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Bichromatic and Equichromatic Lines in $\mathbb{C}^2$ and $\mathbb{R}^2$

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Abstract Let G and R each be a finite set of green and red points, respectively, such that $|G| = n$, $|R| = n - k$, $G \cap R = \emptyset$, and the points of $G \cup R$ are not all collinear. Let t be the total number of lines determined by $G \cup R$. The number of equichromatic lines (a subset of bichromatic) is at least $(t + 2n + 3 - k(k + 1))/4$. A slightly weaker lower bound exists for bichromatic lines determined by points in $\mathbb{C}^2$. For sufficiently large point sets, a proof of a conjecture by Kleitman and Pinchasi is provided. A lower bound of $(2t + 14n - k(3k + 7))/14$ is demonstrated for bichromatic lines passing through at most six points. Lower bounds are also established for equichromatic lines passing through at most four, five, or six points.

1 Introduction

Questions concerning "how many?" or "what types?" of lines are determined by a set of points have been asked since (no later than) 1893 when J.J. Sylvester, in [14], essentially asked whether any non-collinear set of points necessarily determines an ordinary line (i.e., a line that passes through exactly two points). Unaware that it had previously been raised, Erdös arrived at the same question (about 40 years later) while attempting to prove by induction that there are at least n lines determined by a set of n points. T. Gallai provided a proof in the affirmative which was published by Steinberg in [13], hence it is now called the Sylvester–Gallai Theorem. Interestingly, an inequality due to Melchior also provides a proof of this theorem and was published four years earlier in [11].

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More directly related to this article is a question first asked by R.L. Graham around 1965 (see [5]) of whether a bichromatic arrangement of lines necessarily determines a monochromatic point (the dual of the context for the present article). A few years passed before the first proof was published, again in the affirmative, by G.D. Chakerian in [1]. However, Chakerian is not credited with the first proof. That honor belongs to T.S. Motzkin and M. Rabin (and in his article, Chakerian acknowledges this). The theorem is now commonly called the Motzkin–Rabin Theorem.

Motivated by a conjecture of Fukuda [2] (now known to be false for a specific case with nine points [4]), in their groundbreaking paper Pach and Pinchasi demonstrated several lower bounds on the number of bichromatic lines incident to few points [12] (see Sects. 2.3.3 and 3.2 for corrections and improvements to a result from this paper). A line is bichromatic when it passes through at least one point of each color, and monochromatic otherwise. (Note that only lines incident to at least two points are considered.) The best lower bounds established in [12] concern the case where there is an equal number of points of each color, i.e., $|G| = |R|$. The present article provides improved lower bounds for a subset of the bichromatic lines, called equichromatic. Several lower bounds are shown to also be true for lines determined by points in the complex plane ($\mathbb{C}^2$).

In [10], Kleitman and Pinchasi consider the case where neither color class, G nor R, is collinear. They conjecture that among all such arrangements there are at least $|G \cup R| - 1$ bichromatic lines, assuming that $|G| = n$ and $n - 1 \le |R| \le n$, and they prove a lower bound of $|G \cup R| - 3$ bichromatic lines. The present article will prove their conjecture for sufficiently large n.

Pach and Pinchasi derive their results from two identities, which are used again in [10]. Let $t_{i,j}$ be the number of lines which pass through exactly i green and j red points. With the assumption that $|G| = |R| = n$, they show that the number of bichromatic point pairs (i.e., n^2) is equal to a summation over all lines determined by the sets:

$$\sum_{\substack{i,j\ge 0\\ i+j\ge 2}} ij t_{i,j} = n^2.$$

Similarly, monochromatic pairs of points can be counted as

$$\sum_{\substack{i,j\ge 0\\ i+j\ge 2}} \left[\binom{i}{2} + \binom{j}{2}\right] t_{i,j} = 2\binom{n}{2} = n^2 - n.$$

More generally, if one assumes that $|G| = n$ and $|R| = n - k$, these identities become

$$\sum_{\substack{i,j\ge 0\\ i+j\ge 2}} ij t_{i,j} = n(n-k) = n^2 - nk \tag{1}$$

and

$$\sum_{\substack{i,j\ge 0\\ i+j\ge 2}} \left[\binom{i}{2} + \binom{j}{2}\right] t_{i,j}$$

$$
\begin{aligned}
&= \frac{n(n-1)}{2} + \frac{(n-k)(n-k-1)}{2} \\
&= n^2 - n - nk + \frac{k^2+k}{2}. \qquad (2)
\end{aligned}
$$

By subtracting (1) from (2) and then splitting the summation, the following identity is found:

$$\sum_{\substack{i,j\geq 0\\ i+j\geq 2}} (i+j)t_{i,j} = \sum_{\substack{i,j\geq 0\\ i+j\geq 2}} (i-j)^2 t_{i,j} + 2n - \bigl(k^2+k\bigr). \tag{3}$$

This identity will be used throughout the present article.

