Discrete Comput Geom (2009) 42: 705–721
DOI 10.1007/s00454-008-9131-5

Delannoy Orthants of Legendre Polytopes

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Received: 6 December 2007 / Revised: 8 December 2008 / Accepted: 10 December 2008 /
Published online: 24 December 2008

Abstract We construct an n-dimensional polytope whose boundary complex is compressed and whose face numbers for any pulling triangulation are the coefficients of the powers of $(x-1)/2$ in the nth Legendre polynomial. We show that the non-central Delannoy numbers count all faces in the lexicographic pulling triangulation that contain a point in a given open generalized orthant. We thus provide a geometric interpretation of a relation between the central Delannoy numbers and Legendre polynomials, observed over 50 years ago (Good in Proc. Camb. Philos. Soc. 54:39–42, 1958; Lawden in Math. Gaz. 36:193–196, 1952; Moser and Zayachkowski in Scr. Math. 26:223–229, 1963). The polytopes we construct are closely related to the root polytopes introduced by Gelfand et al. (Arnold–Gelfand mathematical seminars: geometry and singularity theory, pp. 205–221. Birkhauser, Boston, 1996).

Keywords Legendre polynomials · Delannoy numbers · Root polytopes · Compressed triangulations · Catalan numbers · Central binomial coefficients · Centrally symmetric polytopes

1 Introduction

The Delannoy numbers, introduced by Henri Delannoy [5] more than a hundred years ago, became recently a subject of renewed interest. More than 50 years ago, a somewhat mysterious connection was noted between the central Delannoy numbers and Legendre polynomials [8, 14, 15]. Until recently, this relation was mostly dismissed as a "coincidence". The first interpretation was given by the present author [11] who noted that the central Delannoy numbers also form the diagonal in an asymmetric

This work was supported by the NSA grant # H98230-07-1-0073.

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variant of the Delannoy table where all elements are obtainable via substitution into Jacobi polynomials, generalizing Legendre polynomials.

The subject of the present paper is a geometric interpretation of the relation between the central Delannoy numbers and the Legendre polynomials. For that purpose, we construct an n-dimensional polytope $\mathcal{L}_n$ for each n, such that $\mathcal{L}_n$ has a compressed boundary complex, thus all its pulling triangulations have the same face numbers. Multiplying the number of $(j-1)$-dimensional faces by $((x-1)/2)^j$ and summing over j yields the nth Legendre polynomial, and substituting $x=3$ gives the number of all faces. This number is also the central Delannoy number $d_{n,n}$. The main result of this paper is that, for the lexicographic pulling triangulation of $\mathcal{L}_n$, the noncentral Delannoy number $d_{n,n-i}$ counts the number of all faces that contain at least one point in the open generalized orthant defined by requiring the first i coordinates to be negative.

The Legendre polytope $\mathcal{L}_n$ is defined as the intersection of an n-dimensional cross-polytope and a hyperplane in Sect. 4. To prepare this definition, Sect. 3 explores a few facts that always hold when we consider the intersection of a centrally symmetric polytope and a hyperplane that contains the origin but no vertex. The graphs we introduce are directed generalizations of the variants of some graphs that appear in the work of Gelfand, Graev, and Postnikov [7], the key Lemma 3.4 is a generalization of a result originally due to Kapranov, Postnikov, and Zelevinski (see the first half of Lemma 12.5 in [16]).

The Legendre polytope $\mathcal{L}_n$ may be represented as the convex hull of the root polytope $P_{A_n^+}$ and its negative. The root polytopes $P_{A_n^+}$ were first studied by Gelfand, Graev and Postnikov [7]. Results on these polytopes were generalized by Postnikov [16] and Wungkum Fong [6]. The main difference between the root polytopes and Legendre polytopes is that $P_{A_n^+}$ contains the origin as a vertex whereas $\mathcal{L}_n$ contains the origin as an interior point, to which it is symmetric. In Sect. 4, we observe that some results in [7] may be restated as saying that all pulling triangulations of a root polytope that contain the origin as the least vertex are compressed. In this form, the statement is a direct consequence of a statement of Stanley [18] and the well-known result [9] stating that the incidence matrix of every directed graph is totally unimodular. This approach easily generalizes to the directed graphs we use to model the faces in the Legendre polytope; thus, we are able to show that every pulling triangulation of the boundary of the Legendre polytope that uses only the vertices is also compressed.

