

On the Maximum Number of Edges in Topological Graphs with no Four Pairwise Crossing Edges

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Abstract A topological graph is called *k-quasi-planar* if it does not contain *k* pairwise crossing edges. It is conjectured that for every fixed *k*, the maximum number of edges in a *k*-quasi-planar graph on *n* vertices is $O(n)$. We provide an affirmative answer to the case $k = 4$.

Keywords Geometric graphs · Topological graphs · Discharging method · Quasi-planar graphs

1 Introduction

A *topological graph* is a graph drawn in the plane with its vertices as points and its edges as Jordan arcs that connect corresponding points and do not contain any other vertex as an interior point. Throughout this paper we use the term *arc* when referring to an edge of a topological graph. We only consider graphs that do not contain loops or parallel arcs. We also assume that any two arcs of a topological graph have a finite number of intersection points that are either endpoints or crossing points. A *planar graph* is a graph that can be drawn in the plane without any pair of crossing arcs. One possible generalization of the notion of planar graphs are *k-quasi-planar graphs*. A *k-quasi-planar graph* is a topological graph with no *k* pairwise crossing arcs. We denote by $f_k(n)$ the maximum number of arcs in such a graph on *n* vertices.

Since 2-quasi-planar graphs are planar graphs, it follows from Euler's Polyhedral Formula that $f_2(n) \leq 3n - 6$. It is a well-known conjecture [3, Problem 3.3] that for any fixed *k*, there is a constant C_k such that $f_k(n) \leq C_k n$. Agarwal et al. [2] were the first to prove this conjecture for $k = 3$. Later, Pach et al. [4] simplified their proof and showed that $f_3(n) \leq 65n$. Recently, Ackerman and Tardos [1] proved that $7n -$

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$O(1) \leq f_3(n) \leq 8n - \Omega(1)$ and provided a tight bound of $6.5n - \Omega(1)$ for *simple* topological graphs (graphs in which every pair of arcs intersect at most once). For a fixed $k \geq 4$, the best upper bound for topological graphs is $O(n \log^{4k-12} n)$ [4], while for topological graphs with x -monotone arcs, Valtr [5] showed an upper bound of $O(n \log n)$. In this paper we extend the work of Ackerman and Tardos [1] to provide the first proof that $f_4(n) = O(n)$.

2 Proof of the Main Theorem

Since the (underlying abstract) graphs we consider contain no loops or parallel arcs, we have $f_4(1) = 0$ and $f_4(2) = 1$. For greater values of n , we prove the following theorem.

Theorem 1 *For any integer $n > 2$, every topological graph on n vertices with no four pairwise crossing arcs has at most $72(n - 2)$ arcs.*

Proof It is easy to see that $f_4(3) = 3 < 72(3 - 2)$. Let G be a 4-quasi-planar graph on $n > 3$ vertices. We denote by $V(G)$ the vertex set of G , and by $E(G)$ the arc set of G . Given an arc $e \in E(G)$ and two points p and q on e , we use the notation $e|_{p,q}$ to denote the segment of e between p and q . For a vertex v , we denote by $d(v)$ the degree of v . If there is a vertex $v \in V(G)$ such that $d(v) < 3$, then we can conclude the theorem by induction. Hence, we assume that the degree of every vertex in G is at least three. Assume, w.l.o.g., that G is drawn with the least possible number of crossings such that there are no four pairwise crossing arcs and that there are no three arcs crossing at the same point. Let a_1 and a_2 be two arcs of G that intersect at least twice. A region bounded by segments of a_1 and a_2 that connect two consecutive intersection points is called a *lens*. We observe, as in [4], that G has no *empty* lenses, that is, lenses that do not contain a vertex of G . If there were empty lenses, then the number of crossings in G could be reduced. For the same reason, G does not contain self-intersecting arcs.

