

# The Least-Perimeter Partition of a Sphere into Four Equal Areas

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**Abstract** We prove that the least-perimeter partition of the sphere into four regions of equal area is a tetrahedral partition.

**Keywords** Minimal partitions · Isoperimetric problem · Tetrahedral partition

## 1 Introduction

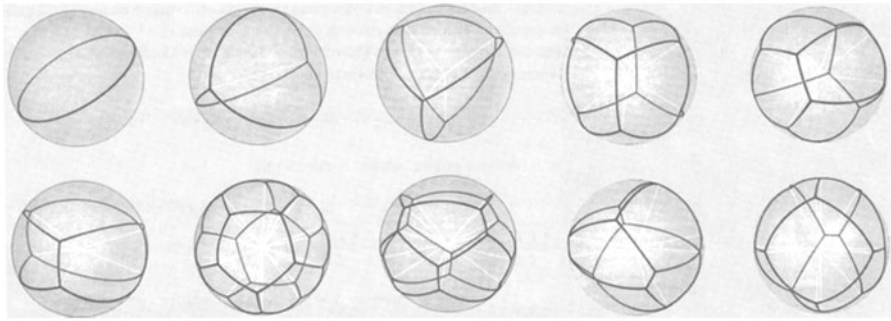
The spherical partition problem asks for the least-perimeter partition of  $\mathbb{S}^2$  into  $n$  regions of equal area. The corresponding planar “Honeycomb Conjecture,” open since antiquity and finally proven by Hales [5] in 2001, states that the regular hexagonal tiling provides a least-perimeter way to partition the plane into unit areas. There are five analogous partitions of the sphere into congruent, regular spherical polygons meeting in threes (see Fig. 1; for why the edges must meet in threes, see Theorem 2.1), three of which have already been proven minimizing:  $n = 2$ , a great circle (Bernstein [2]),  $n = 3$ , three great semi-circles meeting at 120 degrees at antipodal points (Masters [9]), and  $n = 12$ , a dodecahedral arrangement (Hales [6]). The other two, the  $n = 4$  tetrahedral and the  $n = 6$  cubical partitions, were conjectured to be minimizing. In this paper we prove the  $n = 4$  conjecture:

**Theorem 5.2** *The least-perimeter partition of the sphere into four equal areas is the tetrahedral partition.*

The main difficulty is that in principle each region may have many components. Earlier results by Fejes Tóth [4], Quinn [12], and Engelstein et al. [3] required additional assumptions to avoid a proliferation of cases.

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**Fig. 1** The ten partitions of the sphere by geodesics meeting in threes at 120 degrees (picture originally from Almgren and Taylor [1], ©1976 Scientific American)

Our approach starts with easy estimates to show that each region must have one component that encloses the bulk of the area in that region (Proposition 3.5). Examination of the curvature of the interfaces leads to the result that three of the four regions must contain a triangle with large area (Corollary 3.7, Propositions 4.2 and 4.5). Finally in Sect. 5 we examine the fourth region and conclude that the tetrahedral partition is minimizing (Theorem 5.2).

Our proof requires little background knowledge beyond what is discussed in Sect. 2. With the exception of the Gauss–Bonnet theorem, all the ideas and techniques presented are covered by an introductory calculus course.

## 2 Background and Definitions

Before delving into the particulars of the  $n = 4$  case we recall more general results on the existence and regularity of minimizers.

**Theorem 2.1** (Existence: [11], Theorem 2.3 and Corollary 3.3) *Given a smooth compact Riemannian surface  $M$  and finitely many positive areas  $A_i$  summing to the total area of  $M$ , there is a least-perimeter partition of  $M$  into regions of area  $A_i$ . It is given by finitely many constant-curvature curves meeting in threes at 120 degrees at finitely many points.*

It is important to note here that the edges of a minimizing partition are not assumed to be geodesic. In fact, Lamarle [8] and Heppes [7] proved that there are only ten nets of geodesics meeting in threes at 120 degrees on the sphere. These nets are depicted in Fig. 1 and include the previously proved minimizers for  $n = 2$ ,  $n = 3$ , and  $n = 12$  and the conjectured minimizers for  $n = 4$  and  $n = 6$ . For other values of  $n$  the solution cannot be geodesic polygons. However, Maurmann et al. [10] did show that, asymptotically, the perimeter of the solution to the spherical partition problem approaches that of the hexagonal tiling on the plane as  $n$  approaches infinity.

