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Simultaneous Packing and Covering in Sequence Spaces

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Abstract We adapt a construction of Klee (1981) to find a packing of unit balls in ℓ_p ($1 \le p < \infty$) which is efficient in the sense that enlarging the radius of each ball to any $R > 2^{1-1/p}$ covers the whole space. We show that the value $2^{1-1/p}$ is optimal.

Keywords Packings · Coverings · Simultaneous packing and covering constant · ℓ_p sequence space · Closest packing · Loosest covering

1 Introduction

The so-called simultaneous packing and covering constant of a convex body C in Euclidean space is a certain measure of the efficiency of a packing or a covering by translates of C. This notion was introduced in various equivalent forms by Rogers [14], Ryškov [16], and L. Fejes Tóth [8] and in the lattice case can be traced back to Delone [5]. Its study has recently been given renewed attention by Zong [19–23] and others [9, 17]. Important contributions to the non-lattice case have also been made by Linhardt [13], Böröczky [2], and Doyle, Lagarias, and Randall [7]. Since this notion avoids the use of density, it can be used to study packings and coverings in hyperbolic spaces or infinite-dimensional spaces. Rogers [15] considered the infinite-dimensional lattice case for the unit balls of general Banach spaces. In this paper we determine the exact value of this constant for the ℓ_p spaces where $1 \le p < \infty$. The main ingredient in the proof is an adaptation of a construction of Klee [10].

Dedicated to the memory of Victor Klee.

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2 The Simultaneous Packing and Covering Constant

Let $(X, \|\cdot\|)$ be any normed space. Denote by $B(x,r)$ the closed ball with center $x \in X$ and radius r. A subset $P \subseteq X$ is (the set of centers of) *an r-packing* if the collection of balls $\{B(x,r) : x \in P\}$ are pairwise disjoint. Equivalently, P is $2r$*-dispersed*, i.e., $d(x,y) > 2r$ for all distinct $x, y \in P$. For any $P \subseteq X$, define

$$r(P) := \sup\{r : P \text{ is an } r\text{-packing}\}.$$

A subset $P \subseteq X$ is (the set of centers of) *an R-covering* (or *R-net*) if the collection of balls $\{B(x,R) : x \in P\}$ cover X, i.e., $X = \bigcup_{x\in P} B(x,R)$. For any $P \subseteq X$, define

$$R(P) := \inf\{R : P \text{ is an } R\text{-covering}\}.$$

Then $R(P)$ is the supremum of the radii of balls disjoint from P:

$$R(P) = \sup\big\{R : \text{for some } x \in X,\ B(x,R) \cap P = \emptyset\big\}. \tag{1}$$

If P is an r-packing, then $R(P) - r$ is the supremum of the radii of balls that are disjoint from $\bigcup_{p\in P} B(p,r)$, and if P is an R-covering, then $R - r(P)$ is the supremum of the radii of balls that are contained in more than one of $B(p,R)$, $p \in P$ [8].

Definition The *simultaneous packing and covering constant* of (the unit ball of) X is

$$\gamma(X) := \inf\big\{R(P) : P \text{ is a 1-packing}\big\}.$$

We could also use 1-coverings to define this constant, as shown by the identity

$$\gamma(X)^{-1} = \sup\big\{r(P) : P \text{ is a 1-covering}\big\}.$$

It is clear that $R(P) \geq 1$ for any 1-packing P. By Zorn's lemma there always exists a maximal 1-packing, which is necessarily a 2-covering. Therefore,

$$1 \leq \gamma(X) \leq 2.$$

If X is finite-dimensional, then $\gamma(X)$ is exactly the simultaneous packing and covering constant of the unit ball of X, as discussed in the introduction.

3 The Main Theorem

The main result of the paper concerns the case $X = \ell_p$, $1 \leq p < \infty$, which we recall is the space of real sequences $x = (x_i)_{i\in\mathbb{N}}$ such that $\sum_{i=1}^{\infty} |x_i|^p < \infty$ with norm

$$\|x\|_p := \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}.$$

Theorem *For each $p \in [1, \infty)$, $\gamma(\ell_p) = 2^{1-1/p}$.*

In particular, if p is close to 1, then $\gamma(\ell_p)$ is close to 1, which means that there are very good packings of unit balls in ℓ_p. Perhaps more surprisingly, if p is very large, $\gamma(\ell_p)$ is close to 2, i.e., any packing by unit balls has large holes.

In the next section we use a result of Burlak, Rankin, and Robertson [3] to show the lower bound $\gamma(\ell_p) \geq 2^{1-1/p}$. It is more difficult to find good packings. In Sect. 5 we adapt a construction of Klee [10] to give packings that demonstrate the upper bound $\gamma(\ell_p) \leq 2^{1-1/p}$. In fact, Klee already obtained this bound for $\ell_p(\kappa)$, where κ is a regular cardinal such that $\kappa^{\aleph_0} = \kappa$. In our case, $\kappa = \aleph_0$, and then his construction has to be modified substantially.