Let G' be the (drawing of the) planar graph *induced* by G . That is, $V(G') = V(G) \cup X(G)$, where $X(G)$ is the set of crossing points in G ; and $e' \in E(G')$ if e' is a segment of an arc of G that connects two vertices in $V(G')$ and contains no other vertex from $V(G')$. Henceforth, the term *edge* refers to an edge of G' , while the term *arc* refers to an edge of G . Denote by $F(G')$ the set of faces of G' , and let $|f|$ be the number of edges along the boundary of a face $f \in F(G')$.¹ Given a face f , we denote by $v(f)$ the number of vertices from $V(G)$ along the boundary of f (we call these vertices *original* vertices). We will use the terms *triangles*, *quadrilaterals*, *pentagons*, and *hexagons* to refer to faces of size 3, 4, 5, and 6, respectively. An integer m before the name of a face, denotes the number of original vertices on its boundary. For example, a 2-pentagon is a face of size 5 that has 2 original vertices on its boundary.

Following [1], we use the *discharging method* to prove the theorem. We begin by assigning a certain amount of *charge* to every face of G' such that the overall

¹Note that an edge can sometimes appear twice along the boundary of a face.

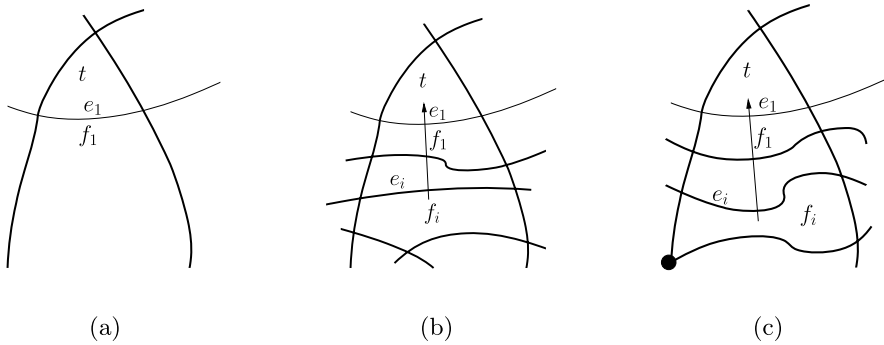


Fig. 1 Charging a 0-triangle

assigned charge is $O(n)$. Then, we redistribute the charges (*discharging phase*) such that every face has a nonnegative charge and the charge of every original vertex v is proportional to $d(v)$. This implies that the overall charge is $\Omega(|E(G)|)$, and therefore, $|E(G)| = O(n)$.

Let us begin then by assigning charges to the faces of G' such that each face f receives a charge of $|f| + v(f) - 4$. Summing the total charges over all the faces of G' , we have

$$\sum_{f \in F(G')} (|f| + v(f) - 4) = 2|E(G')| + \sum_{f \in F(G')} v(f) - 4|F(G')| = 4n - 8, \quad (1)$$

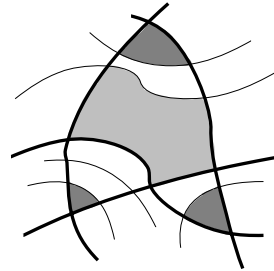
where the last equality follows from Euler’s formula and from the next equalities:

$$\begin{aligned} \sum_{f \in F(G')} v(f) &= \sum_{u \in V(G)} d(u) = \sum_{u \in V(G')} d(u) - \sum_{u \in X(G)} d(u) \\ &= 2|E(G')| - 4(|V(G')| - |V(G)|). \end{aligned} \quad (2)$$

Our plan is to redistribute the charges such that there will be no faces with a negative charge and every original vertex v will be charged with at least $\frac{d(v)}{36}$ units of charge. Then, the total charge over all the original vertices will be $\frac{2|E(G')|}{36} \leq 4n - 8$, and the theorem will follow. Since a face of size one yields a self-intersecting arc and a face of size two yields an empty lens or two parallel arcs, the only faces with a negative charge are 0-triangles. We proceed by describing a method to charge these faces. Then, we will show how to charge the original vertices.