A further regularity condition involves the concept of pressure:

**Theorem 2.2** ([12], Proposition 2.5) *In a perimeter-minimizing partition each region has a pressure, defined up to addition of a constant, so that the difference in pressure between regions  $A$  and  $B$  is the sum of the (signed) curvatures crossed by any path from the interior of  $B$  to the interior of  $A$ .*

**Definition 2.3** Following Quinn [12], we refer to one highest-pressure region as  $R_1$  and then in order of decreasing pressure  $R_2, R_3$ , and  $R_4$ . Let  $\kappa_{ij}$  be equal to the pressure of  $R_i$  minus the pressure of  $R_j$ . Note that  $\kappa_{ij} \geq 0$  if  $i < j$  and  $\kappa_{ij} = -\kappa_{ji}$ . Theorems 2.1 and 2.2 imply that every edge between  $R_i$  and  $R_j$  has (signed) curvature  $\kappa_{ij}$ .

With such strong combinatorial and geometric restrictions on perimeter-minimizing partitions it may be tempting to dismiss the spherical partition problem as a simple exercise in case analysis. The crux of the difficulty (as we mentioned in the introduction) is that disconnected regions are allowed. That regions can, *a priori*, have a finite arbitrary number of components renders a naïve case analysis almost impossible. On the other hand, under the strong assumption that each region is convex, Fejes Tóth [4] proved that each of the partitions in Fig. 1 is minimizing for the areas that it encloses (this also follows easily from the classification of geodesic nets of Fig. 1). For the case of  $n = 4$ , Quinn proved the following, stronger result:

**Theorem 2.4** ([12], Theorem 5.2) *In a perimeter-minimizing partition of the sphere into four equal areas, if  $R_1$  is connected, then that partition is tetrahedral.*

This suggests suggest a more focused analysis on the components of  $R_1$ . In order to avoid confusion we clarify some of our terminology in this manner.

**Definition 2.5** In this paper an  $m$ -gon refers to a spherical polygon with  $m$  edges, each with constant curvature. We write *digon* instead of 2-gon and often use the colloquial triangle, quadrilateral, or pentagon for 3-gon, 4-gon, or 5-gon. Finally we may abuse terminology and use  $m$ -gon to refer to both the polygon and the region bounded by that polygon (allowing us to refer to the “area” of an  $m$ -gon).

Before we delve into the analysis let us recall two more results. The first is due to Quinn [12] and is a corollary of Theorem 2.1.

**Corollary 2.6** ([12], Lemma 2.11) *A perimeter-minimizing partition of the sphere does not contain a set of components whose union is a digon, with distinct incident edges.*

From this it easily follows that in a non-tetrahedral partition no two triangles share an edge. Our second result hinges on the easy observation that no two components of the same region may share an edge.

**Lemma 2.7** *In a perimeter-minimizing partition of the sphere into four equal areas, any component with an odd number of sides is incident to at least one component from every other region.*

Specifically a triangle is adjacent to exactly one component from every other region. With all this in mind we can now move on to the numerical analysis of Sect. 3.

### 3 Area Bounds

In this section we show that every region must consist of one large component and then perhaps several small components (Proposition 3.5). Using the isoperimetric inequality on the sphere and the length of the tetrahedral partition, we are able to establish strict upper bounds on the perimeter of any one region in a potential minimizer. Our starting point is the famous isoperimetric inequality of Bernstein.

**Lemma 3.1** [2] *For given area  $0 < A < 4\pi$ , a curve enclosing area  $A$  on the unit sphere has perimeter  $P \geq B(A) = \sqrt{A(4\pi - A)}$ , with equality only for a single circle.*

Note that Lemma 3.1 gives a lower bound for the perimeter of a region with area  $A$  even when the region is comprised of several connected components.