2 Equichromatic Lines

2.1 Lower Bound in $\mathbb{R}^2$

Definition 2.1 Any line passing through i green and j red points such that $i+j\geq 2$ and $|i-j|\leq 1$ is called *equichromatic*.

An *equichromatic line* can be thought of as one in which the number of points of each color, lying on the line, are as "equal" as possible, i.e., equal if the line contains an even number of points, and otherwise differing by only one. Let Q be the set of equichromatic lines determined by $G\cup R$, and let B be the set of bichromatic lines. A lower bound on the number of equichromatic lines is demonstrated below, and since $Q\subseteq B$, this also applies to bichromatic lines.

Let t_k be the number of lines which pass through exactly k points. In [11], Melchior published the following inequality (which follows from Euler's polyhedral formula):

$$\sum_{k\geq 2}(k-3)t_k \leq -3.$$

Within our context, this can be rewritten as

$$\sum_{\substack{i,j\geq 0\\ i+j\geq 2}} (i+j)t_{i,j} \leq 3\cdot \sum_{\substack{i,j\geq 0\\ i+j\geq 2}} t_{i,j} - 3. \tag{4}$$

By applying to this the identity above, (3), and then reuniting the two summations this becomes

$$\sum_{\substack{i,j\geq 0\\ i+j\geq 2}} \bigl((i-j)^2-3\bigr)t_{i,j} \leq -2n-3+\bigl(k^2+k\bigr).$$

Let q_m be the sum of all $t_{i,j}$ such that $|i-j|=m$, that is, $q_m=\sum_{|i-j|=m} t_{i,j}$. Note that the number of equichromatic lines (i.e., $|Q|$) is equal to q_0+q_1. Let t be the total number of lines determined by $G \cup R$. Now, by partially unwinding the summation and then negating the inequality, it becomes

$$3q_0 + 2q_1 \geq 2n + 3 + q_2 + 6q_3 + \sum_{m\geq 4} \big(m^2 - 3\big) q_m - \big(k^2 + k\big),$$

$$3 \cdot |Q| \geq 2n + 3 + \big(t - |Q|\big) - \big(k^2 + k\big).$$

This gives us the following:

Theorem 2.2 *Let G and R each be a finite set of green or red points, respectively, in $\mathbb{R}^2$ such that $|G| = n$, $|R| = n - k$, $G \cap R = \emptyset$, and the points of $G \cup R$ are not all collinear. Let t be the total number of lines determined by $G \cup R$. The number of equichromatic lines is at least $\frac{1}{4}(t + 2n + 3 - k(k + 1))$.*

Using the Erdös–de Bruijn Theorem, i.e., $t \geq 2n - k$, one can see the following corollary:

Corollary 2.3 *The number of equichromatic lines, $|Q|$, is at least $n + \frac{1}{4}(3 - k(k + 2))$. If $k \in \{0, 1\}$, then $|Q| \geq n + 1 - k$.*

2.2 Proof of the Kleitman–Pinchasi Conjecture

In [10], Kleitman and Pinchasi conjectured that when neither color class is collinear, there exist at least $2n - k - 1$ bichromatic lines, assuming that $|G| = n$, $|R| = n - k$, and $k \in \{0, 1\}$. Since equichromatic lines are a subset of bichromatic, our lower bound (i.e., Theorem 2.2) is better than the Kleitman–Pinchasi conjecture whenever $t \geq 6n - 7$ (or $t \geq 6n - 9$ if $k = 1$). We now prove that when n is sufficiently large, their conjecture is true.

When all but a few points lie on a line, one can verify the conjectured bound directly. We give the following lemmas:

Lemma 2.4 *If neither color class is collinear and $2n - k - 2$ points are incident to one line, then $|Q| \geq 2n - k - 1$.*

Lemma 2.5 *If neither color class is collinear and $2n - k - 3$ points are incident to one line, then $|Q| \geq 3n - k - 4$.*

(Note that by Lemma 2.4 the conjectured $2n - k - 1$ lower bound is sharp, i.e., it cannot be improved.)