Charging 0-triangles Let t be a 0-triangle, let e_1 be one of its edges, and let f_1 be the other face incident to e_1 (see Fig. 1(a)). It must be that $|f_1| > 3$, for otherwise there would be an empty lens. If $v(f_1) > 0$ or $|f_1| > 4$, we move $\frac{1}{3}$ units of charge from f_1 to t and say that f_1 contributed $\frac{1}{3}$ units of charge to t through e_1 . Otherwise, f_1 must be a 0-quadrilateral. Let e_2 be the opposite edge to e_1 in f_1 , and let f_2 be the other face incident to e_2 . Applying the same arguments as above, we conclude that either f_2 contributes $\frac{1}{3}$ units of charge to t through e_2 , or f is also a 0-quadrilateral. In

Fig. 2 A pentagon contributing to three 0-triangles through non-consecutive edges implies four pairwise crossing arcs



the second case we continue to the next face, that is, to the other face that is incident to the opposite edge to e_2 in f_2 . However, at some point we must encounter a face that is not a 0-quadrilateral. Denote by f_i this face, by f_{i-1} the face preceding f_i , and by e_i the edge incident to both of these faces. Then f_i will contribute $\frac{1}{3}$ units of charge to t through e_i (see Figs. 1(b,c)). Note that if f_i is a 1-triangle, then it implies that G has an empty lens. Furthermore, both endpoints of e_i are crossing points. Thus, $|f_i| > 3$.

In a similar way, t obtains $\frac{2}{3}$ units of charge from its other edges. Thus, after redistributing charges this way, the charge of every 0-triangle is 0. Note that a face can contribute charges through each of its edges at most once. Therefore, every face f such that $|f| + v(f) \geq 6$ still has a nonnegative charge. Since triangles do not contribute charge, it remains to verify that 1-quadrilaterals and 0-pentagons, which had only one unit of charge to contribute, also have a nonnegative charge. Indeed, a 1-quadrilateral contributes charge to at most two 0-triangles, since the endpoints of an edge through which it contributes charge must be vertices in $X(G)$. A 0-pentagon can contribute charge to at most three 0-triangles by the following easy observation.

Observation 2.1 *A 0-pentagon contributes charge to at most three 0-triangles. Moreover, if it contributes charge to three 0-triangles, then it must be done through consecutive edges.*

Proof One can easily check that a contribution to three 0-triangles through nonconsecutive edges implies four pairwise crossing arcs (see Fig. 2). In case a 0-pentagon contributes charge to more than three 0-triangles, then there must be three nonconsecutive edges through which it contributes charge. \square

We charge an original vertex v by charging each of the $d(v)$ wedges incident to v . Draw a small circle C around v such that v is the only vertex of G' in the region bounded by C . Then the crossing points of C with the arcs incident to v define a clockwise order of the arcs that are incident to v .

Definition 2.2 (Wedge) *A wedge is a triplet $w = (v, a_l, a_r)$ such that: $v \in V(G)$; a_l and a_r are arcs incident to v ; and a_l immediately follows a_r in a clockwise order of the arcs that are incident to v .*

We say that a face $f \in F(G')$ is the *first face* of a wedge $w = (v, a_l, a_r)$ if there are points $p_l \in a_l$ and $p_r \in a_r$ such that $a_l|_{v,p_l} \cup a_r|_{v,p_r}$ is on the boundary of f .