**Corollary 3.2** *Given a partition of the sphere into  $n$  equal areas, the total perimeter of the partition is greater than  $2\pi\sqrt{n-1}$ .*

*Proof* Each region contains area  $4\pi/n$ . By Lemma 3.1 each region must have perimeter at least  $B(4\pi/n)$ . Multiply by  $n$  for the number of regions and divide by two (as each edge is incident to at most two regions). Simplifying yields the desired result.  $\square$

For  $n = 4$ , Corollary 3.2 yields that the least-perimeter way to partition a sphere into four equal areas must have perimeter at least  $2\pi\sqrt{3} > 10.88$ , whereas the tetrahedral partition has perimeter  $6 \arccos(-1/3) < 11.47$  (given by trigonometry). This yields an immediate upper bound on the size of any one region.

**Corollary 3.3** *In a perimeter-minimizing partition of the sphere into four equal areas every region has perimeter less than 6.62.*

*Proof* Let  $x$  be the perimeter of some region. By Lemma 3.1 the perimeter  $P$  of the entire partition satisfies  $P > (1/2)(x + 3\pi\sqrt{3})$ . Yet if the partition is minimizing, then we have  $P < 11.47$ . Numerics yield  $x < 6.62$ , the desired result.  $\square$

We will now prove and apply an inequality which will force any region to have one large component (Proposition 3.5).

**Lemma 3.4** *The function (for fixed  $0 < k \leq 2\pi$ )*

$$f_k(t) = \sqrt{t(4\pi - t)} + \sqrt{(k - t)(4\pi - k + t)}$$

*defined on the interval  $[0, k]$  is symmetric about the point  $t = k/2$ , and  $f'_k(t) > 0$  for all  $0 < t < k/2$ .*

*Proof* It is evident that the function is symmetric about  $t = k/2$ . The radicands are downward parabolas (in  $t$ ), so the sum of their square roots is a concave down function. Symmetry implies the desired result.  $\square$

**Proposition 3.5** *In a perimeter-minimizing partition of the sphere into four equal areas, every region must contain a component with area at least  $23\pi/25$ .*

*Proof* Let  $t$  be the area of the largest component in the given region. By Lemma 3.1 we have the inequality  $P(t) \geq B(t) + B(\pi - t)$ , where  $P$  is the perimeter of the region. Corollary 3.3 yields  $6.62 > B(t) + B(\pi - t)$ . On the other hand, setting  $t = 23\pi/25$  gives

$$P\left(\frac{23\pi}{25}\right) \geq B\left(\frac{23\pi}{25}\right) + B\left(\frac{2\pi}{25}\right) = \frac{\pi}{25}(\sqrt{23 \cdot 77} + 14) \approx 7 > 6.62.$$

So Lemma 3.4 says  $t > 23\pi/25$  or  $t < 2\pi/25$ .

Suppose  $t < 2\pi/25$ . By Lemma 3.1 when  $A = 2\pi/25$ , the region must have perimeter greater than  $14\pi/25 = 7A$ . As  $B(x)$  is concave down we have  $B(x) \geq 7x$  for  $x < 2\pi/25$ . Therefore the perimeter of the region is greater than 7 times the area of the region. So  $t < 2\pi/25$  implies that the perimeter of the region is at least  $7\pi \approx 21.99 > 6.62$ , a clear contradiction of Corollary 3.3.  $\square$

The following Lemma 3.6 due to Quinn [12] will produce a large triangle in  $R_1$  (Corollary 3.7).

**Lemma 3.6** ([12], Lemma 5.12) *In the highest-pressure region of a perimeter-minimizing partition, (1) a triangle must have area less than or equal to  $\pi$ , (2) a square must have area less than or equal to  $2\pi/3$ , (3) a pentagon must have area less than or equal to  $\pi/3$ , and (4) all other polygons cannot exist. Equality can only occur when the polygon is geodesic.*

*Proof* The result follows directly from Gauss–Bonnet and the convexity of the components of  $R_1$ .  $\square$

**Corollary 3.7** *In a perimeter-minimizing partition of the sphere into four equal areas,  $R_1$  must contain a triangle, and this triangle must have area at least  $23\pi/25$ .*

*Proof* The result is immediate from Lemma 3.6 and Proposition 3.5.  $\square$

#### 4 $R_2$ and $R_3$ Contain Large Triangles

The goals of this section are Propositions 4.2 and 4.5: that  $R_2$  and  $R_3$  each contains a triangle with area at least  $23\pi/25$ . The possibility that the components of  $R_2$  or  $R_3$  are not convex prohibits us from using Lemma 3.6 and necessitates a closer look at the curvature of the interfaces. We start off by bounding  $\kappa_{12}$ .

