Discrete Comput Geom (2010) 43: 951–954
DOI 10.1007/s00454-009-9153-7

An Elementary Deduction of the Topological Radon Theorem from Borsuk–Ulam

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Received: 31 October 2008 / Revised: 20 February 2009 / Accepted: 3 March 2009 /
Published online: 17 March 2009

Abstract The Topological Radon Theorem states that, for every continuous function from the boundary of a $(d+1)$-dimensional simplex into $\mathbb{R}^n$, there exists a pair of disjoint faces in the domain whose images intersect in $\mathbb{R}^n$. The similarity between that result and the classical Borsuk–Ulam Theorem is unmistakable, but a proof that the Topological Radon Theorem follows from Borsuk–Ulam is not immediate. In this note we provide an elementary argument verifying that implication.

Keywords Borsuk–Ulam theorem · Radon's theorem · Topological Radon theorem

1 Introduction

The classical Radon Theorem states that any collection $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{d+2}\}$ of $d+2$ points in $\mathbb{R}^d$ can be divided into two disjoint sets whose convex hulls intersect. The proof is a straightforward application of elementary linear algebra. See, for example, [2, p. 90]. An equivalent formulation of this theorem, with Δ^{d+1} denoting the $(d+1)$-dimensional simplex, is the following.

Theorem 1.1 (Radon's Theorem) *For every affine map $f : \Delta^{d+1} \to \mathbb{R}^d$, there exists a pair of disjoint faces F_A and F_B of Δ^{d+1} such that $f(F_A) \cap f(F_B) \neq \varnothing$.*

The equivalence of these two statements is easily deduced from the fact that every set $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{d+2}\} \subseteq \mathbb{R}^d$ determines an affine map $f : \Delta^{d+1} \to \mathbb{R}^d$ taking the vertices of Δ^{d+1} to the elements of X. Under this map, the image of each face is the convex hull of the images of its vertices.

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The "topological version" of the above theorem relaxes the requirements on the function f.

Theorem 1.2 (The Topological Radon Theorem) *For every continuous function $f : \Delta^{d+1} \to \mathbb{R}^d$, there exists a pair of disjoint faces F_A and F_B of Δ^{d+1} such that $f(F_A) \cap f(F_B) \neq \varnothing$.*

Several proofs of this theorem may be found in the literature—each depending on an application of the Borsuk–Ulam Theorem. See, for example, [1, 3 and 2, Chap. 5]. The goal of this paper is to present a new and particularly elementary method for deducing the Topological Radon Theorem from Borsuk–Ulam.

2 Background and Notation

Recall that the Borsuk–Ulam Theorem guarantees that, for any continuous $g : S^d \to \mathbb{R}^d$, there exists $\mathbf{x} \in S^d$ such that $g(\mathbf{x}) = g(-\mathbf{x})$. Here S^d denotes the standard d-sphere $\{\mathbf{x} \in \mathbb{R}^{d+1} \mid \|\mathbf{x}\| = 1\}$. (Points $\mathbf{x}$ and $-\mathbf{x}$ from S^d are called *antipodal points*.)

Let $\mathbf{N} = (0, \dots, 0, 1)$ and $\mathbf{S} = (0, \dots, 0, -1)$ denote the *north* and *south poles* of S^d and view S^{d-1} as a subset of S^d—the intersection of S^d with the hyperplane $\mathbb{R}^d \times \mathbf{0}$. We may then view S^d as the union $S^d = \bigcup_{\mathbf{y} \in S^{d-1}} G_{\mathbf{y}}$ where $G_{\mathbf{y}}$ is the great semicircle with endpoints $\mathbf{S}$ and $\mathbf{N}$ intersecting S^{d-1} at the point $\mathbf{y}$. In other words, $G_{\mathbf{y}} = \{(\cos\theta \cdot \mathbf{y}, \sin\theta) \mid \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]\}$. Notice that for distinct $\mathbf{y}_1, \mathbf{y}_2 \in S^{d-1}$, $G_{\mathbf{y}_1}$ intersects $G_{\mathbf{y}_2}$ only in the poles $\{\mathbf{N}, \mathbf{S}\}$.

For convenience, we represent a point $(\cos\theta \cdot \mathbf{y}, \sin\theta)$ in *generalized polar form* by the expression $\langle \mathbf{y}, \theta \rangle$. This representation is unique, provided that $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. In this form antipodal points are easy to recognize—the antipode of $\langle \mathbf{y}, \theta \rangle$ is $\langle -\mathbf{y}, -\theta \rangle$.