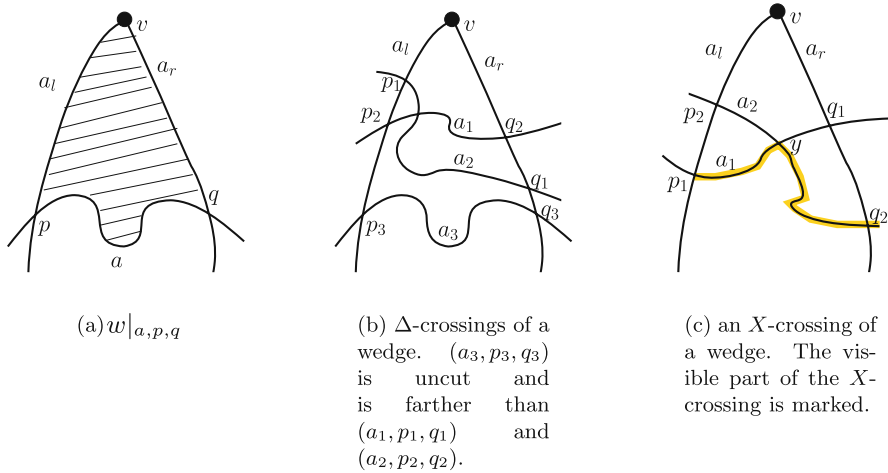


Fig. 3 Crossing patterns of a wedge

Charging Wedges After the previous step, the faces with a zero charge, apart from 0-triangles, 1-triangles, and 0-quadrilateral, are 0-pentagons that contributed charge to three 0-triangles, and 0-hexagons that contributed charge to six 0-triangles. We call such faces *bad* faces. Faces that have a positive charge are called *good* faces. It is easy to verify that a good face has at least $1/3$ units of charge. Our goal now is to find some extra charge for each wedge. This extra charge will be found next to a *farthest uncut Δ -crossing* or *X-crossing* of the wedge. At this point, few new definitions are needed.

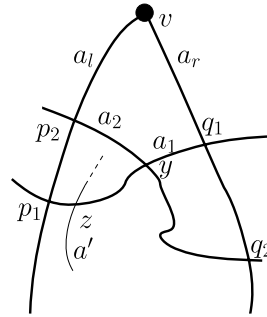
Let $w = (v, a_l, a_r)$ be a wedge, let a be an arc, and let p and q be crossing points of a with a_l and a_r , respectively, such that $a|_{p,q}$ does not cross a_l or a_r in any other point. We denote by $w|_{a,p,q}$ the bounded and closed region whose boundary is the closed curve composed of $a_l|_{v,p}$, $a|_{p,q}$, and $a_r|_{q,v}$ (see Fig. 3(a)).

Definition 2.3 Δ -crossing Let $w = (v, a_l, a_r)$ be a wedge. A Δ -crossing of w is a triplet $\delta = (a, p, q)$ such that a is an arc crossing a_l at p and a_r at q such that: (1) $a|_{p,q}$ does not intersect a_l or a_r ; and (2) there are no original vertices in the interior of $w|_{a,p,q}$. We say that δ is an *uncut Δ -crossing* of w if $a|_{p,q}$ is an edge of G' . For illustrations, refer to Fig. 3(b).

Let $\delta = (a, p, q)$ be a Δ -crossing of a wedge $w = (v, a_l, a_r)$. We use the notation $w|_\delta$ as an abbreviation for $w|_{a,p,q}$. Given another Δ -crossing of w , $\delta' = (a', p', q')$, we say that δ is *farther* than δ' if $p' \in a_l|_{v,p}$ and $q' \in a_r|_{v,q}$. Clearly, not every two Δ -crossings of a wedge are comparable, but uncut Δ -crossings are.

Definition 2.4 (*X-crossing*) Let w be a wedge, and let $\delta_1 = (a_1, p_1, q_1)$ and $\delta_2 = (a_2, p_2, q_2)$ be two Δ -crossings of w . Then $x = (\delta_1, \delta_2)$ is an *X-crossing* of w if $a_1|_{p_1,q_1}$ and $a_2|_{p_2,q_2}$ intersect exactly once. We use the notation $w|_x$ to denote the region $w|_{\delta_1} \cup w|_{\delta_2}$.

Fig. 4 An illustration for the proof of Observation 2.6



Let x and x' be two X -crossings of a wedge w . We say that x is *farther* than x' if one Δ -crossing of x is farther than one Δ -crossing of x' and the other Δ -crossing of x' is not farther than the other Δ -crossing of x (note that the last two Δ -crossings might be identical). In a similar way, we say that an uncut Δ -crossing is *farther* (resp., *closer*) than an X -crossing of the same wedge if it is farther (resp., closer) than both Δ -crossings of the X -crossing.