**Lemma 4.1** *In a least-perimeter partition of the sphere into four equal areas,  $\kappa_{12} < 1/21$ .*

*Proof* By Corollary 3.7,  $R_1$  has a triangle,  $T$ , of area  $A_T \geq 23\pi/25$ . By Gauss–Bonnet, the perimeter  $P$  and exterior angles  $\alpha_i$ , of this triangle satisfy

$$2\pi = A_T + \int_{\partial T} \kappa ds + \sum \alpha_i \geq \frac{23\pi}{25} + P\kappa_{12} + \pi,$$

so  $P\kappa_{12} \leq 2\pi/25$ . By Lemma 3.1,  $P \geq B(23\pi/25)$ . Therefore  $\kappa_{12} < 1/21$ .  $\square$

Now we are able to establish an analogue to Corollary 3.7 for  $R_2$ .

**Proposition 4.2** *In a least-perimeter partition of the sphere into four equal areas,  $R_2$  contains a triangle of area at least  $23\pi/25$ .*

*Proof* Proposition 3.5 says that  $R_2$  must have a component with area no less than  $23\pi/25$ . Assume by way of contradiction that this component has at least four sides. Then Gauss–Bonnet gives

$$\frac{23\pi}{25} - \kappa_{12}P_{12} + \kappa_r P_r + \frac{4\pi}{3} \leq 2\pi,$$

where  $P_{12}$  is the perimeter between  $R_1$  and  $R_2$ .  $\kappa_r P_r$  represents the (positive) contribution of curvature from lower pressure regions. Combining terms we get:  $\pi(23/25 - 2/3) \leq \kappa_{12}P_{12}$ . Bounding  $\kappa_{12}$  using Lemma 4.1 yields  $19\pi/75 \leq P_{12}/21$ . Isolating  $P_{12}$  gives  $P_{12} \geq 133\pi/25 > 12$ , obviously contradicting Corollary 3.3.  $\square$

In order to establish an analogue to Lemma 4.1 for  $R_3$ , we must insure that  $R_2$  does not occupy too much of the perimeter of  $R_1$ 's large triangle. A quick corollary bounds the length of the side of  $R_1$ 's large triangle which is incident to  $R_2$  (a side we know exists by Lemma 2.7).

**Corollary 4.3** *In a least-perimeter partition of the sphere into four equal areas, let  $P$  be the perimeter of the large triangle in  $R_1$ , and let  $l$  be the length of the side incident to  $R_2$  in that triangle. Then  $l \leq P/3$ .*

*Proof* If the partition in question is tetrahedral, then the statement is trivial. Assume that it is not tetrahedral and, to obtain a contradiction, that  $l > P/3$ . By Corollary 2.6 no two triangles are incident to one another in a non-tetrahedral minimizing partition. Therefore the perimeter of  $R_2$  is at least  $P_2 + l$ , where  $P_2$  is the perimeter of the large triangle in  $R_2$  (whose existence is established in Proposition 4.2). By Lemma 3.1 we have that the perimeter of  $R_2$  is at least  $(4/3)B(23\pi/25) > 7$ , which contradicts the partition's minimality by Corollary 3.3.  $\square$

Now we proceed as in the  $R_2$  case; first we bound  $\kappa_{13}$ .

**Lemma 4.4** *In a least-perimeter partition of the sphere into four equal areas,  $\kappa_{13} < 1/14$ .*

*Proof* By Corollary 3.7  $R_1$  has a large triangle. Let  $P$  be the perimeter of this triangle and  $P'$  the lengths of the side of the triangle which are not incident to  $R_2$ . Then Corollary 4.3 states that  $P \leq 3P'/2$ . Using Lemma 3.1 to obtain a lower bound for  $P$ , we write  $P' \geq (2/3)B(23\pi/25)$ . Applying Gauss–Bonnet to this large triangle gives the inequality  $23\pi/25 + \kappa_{13}P' \leq \pi$ . Substituting the bound on  $P'$  and simplifying results in the desired inequality  $\kappa_{13} \leq 3/\sqrt{23 \cdot 77} < 1/14$ .  $\square$

In the same vein as Proposition 4.2 we now prove that  $R_3$  must contain a triangle with area at least  $23\pi/25$ .