4 The Lower Bound

For a proof of the following packing property of ℓ_p, see [3, 11], or [18].

Lemma 1 *If the unit ball of ℓ_p contains an infinite α-dispersed set, then $\alpha \leq 2^{1/p}$.*

To prove $\gamma(\ell_p) \geq 2^{1-1/p}$, it is sufficient to show the following:

Proposition 2 *Let P be a 1-dispersed subset of ℓ_p, where $1 \leq p < \infty$. Then $R(P) \geq 2^{-1/p}$.*

Proof Let $0 < \varepsilon < R(P)$. Set $r := R(P) - \varepsilon$ and $\delta := ((r + 2\varepsilon)^p - r^p)^{1/p}$. By (1) there exists $c \in \ell_p$ such that $B(c, r) \cap P = \emptyset$. Translate P by $-c$ so that we may assume without loss of generality that $c = o$. Thus $\|x\| > r$ for all $x \in P$.

We claim that $Q := B(o, r + \delta + 2\varepsilon) \cap P$ is infinite. Suppose to the contrary that Q is finite. As usual, we denote by e_n the sequence which is 1 in position n and 0 in all other positions. For any $n \in \mathbb{N}$ and $x \in Q$,

$$\|x - \delta e_n\|_p^p = \|x\|_p^p - |x_n|^p + |x_n - \delta|^p.$$

Therefore,

$$\lim_{n\to\infty} \|x - \delta e_n\|_p^p = \|x\|_p^p + \delta^p > r^p + \delta^p = (r + 2\varepsilon)^p.$$

Since Q is finite, there exists $n \in \mathbb{N}$ such that for all $x \in Q$, $\|x - \delta e_n\| > r + 2\varepsilon$. On the other hand, for any $x \in P \setminus Q$, $\|x\| > r + \delta + 2\varepsilon$, and then the triangle inequality gives $\|x - \delta e_n\| > r + 2\varepsilon$. Therefore, $B(\delta e_n, r + 2\varepsilon) \cap P = \emptyset$, which gives $R(P) \geq r + 2\varepsilon$, a contradiction.

Thus Q is infinite, and by Lemma 1,

$$R(P) + \delta + \varepsilon = r + \delta + 2\varepsilon \geq 2^{-1/p}.$$

By letting $\varepsilon \to 0$ we obtain $R(P) \geq 2^{-1/p}$, as required. □

5 Constructing an Optimal Packing

To prove the upper bound $\gamma(\ell_p) \le 2^{1-1/p}$, it is sufficient to show the following:

Proposition 3 *For any $1 \le p < \infty$, there exists a $2^{1/p}$-dispersed set $P \subseteq \ell_p$ such that $R(P) \le 1$.*

Proof We recursively construct the set P together with the space, which in the end is isometric to ℓ_p.

If A is any set, we denote by $\ell_p(A)$ the normed space of all real-valued functions f on A with countable support $\operatorname{supp}(f) := \{a \in A : f(a) \neq 0\}$ such that $\sum_{a\in\operatorname{supp}(f)}|f(a)|^p < \infty$, with norm

$$\|f\|_p := \left(\sum_{a\in\operatorname{supp}(f)} \left|f(a)\right|^p\right)^{1/p}.$$

Thus $\ell_p = \ell_p(\mathbb{N})$ is isometric to $\ell_p(A)$ if A is countably infinite. For any $a \in A$, let e_a be the function on A such that $e_a(a) = 1$ and $e_a(b) = 0$ for all $b \in A$, $b \neq a$. If $A \subseteq A'$, then we consider $\ell_p(A)$ to be a subspace of $\ell_p(A')$ in the natural way.

We construct two sequences of countable sets P_n and D_n. Let $P_1 = \emptyset$ and $D_1 = \{0\} = \ell_p(\emptyset)$. If $P_1, \dots, P_n$ and $D_1, \dots, D_n$ have been constructed for some $n \ge 1$, let

$$P_{n+1} := \{x + e_x : x \in D_n\} \subseteq \ell_p\left(\bigcup_{i=1}^n D_i\right),$$

and let D_{n+1} be a countable dense subset of

$$\ell_p\left(\bigcup_{i=1}^n D_i\right) \Big\backslash \bigcup\left\{B(x,1) : x \in \bigcup_{i=1}^{n+1} P_i\right\}.$$

By the definition of P_{n+1} it follows that $D_k \subseteq \bigcup_{x\in P_{k+1}} B(x,1)$ for each $k = 1, \dots, n$; hence, D_{n+1} is disjoint from $\bigcup_{i=1}^n D_i$. It follows that the P_n are also pairwise disjoint.