Definition 2.5 (Visible part of an X -crossing) Let $x = (\delta_1 = (a_1, p_1, q_1), \delta_2 = (a_2, p_2, q_2))$ be an X -crossing of a wedge $w = (v, a_l, a_r)$. Then the part of the boundary of $w|_x$ that is composed of segments of the arcs a_1 and a_2 is called the *visible* part of $w|_x$ and is denoted by $\text{Vis}(x)$ when it is clear to which wedge we refer. We denote by $\text{Vis}(x, a_l)$ (resp., $\text{Vis}(x, a_r)$) the segment of $\text{Vis}(x)$ whose one endpoint lies on a_l (resp., a_r) and whose other endpoint is the intersection point of $a_1|_{p_1, q_1}$ and $a_2|_{p_2, q_2}$. See Fig. 3(c) for an example.

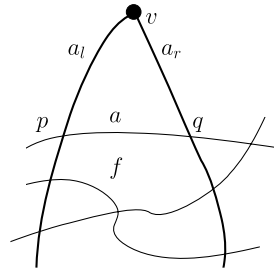
The next observation will be useful later.

Observation 2.6 Suppose that $x = (\delta_1 = (a_1, p_1, q_1), \delta_2 = (a_2, p_2, q_2))$ is an X -crossing of a wedge $w = (v, a_l, a_r)$ such that $\text{Vis}(x, a_l) \subset a_l$. Then an arc a' that crosses $\text{Vis}(x, a_l)$ (resp., $\text{Vis}(x, a_r)$) must cross a_l (resp., a_r) and must not cross a_2 (resp., a_1).

Proof Let y be the crossing point of $a_1|_{p_1, q_1}$ and $a_2|_{p_2, q_2}$, and let z be the crossing point of a' and $a_1|_{p_1, y}$ (see Fig. 4). Since the interior of $w|_x$ contains no original vertex, a' must cross the boundary of $w|_x$ at least one more time. If it crosses $a_1|_{p_1, q_1}$ at any point other than z , we have an empty lens. Therefore, a' cannot cross $\text{Vis}(x)$ at any point but z . Otherwise, if a' crosses $a_r|_{v, q_2}$, then it must also cross $a_2|_{p_2, y}$. This implies four pairwise crossing arcs: a' , a_1 , a_2 , and a_r . Thus a' must cross a_l . Moreover, it must not cross a_2 since this also yields four pairwise crossing edges. The proof for an arc crossing $\text{Vis}(x, a_r)$ is similar and is thus omitted. \square

Rules for Charging a Wedge We are now ready to define the charging rules for wedges. Given a wedge $w = (v, a_l, a_r)$, we look for an uncut Δ -crossing or X -crossing of w such that there is no uncut Δ -crossing or X -crossing farther than it and continue as follows:

Fig. 5 A bad pentagon next to an uncut Δ -crossing implies a farther X -crossing



1. Suppose that there is no such uncut Δ -crossing or X -crossing. Then the first face of w is not a 1-triangle. Thus, its charge is at least $\frac{1}{3}$ units, from which we use $\frac{1}{36}$ units to charge w .
2. Suppose that there is such an uncut Δ -crossing $\delta = (a, p, q)$. Let f be the face incident to $a|_{p,q}$ outside $w|_\delta$. f is not a triangle as this would yield an empty lens or parallel arcs. It also cannot be a 0-quadrilateral since this would imply an uncut Δ -crossing farther than δ . If f is a bad pentagon, then it follows from Observation 2.1 that there is an X -crossing farther than δ (see Fig. 5). f cannot be a bad hexagon, since this would imply that it has contributed charge to a 0-triangle through $a|_{p,q}$. Therefore, f must be a good face that will contribute $\frac{1}{36}$ units of charge to w through $a|_{p,q}$.
3. Suppose that there is such an X -crossing $x = (\delta_1 = (a_1, p_1, q_1), \delta_2 = (a_2, p_2, q_2))$. Denote by y the intersection point of $a_1|_{p_1,q_1}$ and $a_2|_{p_2,q_2}$, and assume w.l.o.g. that $\text{Vis}(x, a_l) = a_1|_{p_1,y}$. Let f_1 be the face that is incident to y and outside $w|_x$ (see Fig. 6). Let us analyze the possible subcases.
 - (a) f_1 is a good face. Let e_i be the edge of f_i that is contained in $\text{Vis}(x, a_l)$. Then f_i contributes $\frac{1}{36}$ units of charge to w through e_i .
 - (b) f_1 is a 0-triangle. Let a_3 be the third arc incident to f_1 (see Fig. 6(a)). It follows from Observation 2.6 that a_3 must cross a_l , thus a_l, a_1, a_2 , and a_3 are pairwise crossing. Hence, this subcase is impossible.
 - (c) If f_1 is a bad pentagon, then we consider the possible cases according to which none, one, or both of $a_1|_{p_1,y}$ and $a_2|_{y,q_2}$ are edges of G' . In case both of them are edges (see Fig. 6(b)), then there is an uncut Δ -crossing of w that is farther than x . In case one of them, say $a_2|_{y,q_2}$, is an edge (see Fig. 6(c)), then there is an X -crossing farther than x . If none of them is an edge (see Fig. 6(d)), then there must be four pairwise crossing arcs. Therefore f_1 cannot be a bad pentagon.
 - (d) By arguments similar to the previous case it follows that if f_1 is a bad hexagon, then there must be four pairwise crossing arcs, or an X -crossing of w farther than x .
 - (e) f_1 is a 0-quadrilateral. Let f_2 be the face outside of $w|_x$ that shares an edge with f_1 and is incident to $\text{Vis}(x, a_l)$ (see Fig. 6(e)). If there is no such face, or there is no face outside $w|_x$ that shares an edge with f_1 and is incident to $\text{Vis}(x, a_r)$, then there is an X -crossing farther than x . Examining f_2 , one can see by inspection that it cannot be a 0-triangle, as this implies four pairwise crossing arcs. If f_2 is a bad pentagon (see Fig. 6(f, g)) or a bad hexagon,

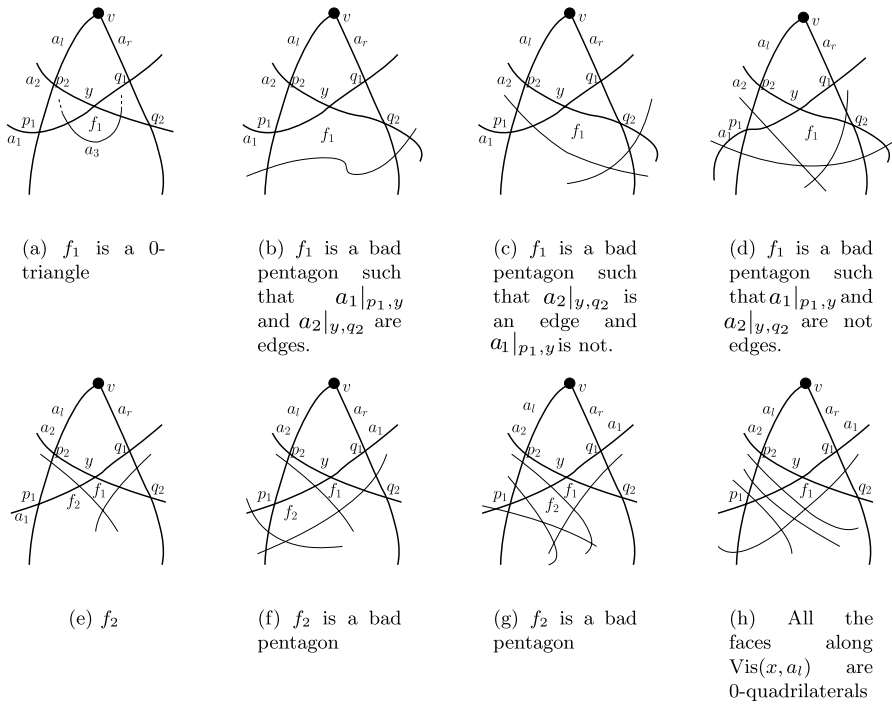


Fig. 6 Obtaining a charge near a farthest X -crossing

then again there must be four pairwise crossing arcs or an X -crossing farther than x . Thus, either f_2 is a good face or it is a 0-quadrilateral.