Although one can count the number of equichromatic lines in the specific cases above, a tool is needed for more general point configurations. Just such a tool can be found in a well-known paper from 1958 by Kelly and Moser.

In [9], Kelly and Moser proved a lower bound on the number of lines, t, assuming that at most $2n - k - r$ points are collinear. (Note that Kelly and Moser originally assumed n points total, but the present article uses the context of $2n - k$ points.) Their lower bound is as follows:

Theorem 2.6 *If at most $2n-k-r$ points are incident to a single line and $2n-k \geq \frac{1}{2}(3(3r-2)^2+3r-1)$, then*

$$t \geq r(2n-k) - \frac{(3r+2)(r-1)}{2}.$$

So by letting $r=4$ in the inequality above, we get the needed lower bound for the total number of lines:

Lemma 2.7 *If $n \geq 78+k$ and no more than $2n-k-4$ points are incident to one line, then $t \geq 8n-4k-21$.*

By Lemmas 2.4, 2.5, and 2.7 we have the following theorem:

Theorem 2.8 *Let G and R each be a finite set of green or red points, respectively, in $\mathbb{R}^2$ such that $|G|=n$, $|R|=n-k$, $k \in \{0,1\}$, $G \cap R = \emptyset$, and neither color class is collinear. If $n \geq 78+k$, then the number of equichromatic lines is at least $2n-k-1$.*

Thus, the Kleitman–Pinchasi conjecture is true for all $n \geq 79$.

2.3 Equichromatic Lines with Few Points

2.3.1 Equichromatic Lines Through at Most Four Points

Although Pach and Pinchasi did not define "equichromatic," Theorem 2(i) in their paper [12] proves a lower bound on the number of equichromatic lines passing through at most four points. For the convenience of the reader, we will show Pach and Pinchasi's derivation, except we will assume that $|G|=n$ and $|R|=n-k$ (i.e., we do not assume $|G|=|R|$ as was originally).

By adding twice Melchior's inequality (4) to the identity (3), one can see that

$$\sum_{\substack{i,j\geq 0\\ i+j\geq 2}} \big(6-(i+j)-(i-j)^2\big)t_{i,j} \geq 2n+6-k(k+1).$$

Let $\gamma_{i,j}$ be the coefficient corresponding to $t_{i,j}$ in the summation above. The only positive coefficients are $\gamma_{1,1}=4$ and $\gamma_{1,2}=\gamma_{2,1}=\gamma_{2,2}=2$. So, it must be the case that

$$2t_{1,1}+t_{1,2}+t_{2,1}+t_{2,2} \geq n+3-\frac{k(k+1)}{2}.$$

From this we get the following theorem:

Theorem 2.9 *Let G and R each be a finite set of green or red points, respectively, in $\mathbb{R}^2$ such that $|G|=n$, $|R|=n-k$, $G \cap R = \emptyset$, and the points of $G \cup R$ are not all collinear. The number of equichromatic lines determined by at most four points is at least $\frac{1}{4}(2n+6-k(k+1))$.*

2.3.2 Equichromatic Lines Through at Most Five Points

To extend these results to equichromatic lines in the complex plane, Hirzebruch's "Improved" Inequality will be used. The Hirzebruch Inequalities are derived from the Miyaoka–Yau Inequality (known in the field of algebraic geometry), and they are true in the complex plane (i.e., $\mathbb{C}^2$). Unlike Melchior's Inequality, both of Hirzebruch's Inequalities show that among a set of points in $\mathbb{C}^2$ there must exist a line determined by at most *three* points (see also [3] and [8]). One must note that these inequalities were originally proven for the dual of the present context, i.e., an arrangement of lines in the complex plane (such that t_k was the number of intersection points at which k lines cross). The present article will instead remain consistent with the context used by Kleitman, Pach, and Pinchasi. Of course, the lower bound established below for points in $\mathbb{C}^2$ also applies to points in $\mathbb{R}^2$.

