 9), we also obtain

$$|U_i| + |U_j| + |U_k| + |U_l| \le \frac{2}{3}\big(|T_i| + |T_j| + |T_k| + |T_l|\big).$$

5 The Proof

Proof Since the problem is invariant under affine transformations of the plane, we can consider coverings with translates of the right isosceles triangle T_0.

Obviously, $\vartheta_T(T_0) \le \vartheta_L(T_0) = \frac{3}{2}$. We show that $\vartheta_T(T_0) \ge \frac{3}{2}$.

Let $\mathcal{T}$ be an arbitrary covering of the plane with translates of T_0. We can assume that no triangles in $\mathcal{T}$ coincide (we are looking for the minimum covering density).

Let r be an arbitrary number greater than 4. We can assume that only finitely many of the triangles in $\mathcal{T}$ intersect $B^2(r)$, otherwise the lower density of $\mathcal{T}$ would be infinite. We define parts used for the covering and types of triangles as in Sects. 2 and 3.

Denote by A_h the set of integers i such that T_i is of the hth type for $h = 1, \dots, 5$ and let $A = A_1 \cup \dots \cup A_5$. From the consideration presented in Sect. 4 we have

$$\sum_{i\in A} |T_i| \ge \frac{3}{2} \sum_{i\in A} |U_i|.$$

Obviously, $T_i \subset B^2(r - 4 + \sqrt{2})$ for $i \in \{1, \dots, s\}$. Moreover, each triangle for which a type has been defined is contained in $B^2(r - 2 + \sqrt{2})$.

Since

$$B^2(r) \supset \bigcup_{i\in A} T_i \supset \bigcup_{i\in A} U_i \supset B^2(r-4),$$

it follows that

$$\sum_{i=1}^{z} \left|T_i \cap B^2(r)\right| \ge \sum_{i\in A} |T_i| \ge \frac{3}{2} \sum_{i\in A} |U_i| \ge \frac{3}{2}\pi (r-4)^2.$$

Consequently,

$$\liminf_{r\to\infty} d\left(\mathcal{T}, B^2(r)\right) \ge \frac{3}{2} \lim_{r\to\infty} \frac{(r-4)^2}{r^2} = \frac{3}{2}. \qquad \square$$

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