In the second case, we examine the “next” face, that is, the face $f_3 \neq f_1$ such that f_3 is outside $w|_x$, shares an edge with f_2 , and is incident to $\text{Vis}(x, a_l)$. If there is no such face, then we have an X -crossing farther than x . Otherwise, we can apply the same arguments we used for f_2 on f_3 , proceed to the next face if f_3 is not a good face, and so on. At some point we must encounter a good face, for otherwise we have an X -crossing farther than x (see Fig. 6(h)). Let f_i be the first good face we encounter along $\text{Vis}(x, a_l)$, and let e_i be the edge of f_i that is contained in $\text{Vis}(x, a_l)$. Then f_i contributes $\frac{1}{36}$ units of charge to w through e_i .

Analyzing the Final Charges Next, we prove that after applying the above charging rules there are no faces with a negative charge. For that, we need to show that a face cannot contribute charge to “too many” wedges. Apart from Rule 3(e), whenever a face contributes charge to a wedge, the edge through which the contribution is made lies on one arc of the wedge. Therefore, it is easy to see that, following those rules, a face contributes a charge that is proportional to its size. We will also show that Rule 3(a) cannot be applied more than once for a certain edge of a certain face. For this, we introduce the following definition.

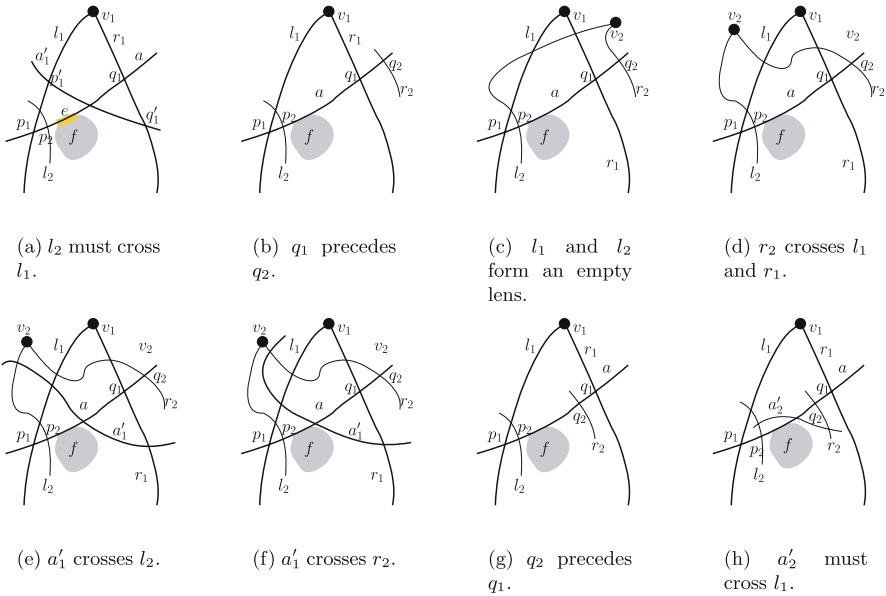


Fig. 7 Illustrations for the proof of Observation 2.8

Definition 2.7 (Possible X -contributor) For a (good) face f and one of its edges e , we say that f is a possible X -contributor to a wedge $w = (v, a_l, a_r)$ through e if there is an X -crossing of w, x , such that f is outside $w|_x$ and $e \subset \text{Vis}(x, a_l)$.

Observation 2.8 Let f be a face, and let e be one of its edges. Then f is a possible X -contributor through e to at most one wedge.