The first of Hirzebruch's two inequalities, as used in the present article, was published in [6]. It states that, among a set of n points in $\mathbb{C}^2$ and assuming $t_n = t_{n-1} = 0$,

$$t_2 + t_3 \geq n + \sum_{k\geq 5}(k-4)t_k. \tag{5}$$

In the slightly more restrictive case of $t_n = t_{n-1} = t_{n-2} = 0$, published in [7], there exists the improved inequality

$$t_2 + \frac{3}{4}t_3 \geq n + \sum_{k\geq 5}(2k-9)t_k. \tag{6}$$

In the present context (i.e., $|G| = n$, $|B| = n-k$), the improved inequality can be rewritten:

$$\begin{aligned}&-(t_{0,2}+t_{2,0}) - t_{1,1} - \frac{3}{4}(t_{0,3}+t_{3,0}) - \frac{3}{4}(t_{1,2}+t_{2,1})\\&\quad+\sum_{\substack{i,j\geq 0\\ i+j\geq 5}}\big(2(i+j)-9\big)t_{i,j} \leq -(2n-k).\end{aligned} \tag{7}$$

Similarly, subtracting (1) from (2) and unwinding the first few terms of the summation produces

$$\begin{aligned}&(t_{0,2}+t_{2,0}) - t_{1,1} + 3(t_{0,3}+t_{3,0}) - (t_{1,2}+t_{2,1}) + 6(t_{0,4}+t_{4,0}) - 2t_{2,2}\\&\quad+\sum_{\substack{i,j\geq 0\\ i+j\geq 5}}\left(\binom{i}{2}+\binom{j}{2}-ij\right)t_{i,j} = -n + \frac{k^2+k}{2}.\end{aligned} \tag{8}$$

By adding two times (8) and $(1+\epsilon)$ times (7), it becomes

$$(1-\epsilon)(t_{2,0}+t_{0,2}) - (3+\epsilon)t_{1,1} + \left(\frac{21}{4}-\frac{3}{4}\epsilon\right)(t_{0,3}+t_{3,0})$$

$$
\begin{aligned}
&- \left(\frac{11}{4} + \frac{3}{4}\epsilon\right)(t_{1,2} + t_{2,1}) + 12(t_{0,4} + t_{4,0}) - 4t_{2,2} \\
&+ \sum_{\substack{i,j\geq 0 \\ i+j\geq 5}} \big((i-j)^2 + \epsilon\big(2(i+j) - 9\big) + i + j - 9\big)t_{i,j} \\
\leq{}& -2n(2+\epsilon) + k(k+2+\epsilon). \qquad (9)
\end{aligned}
$$

Let $\epsilon = 1$, and let $\gamma_{i,j}$ be the coefficient for $t_{i,j}$ produced by the left side of the inequality above. One can see that the only negative coefficients are $\gamma_{1,1} = \gamma_{2,3} = \gamma_{3,2} = -2$, $\gamma_{1,2} = \gamma_{2,1} = -\frac{7}{2}$, and $\gamma_{2,2} = -4$, and so

$$
-4 \cdot (t_{1,1} + t_{1,2} + t_{2,1} + t_{2,2} + t_{2,3} + t_{3,2}) \leq -6n + k(k+3).
$$

This gives us the following:

Theorem 2.10 *Let G and R each be a finite set of green or red points, respectively, in $\mathbb{C}^2$ such that $|G| = n$, $|R| = n - k$, $G \cap R = \emptyset$, and no $2n - k - 2$ points of $G \cup R$ are collinear. The number of equichromatic lines determined by at most five points is at least $\frac{1}{4}(6n - k(k+3))$.*

For the cases where at most $2n - k - 2$ points are collinear, we provide the following two lemmas:

Lemma 2.11 *If exactly $2n-k-2$ points of $G \cup R$ are collinear in $\mathbb{C}^2$, the number of equichromatic lines, $|Q|$, determined by at most three points is at least $2(n-k-1)$. Since the total number of lines, t, is at least $2(2n-k-2)$ and at most $2(2n-k-1)$, $|Q|$ is at least one-half of t, i.e., $\frac{t}{2}$.*

Lemma 2.12 *If exactly $2n - k - 1$ points of $G \cup R$ are collinear in $\mathbb{C}^2$, the number of equichromatic lines, $|Q|$, determined by at most two points is at least $n - k$. Since the total number of lines, t, is $2n - k$, $|Q|$ is at least one-half of t, i.e., $\frac{t}{2}$.*

Note that the bound given in Lemma 2.11 is better than that of Theorem 2.10 for all $n \geq 7$. Thus, if at most $2n-k-2$ points are collinear and $n \geq 7$, then the number of equichromatic lines is at least $\frac{1}{4}(6n - k(k+3))$. (These lemmas will be referred to again to augment results in the complex plane.)