Next we discuss simplexes, their faces, and their boundaries. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{d+1}$ be the points $(1, 0, 0, \dots, 0)$, $(0, 1, 0, \dots, 0), \dots, (0, 0, 0, \dots, 1)$ in $\mathbb{R}^{d+1}$. The d-dimensional simplex Δ^d is the convex hull of $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{d+1}\}$. Thus,

$$\Delta^d = \left\{ \sum_{i=1}^{d+1} a_i \mathbf{v}_i \mid a_i \geq 0 \text{ and } \sum_{i=1}^{d+1} a_i = 1 \right\}.$$

We call $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{d+1}$ the *vertices* of Δ^d. The coefficient a_i of a given point is called its ith *barycentric coordinate*. The point in Δ^d with barycentric coordinates uniformly equal to $\frac{1}{d+1}$ is called the *barycenter* of Δ^d; it will be denoted $\mathbf{b}_d$.

Notice that, for any $k \leq d$, the simplex Δ^k may be viewed as a subset of Δ^d. More generally, if $A \subseteq \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{d+1}\}$, we call the convex hull of A, denoted F_A, a *face* of Δ^d. When A contains exactly $k + 1$ elements, then F_A is an isometric copy of Δ^k. Faces F_A and F_B are disjoint if and only if $A \cap B = \varnothing$. The *boundary* of a simplex Δ^d, denoted $\partial\Delta^d$, is the union of all proper faces of Δ^d.

In preparation for our theorem, we express $\partial\Delta^d$ as a union of subsets, each made up of a pair of line segments. Let $\partial^+\Delta^d$ denote the union of all proper faces of Δ^d except for Δ^{d-1}. Then each $\mathbf{p} \in \partial^+\Delta^d$ lies on a line segment $K_{\mathbf{q}}$ connecting a point

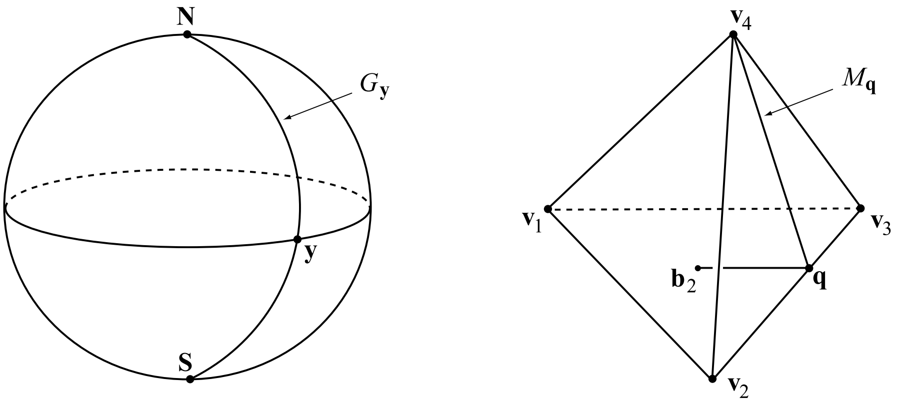

Fig. 1 A great semicircle $G_\mathbf{y}$ in S^2 and a "bent segment" $M_\mathbf{q}$ in $\partial\Delta^3$

$\mathbf{q} \in \partial\Delta^{d-1}$ to the vertex $\mathbf{v}_{d+1}$. That segment is unique, unless $\mathbf{p} = \mathbf{v}_{d+1}$. Similarly, each $\mathbf{p} \in \Delta^{d-1}$ lies on a segment $L_\mathbf{q}$ connecting a point $\mathbf{q} \in \partial\Delta^{d-1}$ to the barycenter $\mathbf{b}_{d-1}$ of Δ^{d-1}. Let $M_\mathbf{q} = K_\mathbf{q} \cup L_\mathbf{q}$, a "bent segment" connecting $\mathbf{b}_{d-1}$ to $\mathbf{v}_{d+1}$.

The $M_\mathbf{q}$'s in $\partial\Delta^d$ are analogous to the great semi-circles $G_\mathbf{y}$ in S^{d-1} with $\{\mathbf{v}_{d+1}, \mathbf{b}_{d-1}\}$ analogous to $\{\mathbf{N}, \mathbf{S}\}$. In particular, $\partial\Delta^d = \bigcup_{\mathbf{q}\in\partial\Delta^{d-1}} M_\mathbf{q}$, with $M_\mathbf{q}$ intersecting $M_{\mathbf{q}'}$ precisely in $\{\mathbf{v}_{d+1}, \mathbf{b}_{d-1}\}$ whenever $\mathbf{q} \neq \mathbf{q}'$. See Fig. 1.