**Proposition 4.5** *In a least-perimeter partition of the sphere into four equal areas,  $R_3$  contains a triangle with area at least  $23\pi/25$ .*

*Proof* By Proposition 3.5  $R_3$  must contain some component with area at least  $23\pi/25$ . For the sake of contradiction, assume that component has at least four sides. Let  $P$  be the perimeter of this component, then Gauss–Bonnet yields

$$\frac{23\pi}{25} - \kappa_{13}P + \frac{4\pi}{3} \leq 2\pi,$$

or  $\kappa_{13}P \geq 19\pi/75$ . By Lemma 4.4 we get that  $\kappa_{13} < 1/14$ , which means that  $P > 266\pi/75 > 11$  which contradicts the partition's minimality by Corollary 3.3.  $\square$

Now we can establish an analogue to Corollary 4.3 for  $R_3$ .

**Corollary 4.6** *In a least-perimeter partition of the sphere into four equal areas, let  $P$  be the perimeter of the large triangle in  $R_1$ , and let  $l$  be the length of the side incident to  $R_3$  in that triangle. Then  $l \leq P/3$ .*

*Proof* If the partition is tetrahedral, then the statement is trivial. Assume that it is not tetrahedral and, to obtain a contradiction, that  $l > P/3$ . By Corollary 2.6 no two triangles are incident to one another in a non-tetrahedral minimizing partition. Therefore the perimeter of  $R_3$  is at least  $P_3 + l$ , where  $P_3$  is the perimeter of the large triangle in  $R_3$  (whose existence is established in Proposition 4.5). By Lemma 3.1 we have that the perimeter of  $R_3$  is at least  $(4/3)B(23\pi/25) > 7$ , which contradicts the partition's minimality by Corollary 3.3.  $\square$

## 5 The Tetrahedral Partition Is Minimizing

In this section we reach our goal in Theorem 5.2, which states that the perimeter-minimizing partition of the sphere into four equal areas is the tetrahedral partition. We require only one lemma, a lower bound for  $\kappa_{14}$ .

**Lemma 5.1** *In a non-tetrahedral perimeter-minimizing partition of the sphere into four equal areas, we have  $\kappa_{14} > 1/2$ .*

*Proof* Let  $P$  be the perimeter of the large triangle in  $R_1$  (which we know exists by Corollary 3.7) and  $P_r$  be the rest of the perimeter of  $R_1$ . Corollary 3.3 then gives

$$6.62 > P + P_r \geq B(23\pi/25) + P_r,$$

where the second inequality is Lemma 3.1. This yields  $P_r < 1.34$ .

Since the partition is non-tetrahedral, by Theorem 2.4  $R_1$  must have another component, and this component has at most five sides (Lemma 3.6). Use Gauss–Bonnet on this component to obtain a second inequality on  $P_r$ :

$$\kappa_{14} P_r \geq \frac{\pi}{3} - \frac{2\pi}{25}$$

and combine the two inequalities to get that  $\kappa_{14} > 1/2$ . □

Now we reach our ultimate goal.

**Theorem 5.2** *The least-perimeter partition of the sphere into four equal areas is the tetrahedral partition.*

*Proof* Assume that  $R_1$  is non-tetrahedral. Let  $P$  be the perimeter of the large triangle in  $R_1$  (which exists by Corollary 3.7), and let  $l_4$  be the length of the side of this large triangle incident to  $R_4$ . By Corollaries 4.3 and 4.6 and Lemma 3.1 we have  $l_4 \geq P/3 \geq (1/3)B(23\pi/25)$ .

Apply Gauss–Bonnet to the large triangle in  $R_1$  to get

$$\frac{23\pi}{25} + \frac{\kappa_{14} B(\frac{23\pi}{25})}{3} \leq \pi.$$

Simplify and isolate  $\kappa_{14}$  to obtain  $\kappa_{14} \leq 6/\sqrt{23 \cdot 77} < 1/7$ , a clear contradiction of Lemma 5.1. □

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