2.3.3 Equichromatic Lines Through at Most Six Points

In [12], Pach and Pinchasi claim "the number of bichromatic lines passing through at most six points is at least one tenth of the total number of connecting lines." Although this statement is true (the present article contains an even better result of $\frac{t}{7} + n$), their derivation contains a small mistake (i.e., Theorem 2(ii) of their paper should state "eight" points instead of "six," since the coefficient $\gamma_{4,4} = 0$, not $\frac{2}{5}$ as would have been needed). Below, we show a fix for their derivation which maintains the "at most six points," applies it to equichromatic lines, but weakens the result to a twelfth of the total number of lines. Also, see Sect. 3.2 for a stronger result for bichromatic lines through at most six points in $\mathbb{C}^2$ (which of course applies to $\mathbb{R}^2$).

Again following the method of Pach and Pinchasi, add the identity (3) to $(1+\epsilon)$ times (4). From this one obtains (the negated form of what was used by Pach and Pinchasi)

$$\sum_{\substack{i,j\geq 0\\ i+j\leq 2}} \bigl(3-\epsilon(i+j-3)-(i-j)^2\bigr)t_{i,j} \geq 2n+3(1+\epsilon)-k(k+1).$$

Let $\epsilon = \frac{2}{3}$ and $\gamma_{i,j}$ be the coefficient for $t_{i,j}$ in the summation above. With careful inspection, one can verify that the only positive coefficients are $\gamma_{2,3} = \gamma_{3,2} = \frac{2}{3}$, $\gamma_{3,3} = 1$, $\gamma_{1,2} = \gamma_{2,1} = 2$, $\gamma_{2,2} = \frac{7}{3}$, and the largest $\gamma_{1,1} = \frac{11}{3}$. Of the nonpositive coefficients, $\gamma_{0,2} = -\frac{1}{3}$ is the largest. So by subtracting $4 \cdot (t_{2,3} + t_{3,2} + t_{3,3} + t_{1,2} + t_{2,1} + t_{2,2})$ from both sides of the inequality one can see that

$$-\frac{1}{3} \cdot \sum_{\substack{i,j\geq 0\\ i+j\geq 2}} t_{i,j} \geq 2n+5-\bigl(k^2+k\bigr)-4\cdot \sum_{\substack{i,j\geq 1\\ |i-j|\leq 1\\ 2\leq i+j\leq 6}} t_{i,j},$$

$$\sum_{\substack{i,j\geq 1\\ |i-j|\leq 1\\ 2\leq i+j\leq 6}} t_{i,j} \geq \frac{t+6n+15-3(k^2+k)}{12}.$$

So we have now proven the following:

Theorem 2.13 *Let G and R each be a finite set of green or red points, respectively, in $\mathbb{R}^2$ such that $|G| = n$, $|R| = n-k$, $G \cap R = \emptyset$, and the points of $G \cup R$ are not all collinear. Let t be the total number of lines determined by $G \cup R$. The number of equichromatic lines determined by at most six points is at least* $(t+6n+15-3k(k+1))/12$.

By letting $k = 0$, Theorem 2.13 becomes the corrected result for Pach and Pinchasi's. The fact that the lines counted are equichromatic, instead of the more general bichromatic, comes for free.

3 Lines in $\mathbb{C}^2$

3.1 A Lower Bound for Bichromatic Lines

In Sect. 2.1, a lower bound on the number of equichromatic lines in $\mathbb{R}^2$ was shown to be $\frac{1}{4}(t+2n+3-(k^2+k))$. Below, a nearly equivalent lower bound is demonstrated for bichromatic lines in $\mathbb{C}^2$, assuming that no $2n-k-2$ points are collinear (note that $|G \cup R| = 2n-k$).