Proof Suppose that there is a face f that is a possible X -contributor through one of its edges, e , to two wedges, $w_1 = (v_1, l_1, r_1)$ and $w_2 = (v_2, l_2, r_2)$. Let a be the arc containing e ; then there are four points, p_1, q_1, p_2, q_2 , such that $\delta_1 = (a, p_1, q_1)$ is a Δ -crossing of w_1 and $\delta_2 = (a, p_2, q_2)$ is a Δ -crossing of w_2 . Denote by $x_1 = ((a, p_1, q_1), (a'_1, p'_1, q'_1))$ the X -crossing of w_1 such that $e \subset \text{Vis}(x_1, l_1)$, and by $x_2 = ((a, p_2, q_2), (a'_2, p'_2, q'_2))$ the X -crossing of w_2 such that $e \subset \text{Vis}(x_2, l_2)$. Suppose that we sort p_1, q_1, p_2, q_2 by the order in which they appear when traversing a from one of its endpoints to the other, so that when traversing e the face f is to our right. Then, p_i must precede q_i for $i = 1, 2$, since f is outside of $w_i|_{x_i}$. Assume, w.l.o.g., that p_1 precedes p_2 . It follows from Observation 2.6 that l_2 crosses l_1 (see Fig. 7(a)). Since $e \subset a|_{p_1, q_1} \cap a|_{p_2, q_2}$, the order of the four points is either p_1, p_2, q_1, q_2 or p_1, p_2, q_2, q_1 . Let us consider these cases:

Case 1: Suppose that q_1 precedes q_2 (see Fig. 7(b)). Since $w_1|_{x_1}$ and $w_2|_{x_2}$ do not contain any original vertex, either (1) l_2 must cross l_1 (one more time) and r_1 (see Fig. 7(c)) or (2) r_2 must cross l_1 and r_1 (see Fig. 7(d)). The first case yields an empty lens. In the second case, note that a'_1 must cross either l_2 (see Fig. 7(e)) or r_2 (see Fig. 7(f)), yielding four pairwise crossing arcs.

Case 2: Suppose that q_2 precedes q_1 (see Fig. 7(g)). Then the arc a'_2 must cross a twice, creating an empty lens, or cross l_1 , yielding four pairwise crossing arcs (see Fig. 7(h)).

Since all the cases imply forbidden configurations (an empty lens or four pairwise crossing arcs), f cannot be a possible X -contributor through e to more than one wedge. \square

Lemma 2.9 *The final charge of every face is nonnegative.*

Proof Let e be an edge of a face f . Recall that f contributes at most $1/3$ units of charge through e when 0-triangles are being charged. When charging the wedges, Rules 2 and 3 can be applied on f and e at most once each. Note that if Rule 1 is applied on f to charge a wedge with a vertex v , then f does not contribute charge through its two edges that are incident to v (the endpoints of an edge through which charge is contributed are always in $X(G)$). Thus, f contributes at most $|f| - 4 - |f|(\frac{1}{3} + \frac{1}{36} + \frac{1}{36})$ units of charge. Therefore, if $|f| \geq 7$, its final charge is nonnegative.

Assume that $|f| \leq 6$. If f is a bad face, then it does not contribute any charge to a wedge, and so its final charge is zero. If f is a good face, then recall that its charge after the 0-triangles were charged is at least $1/3$. When the wedges are being charged, f contributes at most $\frac{2|f|}{36}$ units of charge, thus it remains with a nonnegative charge. \square

Summing up the charges over all the wedges, we have $\frac{2|E(G)|}{36} \leq 4n - 8$; hence $|E(G)| \leq 72(n - 2)$. \square

3 Discussion

We have shown that the maximum number of arcs in a topological graph on n vertices with no four pairwise crossing arcs is $O(n)$. The constant in Theorem 1 is certainly not optimal. One can reduce it by noticing that it is impossible that all the faces incident to a vertex of G are 1-triangles (as done in [1]), or by further analyzing the ways a small-sized face contributes charge. The bound we found, combined with the analysis in [2] and [4], yields the following corollaries.

Corollary 3.1 *For any fixed integer $k > 4$, a simple topological graph on n vertices with no k pairwise crossing arcs has $O(n \log^{2k-8} n)$ arcs.*

Corollary 3.2 *For any fixed integer $k > 4$, a topological graph on n vertices with no k pairwise crossing arcs has $O(n \log^{4k-16} n)$ arcs.*

This improves the previous bounds by a factor of $\Theta(\log^2 n)$ and $\Theta(\log^4 n)$, respectively. However, the conjecture that $f_k(n) = O(n)$ for any fixed $k > 4$ remains open. It might be possible to settle this conjecture for $k = 5$ using our method, but it seems that for greater values, one should come up with new ideas.

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