Let $P := \bigcup_{n\in\mathbb{N}} P_n$. Then P is a subset of the space $\ell_p(\bigcup_{n\in\mathbb{N}} D_n)$, which is isometric to ℓ_p (note that already D_2 is infinite). We now show that P is $2^{1/p}$-dispersed and is a $(1+\varepsilon)$-covering for all $\varepsilon > 0$.

Choose two arbitrary elements $x + e_x, y + e_y \in P$, where $x \in D_n$ and $y \in D_m$, $x \neq y$, and $n \le m$. Since $\operatorname{supp}(x), \operatorname{supp}(y) \subseteq \bigcup_{i=1}^{m-1} D_i$, which is disjoint from D_m, it follows that $\operatorname{supp}(x-y)$ and $\operatorname{supp}(e_y) = \{y\}$ are disjoint. We distinguish between two cases.

If $n = m$, then $\operatorname{supp}(e_x) = \{x\}$ is also disjoint from $\operatorname{supp}(x-y)$ and $\operatorname{supp}(e_y)$; hence,

$$\left\|(x+e_x) - (y+e_y)\right\|_p^p = \|x - y + e_x - e_y\|_p^p = \|x-y\|_p^p + 1 + 1 > 2.$$

In the second case, $n < m$. Since $y \in D_m$ and $x + e_x \in P_{n+1}$, it follows that $y \notin B(x + e_x, 1)$; hence, $\|x + e_x - y\|_p^p > 1$. Since $\operatorname{supp}(x + e_x - y)$ and $\operatorname{supp}(e_y)$ are now disjoint,

$$\left\|(x+e_x)-(y+e_y)\right\|_p^p = 1 + \|x+e_x-y\|_p^p > 1+1.$$

It follows that P is $2^{1/p}$-dispersed.

Let $\varepsilon > 0$ and choose an arbitrary $x \in \ell_p(\bigcup_{n\in\mathbb{N}} D_n)$. Choose $N \in \mathbb{N}$ large enough such that $\|x-y\|_p < \varepsilon/2$ for some $y \in \ell_p(\bigcup_{i=1}^{N-1} D_i)$. If $y \in \bigcup\{B(z,1) : z \in \bigcup_{i=1}^{N} P_i\}$, then for some $z \in \bigcup_{i=1}^{N} P_i$,

$$\|x-z\|_p \le \|x-y\|_p + \|y-z\|_p < 1+\varepsilon/2.$$

If, on the other hand, $y \notin \bigcup\{B(z,1) : z \in \bigcup_{i=1}^{N} P_i\}$, then there exists $z \in D_N$ such that $\|y-z\|_p^p < (1+\varepsilon/2)^p - 1$. Note that $z + e_z \in P_{N+1}$. Since $z \in D_N$ and $y, z \in \ell_p(\bigcup_{i=1}^{N} D_i)$, $\operatorname{supp}(e_z) = \{z\}$ is disjoint from $\operatorname{supp}(y-z)$. Thus,

$$\|z+e_z-y\|_p^p = 1 + \|z-y\|_p^p < (1+\varepsilon/2)^p,$$

and by the triangle inequality, $\|z+e_z-x\| < 1+\varepsilon$. It follows that P is a $(1+\varepsilon)$-covering. □

6 Closing Remarks and Questions

6.1

Define the *lattice simultaneous packing and covering constant* of (the unit ball of) a normed space X to be

$$\gamma^*(X) := \inf\bigl\{R(P) : P \text{ is a 1-packing and a subgroup of } (X,+)\bigr\}.$$

Since an arbitrary maximal 1-packing cannot be expected to be closed under addition, it becomes nontrivial to find an upper bound to $\gamma^*(X)$. Rogers [15] showed that $\gamma^*(X) \le 3$ for any Banach space X. Can this upper bound be improved to $\gamma^*(X) \le 2$?

6.2

Let

$$\beta(X) := \sup\{\alpha : \text{the unit ball of } X \text{ contains an infinite } \alpha\text{-dispersed set}\}.$$

This is also known as the separation measure of noncompactness of the unit ball of X [1]. Elton and Odell proved that $\beta(X) > 1$ for any Banach space X. For more information, see [12] and references therein. Using a result of Corson [6], Casini, Papini, and Zanco [4] generalized Proposition 2 by showing that $\gamma(X) \ge 2/\beta(X)$ if X has an infinite-dimensional reflexive subspace. This, together with Lemma 1, immediately implies Proposition 2. However, our proof for ℓ_p is direct.

The following question is essentially stated in [4]: Is $\gamma(X) = 2/\beta(X)$ or perhaps even $\gamma^*(X) = 2/\beta(X)$?

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