Hirzebruch's Improved Inequality (6) can be rewritten as (assuming n points)

$$9\cdot\sum_{k\geq 2} t_k \geq n + 2\cdot\sum_{k\geq 2} k t_k + 4t_2 + \frac{9}{4}t_3 + t_4.$$

In our context, this would be

$$9\cdot\sum_{\substack{i,j\geq 0\\ i+j\geq 2}} t_{i,j} \geq 2n - k + 2\cdot\sum_{\substack{i,j\geq 0\\ i+j\geq 2}} (i+j)\cdot t_{i,j} + 4t_2 + \frac{9}{4}t_3 + t_4.$$

Using our identity (3), we have

$$9\cdot\sum_{\substack{i,j\geq 0\\ i+j\geq 2}} t_{i,j} - 6n + 2\bigl(k^2+k\bigr) \geq 2\cdot\sum_{\substack{i,j\geq 0\\ i+j\geq 2}} (i-j)^2\cdot t_{i,j} + 4t_2 + \frac{9}{4}t_3 + t_4.$$

On the right-hand side of this inequality one can see that every monochromatic line is counted at least twelve times. Therefore,

$$m \leq \frac{3}{4}t - \frac{n}{2} + \frac{k^2+k}{6}.$$

We get the following theorem:

Theorem 3.1 *Let G and R each be a finite set of green or red points, respectively, in $\mathbb{C}^2$ such that $|G| = n$, $|R| = n-k$, $G\cap R = \emptyset$, and no $2n-k-2$ points of $G\cup R$ are collinear. Let t be the total number of lines determined by $G\cup R$. The number of bichromatic lines is at least* $(3t + 6n - 2k(k+1))/12$.

Similar attempts to place a lower bound for equichromatic lines (Q) in $\mathbb{C}^2$ yield only $|Q| \geq n$. We refer the reader to Lemmas 2.11 and 2.12 for the cases where there are $2n-k-2$ or $2n-k-1$ collinear points.

3.2 Bichromatic Lines Through at Most Six Points

Returning to the inequality derived in Sect. 2.3.2, let $\epsilon = \frac{1}{3}$ in (9). The inequality becomes

$$\begin{aligned}
&\frac{2}{3}(t_{2,0}+t_{0,2}) - \frac{10}{3}t_{1,1} + 5(t_{0,3}+t_{3,0})\\
&\quad - 3(t_{1,2}+t_{2,1}) + 12(t_{0,4}+t_{4,0}) - 4t_{2,2}\\
&\quad + \sum_{\substack{i,j\geq 0\\ i+j\geq 5}} \left((i-j)^2 + \frac{1}{3}\bigl(2(i+j)-9\bigr) + i + j - 9\right)t_{i,j}\\
&\leq -\frac{14n}{3} + k\left(k+\frac{7}{3}\right).
\end{aligned}$$

Let $\gamma_{i,j}$ be the coefficient for $t_{i,j}$ given in the inequality above. Of all coefficients for the bichromatic lines containing fewer than six points, the smallest is $\gamma_{2,2} = -4$. Of all other coefficients (i.e., for lines either not bichromatic or containing more than six points), the smallest are $\gamma_{0,2} = \gamma_{3,4} = \gamma_{4,3} = \frac{2}{3}$.

By adding $\frac{14}{3} \cdot \sum_{\substack{i,j\geq 1\\ 2\leq i+j\leq 6}} t_{i,j}$ to both sides and rearranging, one can see that

$$\frac{2t}{3} + \frac{14n}{3} - k\left(k + \frac{7}{3}\right) \leq \frac{14}{3} \cdot \sum_{\substack{i,j\geq 1\\ 2\leq i+j\leq 6}} t_{i,j},$$

$$\sum_{\substack{i,j\geq 1\\ 2\leq i+j\leq 6}} t_{i,j} \geq \frac{t}{7} + n - \frac{k(3k+7)}{14}.$$

We get the following theorem:

Theorem 3.2 *Let G and R each be a finite set of green or red points, respectively, in $\mathbb{C}^2$ such that $|G| = n$, $|R| = n - k$, $G \cap R = \emptyset$, and no $2n - k - 2$ points of $G \cup R$ are collinear. Let t be the total number of lines determined by $G \cup R$. The number of bichromatic lines determined by at most six points is at least* $(2t + 14n - k(3k + 7))/14$.

We remark that this result is of course also true in $\mathbb{R}^2$ and is better than the lower bound found by Pach and Pinchasi. Again, we refer the reader to Lemmas 2.11 and 2.12 for the cases where there are $2n - k - 2$ or $2n - k - 1$ collinear points.

3.3 Lower Bound on Total Number of Lines

In Sect. 2.2, we utilized a result from a Kelly–Moser paper [9] to prove a conjecture of Kleitman and Pinchasi. This result of Kelly and Moser (Theorem 4.1 in their paper) showed a lower bound, assuming that certain conditions were met on the total number of lines determined by an arrangement of points. Since their result was derived from Euler's Polyhedral Formula, it does not apply to the complex plane. Below, an analogous result is derived and valid for $\mathbb{C}^2$.