3 Proofs

The Topological Radon Theorem is an easy consequence of the following:

Proposition 3.1 *For every $d \geq 0$, there exists a continuous function $\lambda_d : S^d \to \partial\Delta^{d+1}$ such that, for any $\mathbf{x} \in S^d$, $\lambda_d(\mathbf{x})$ and $\lambda_d(-\mathbf{x})$ lie in disjoint faces of $\partial\Delta^{d+1}$.*

Proof of Theorem 1.2 from Proposition 3.1 Given a continuous function $f : \Delta^{d+1} \to \mathbb{R}^d$, consider $f \circ \lambda_d : S^d \to \mathbb{R}^d$. By the Borsuk–Ulam Theorem, these exists $\mathbf{x} \in S^d$ such that $f \circ \lambda_d(\mathbf{x}) = f \circ \lambda_d(-\mathbf{x})$. By Proposition 3.1, there exist disjoint faces F_A and F_B of $\partial\Delta^{d+1}$ containing $\lambda_d(\mathbf{x})$ and $\lambda_d(-\mathbf{x})$, respectively. Then $f(F_A) \cap f(F_B) \neq \varnothing$. □

Proof of Proposition 3.1 For $d = 0$, $S^0 = \{-1, 1\}$ and $\partial\Delta^1 = \{\mathbf{v}_1, \mathbf{v}_2\}$. Simply define $\lambda_0(-1) = \mathbf{v}_1$ and $\lambda_0(1) = \mathbf{v}_2$.

Proceeding inductively, assume that an acceptable $\lambda_k : S^k \to \partial\Delta^{k+1}$ exists for some k. We show how to obtain $\lambda_{k+1} : S^{k+1} \to \partial\Delta^{k+2}$.

For each $\mathbf{y} \in S^k$, define λ_{k+1} to take $G_\mathbf{y} \subseteq S^{k+1}$ onto $M_{\lambda_k(\mathbf{y})} \subseteq \partial\Delta^{k+2}$ as follows:

$$\lambda_{k+1}(\langle \mathbf{y}, t\rangle) = \begin{cases} \mathbf{v}_{k+3}, & \text{for } \frac{\pi}{4} \leq t \leq \frac{\pi}{2}, \\ (1 - \frac{4t}{\pi}) \cdot \lambda_k(\mathbf{y}) + (\frac{4t}{\pi}) \cdot \mathbf{v}_{k+3}, & \text{for } 0 \leq t \leq \frac{\pi}{4}, \\ \lambda_k(\mathbf{y}) & \text{for } -\frac{\pi}{4} \leq t \leq 0, \\ -(1 + \frac{4t}{\pi}) \cdot \mathbf{b}_{k+1} + (2 + \frac{4t}{\pi}) \cdot \lambda_k(\mathbf{y}), & \text{for } -\frac{\pi}{2} \leq t \leq -\frac{\pi}{4}. \end{cases}$$

In words, λ_{k+1} maps the upper half of a great semicircle $G_{\mathbf{y}}$ onto the segment $K_{\lambda_k(\mathbf{y})}$ by squeezing the $[\frac{\pi}{4}, \frac{\pi}{2}]$-portion to the vertex $\mathbf{v}_{k+3}$ and stretching the $[0, \frac{\pi}{4}]$-portion over the entire segment. On the lower half of $G_{\mathbf{y}}$, λ_{k+1} maps the entire $[-\frac{\pi}{4}, 0]$-portion to the point $\lambda_k(\mathbf{y})$ and stretches the $[-\frac{\pi}{2}, -\frac{\pi}{4}]$-portion over the segment $L_{\lambda_k(\mathbf{y})}$. The continuity of λ_{k+1} follows easily from the continuity of λ_k combined with the obvious continuity of λ_{k+1} on each of the great semicircles $G_{\mathbf{y}}$.

Claim *For any* $\langle \mathbf{y}, t\rangle \in S^{k+1}$, $\lambda_{k+1}(\langle \mathbf{y}, t\rangle)$ *and* $\lambda_{k+1}(\langle -\mathbf{y}, -t\rangle)$ *lie in disjoint faces of* $\partial\Delta^{k+2}$.

Without loss of generality, we may assume that $t \in [0, \frac{\pi}{2}]$.

Case 1: $t \in [\frac{\pi}{4}, \frac{\pi}{2}]$. Then $\lambda_{k+1}(\langle \mathbf{y}, t\rangle) = \mathbf{v}_{k+3}$ and $\lambda_{k+1}(\langle -\mathbf{y}, -t\rangle) = -(1 + \frac{4t}{\pi}) \cdot \mathbf{b}_{k+1} + (2 + \frac{4t}{\pi}) \cdot \lambda_k(\mathbf{y}) \in \Delta^{k+1}$. Since $\{\mathbf{v}_{k+3}\}$ and Δ^{k+1} are disjoint, the claim holds.

Case 2: $t \in [0, \frac{\pi}{4}]$. By the inductive hypothesis, there exist disjoint faces F_A and F_B of $\partial\Delta^{k+1}$ containing $\lambda_k(\mathbf{y})$ and $\lambda_k(-\mathbf{y})$, respectively. Applying the definition of λ_{k+1}, we see that $\lambda_{k+1}(\langle \mathbf{y}, t\rangle) \in F_{A\cup\{\mathbf{v}_{k+3}\}}$ and $\lambda_{k+1}(\langle -\mathbf{y}, -t\rangle) \in F_B \subseteq \partial\Delta^{k+1}$. Since $A \cup \{\mathbf{v}_{k+3}\}$ and B are disjoint, so are the corresponding faces. $\square$

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