Lemma 3.3 *Given r points on a line, L, and s points not on L, in $\mathbb{C}^2$ at least* $1 + rs - s(s-1)/2$ *lines, including L, are formed.*

Proof Using induction on s. If $s = 1$, then there are $r + 1$ lines, and the lemma is true.

Suppose that the lemma is true for s. A new point off of L will determine r lines, of which at most s already exist. Hence, we have at least

$$\big(1 + rs - s(s-1)/2\big) + (r - s) = 1 + r(s+1) - (s+1)s/2$$

lines. The lemma follows by induction. □

The lemma above is quite strong but requires exactly "r points on a line, L, and s points not on L." To get the desired result, we must extend this to point sets where *at most* $n-k$ are collinear.

We now alter our terms. Instead of r and s, we will refer to a set of n points with at most $n-k$ collinear. Furthermore, we define the expression from the lemma above to be a function of k (with n being considered constant), i.e., $f(k) \stackrel{\text{def}}{=} k(n-k) - \frac{k(k-1)}{2} + 1$.

Theorem 3.4 *Given a finite set of points, G, in $\mathbb{C}^2$ such that $|G| = n$, $k \geq 2$, and at most $n-k$ of the points are collinear. Let t be the total number of lines determined by G. If*

$$n \geq \big(3\big(16k^2 - 23k + 9\big) - 1\big)/2, \tag{10}$$

then

$$t \geq f(k) = kn - \frac{(3k+2)(k-1)}{2}.$$

Proof Let r_i denote the number of points that are incident to precisely i lines. (Note that we only consider lines determined by the set of points.) We assume that the points are not all collinear, so $r_1 = 0$. Fix $a > k$ to be a number determined later.

Case 1: Suppose $r_2 + r_3 + \cdots + r_{a+1} \geq 2$. There are two points, P and Q, each incident with at most $a+1$ lines. Let L be the line PQ. There are at most a^2 points not on L, since neither P nor Q can be incident to a line, other than L, with more than $a+1$ points. Let $n-x$ be the exact number of points on L. Now consider the function $f(x)$ defined above. The first and second derivative of this function are $f'(x) = -3x + n + \frac{1}{2}$ and $f''(x) = -3$ (i.e., $f(x)$ is concave down). Since $k \leq x \leq a^2$, we know that $t \geq f(x) \geq \min\{f(k), f(a^2)\}$.

We are going to choose n so that $f(k) \leq f(a^2)$, so that $t \geq f(k)$, and the theorem holds. (Note the ordering of values: $a^2 > a > k \geq 2$.) Thus we want that

$$k(n-k) - \frac{k(k-1)}{2} + 1 \leq a^2\big(n - a^2\big) - \frac{a^2(a^2-1)}{2} + 1,$$

and hence we want that

$$n\big(a^2 - k\big) \geq \frac{3(a^4 - k^2)}{2} - \frac{a^2 - k}{2}.$$

Dividing both sides by $a^2 - k > 0$, we obtain

$$n \geq \frac{3(a^2+k) - 1}{2}. \tag{11}$$

Case 2: Suppose $r_2 + r_3 + \cdots + r_{a+1} \leq 1$. By Hirzebruch's first inequality (5), we know that

$$\begin{aligned} t_2 + t_3 + 4t &\geq 4t + n + t_5 + 2t_6 + 3t_7 + \cdots \\ &\geq n + 4t_2 + 4t_3 + 4t_4 + 5t_5 + 6t_6 + \cdots . \end{aligned}$$

Hence,

$$\begin{aligned}4t &\geq n + 3t_2 + 3t_3 + 4t_4 + 5t_5 + \cdots \\ &\geq n + \sum_{i\geq 2} i \cdot t_i = n + \sum_{i\geq 2} i \cdot r_i \\ &\geq n + 2 + (a+2)\bigl(-1 + (r_2 + r_3 + \cdots + r_{a+1}) + r_{a+2} + \cdots\bigr) \\ &= n + 2 + (a+2)(n-1).\end{aligned}$$

We need to choose a so that $t \geq f(k) = k(n-k) - \frac{k(k-1)}{2} + 1$. We choose $a = 4k-3$, so that $4t \geq 2 + (4k-1)(n-1) + n = 4kn - 4k + 3$, and so $t > kn - k > f(k)$. Combining this with (11) gives us (10). □

3.4 The Kleitman–Pinchasi Conjecture Revisited

In Sect. 3.2 we proved a lower bound on the number of bichromatic lines passing through at most six points in $\mathbb{C}^2$. This result will now be used in conjunction with Theorem 3.4 to extend the proof of the Kleitman–Pinchasi conjecture to the complex plane.

The following lemmas, the same as used in Sect. 2.2, are also true in the complex plane ($\mathbb{C}^2$). (Note that we are now using the following context: Let G and R each be a finite set of green or red points, respectively, in $\mathbb{C}^2$ such that $|G| = n$, $|R| = n-k$, $k \in \{0, 1\}$, and $G \cap R = \emptyset$.)

Lemma 3.5 *If neither color class is collinear and if* $2n-k-2$ *points are incident to one line, then the number of equichromatic lines determined by at most three points is at least* $2n-k-1$.

Lemma 3.6 *If neither color class is collinear and if* $2n-k-3$ *points are incident to one line, then the number of equichromatic lines determined by at most four points is at least* $3n-k-4$.

With little effort, one can see from Theorem 3.2 that whenever $t \geq 7n-7$, the number of bichromatic lines through no more than six points in $\mathbb{C}^2$ is greater than $2n-k-1$. Using Theorem 3.4, we have the following lemma:

Lemma 3.7 *If* $n \geq 130$ *and no more than* $2n-k-4$ *points are incident to one line, then* $t \geq 8n-4k-21$.

By combining Lemmas 3.5, 3.6, and 3.7 with Theorem 3.2 we get the following theorem:

Theorem 3.8 *Let G and R each be a finite set of green or red points, respectively, in* $\mathbb{C}^2$ *such that* $|G| = n$, $|R| = n-k$, $k \in \{0, 1\}$, $G \cap R = \emptyset$, *and neither color class is collinear. If* $n \geq 130$, *then the number of bichromatic lines determined by at most six points is at least* $2n-k-1 = |G| + |R| - 1$.

Table 1 Best general lower bounds

	In $\mathbb{R}^2$	In $\mathbb{C}^2$[a]
Equichromatic	$(t+2n+3-k(k+1))/4$	$(6n-k(k+3))/4$
Bichromatic	$(t+2n+3-k(k+1))/4$[b]	$(3t+6n-2k(k+1))/12$

[a]Results in this column assume that no $2n-k-2$ points are collinear

[b]For $k \geq 3$, $(3t+6n-2k(k+1))/12$ is a better result

Table 2 Best equichromatic lower bounds

# of Points (at most)	In $\mathbb{R}^2$	In $\mathbb{C}^2$[a]
4	$(2n+6-k(k+1))/4$	N/A
5	$(6n-k(k+3))/4$[b]	$(6n-k(k+3))/4$
6	$(t+6n+15-3k(k+1))/12$	$(6n-k(k+3))/4$

[a]Results in this column assume that no $2n-k-2$ points are collinear

[b]Assumes that no $2n-k-2$ points are collinear

Table 3 Best bichromatic lower bounds

# of Points (at most)	In $\mathbb{R}^2$	In $\mathbb{C}^2$[a]
4	$(2n+6-k(k+1))/4$	N/A
5	$(6n-k(k+3))/4$[b]	$(6n-k(k+3))/4$
6	$(2t+14n-k(3k+7))/14$[c]	$(2t+14n-k(3k+7))/14$

[a]Results in this column assume that no $2n-k-2$ points are collinear

[b]Assumes that no $2n-k-2$ points are collinear

[c]Assumes that no $2n-k-2$ points are collinear

4 Conclusion

For the convenience of the reader, Tables 1, 2, and 3 are a collection of the results from this paper.

We ask whether one can prove a tight lower bound on the number of equichromatic or bichromatic lines determined by at most four points in $\mathbb{C}^2$.

Let t be the total number of lines determined by a point set. We conjecture that there exists an $\Omega(t)$ lower bound on the number of equichromatic lines in $\mathbb{C}^2$. We also conjecture that there exists a lower bound on the number of bichromatic lines determined by points in $\mathbb{C}^2$ (or $\mathbb{R}^2$) that is asymptotic to $t/2$. (The example of $|G| = |R| = n$ with points in general position gives $\binom{n}{2} + \binom{n}{2} = n^2 - n$ monochromatic lines and n^2 bichromatic lines, so $\sim t/2$ cannot be made stronger.)

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