We compute the face numbers of a pulling triangulation of a Legendre polytope and of a root polytope in Sect. 5. For the Legendre polytope, this is easily done using the lexicographic order to pull the vertices, but the implied combinatorial enumeration problem does not seem to have a nice variant for root polytopes. On the other hand, the face numbers in the triangulation with respect to the revlex order may be counted with relative ease for the root polytopes, and this argument has a generalization to Legendre polytopes, relying on the result for the root polytopes.

Our main result is in Sect. 6 where we establish the above stated geometric interpretation of the Delannoy numbers.

It is our hope that the approach developed here will be inspiring for researchers of generalized root polytopes, perhaps the centrally symmetric variants of those polytopes will turn out to be as interesting as the Legendre polytopes. We also hope that the results will enhance the interest of experts of lattice path enumeration in polyhedral face enumeration. Further remarks and suggestions may be found in the concluding Sect. 7.

2 Preliminaries

2.1 Delannoy Numbers

The *Delannoy array* $(d_{i,j} : i, j \in \mathbb{Z})$, introduced by Henri Delannoy [5] may be defined by the recursion formula

$$d_{i,j} = d_{i-1,j} + d_{i,j-1} + d_{i-1,j-1} \tag{1}$$

with the conditions $d_{0,0} = 1$ and $d_{i,j} = 0$ if $i < 0$ or $j < 0$. The significance of these numbers is explained within a historic context in the paper "Why Delannoy numbers?"[2] by Banderier and Schwer. The diagonal elements $(d_{n,n} : n \geq 0)$ in this array are the *(central) Delannoy numbers* (A001850 of Sloane [17]). These numbers are known through the books of Comtet [4] and Stanley [19], but it is Sulanke's paper [20] that gives the most complete enumeration of all known uses of Delannoy numbers (a total of 29 configurations). For more information and a detailed bibliography we refer the reader to the above mentioned sources. Perhaps the simplest interpretation of the Delannoy number $d_{i,j}$ is that it is the number of lattice paths from $(0,0)$ to (i,j) using the steps $(1,0)$, $(0,1)$ and $(1,1)$.

2.2 Legendre Polynomials and Their Connection to the Delannoy Numbers

The nth Legendre polynomial $P_n(x)$ is a special case of the nth Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ which may be defined by the *Rodrigues formula*

$$P_n^{(\alpha,\beta)}(x) = (-2)^{-n}(n!)^{-1}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\big((1-x)^{n+\alpha}(1+x)^{n+\beta}\big).$$

Substituting $\alpha = \beta = 0$ yields $P_n(x)$ [3, Chap. V, (2.2)]. In the classical literature, the above definition is of $P_n^{(\alpha,\beta)}(x)$ is usually accompanied by the restriction $\alpha > -1$, $\beta > -1$, "for integrability purposes" [3, Chap. V, (2.1)], but the definition works for any α, β.

Equivalently, we may define the Jacobi polynomials by

$$P_n^{(\alpha,\beta)}(x) = \sum_k \binom{n+\alpha}{k}\binom{n+\beta}{k}\left(\frac{x-1}{2}\right)^{n-k}\left(\frac{x+1}{2}\right)^k.$$

For a nonnegative integer α, a Jacobi polynomial may be also given in the form

$$P_n^{(\alpha,\beta)}(x) = \sum_j \binom{n+\alpha+\beta+j}{j}\binom{n+\alpha}{j+\alpha}\left(\frac{x-1}{2}\right)^j. \tag{2}$$

See, e.g., Wilf [22, Chap. 4, Exercise 15 (b)]. Thus, $P_n(x)$ satisfies

$$P_n(x) = \sum_j \binom{n+j}{j}\binom{n}{j}\left(\frac{x-1}{2}\right)^j. \tag{3}$$

The following connection between the central Delannoy numbers and Legendre polynomials has been known for at least half a century [8, 14, 15]:

$$d_{n,n} = P_n(3), \tag{4}$$

but no combinatorial explanation was found; there even seemed to be a consensus that this equation is rather a "coincidence". Banderier and Schwer [2] note that there is no "natural" correspondence between Legendre polynomials and the original lattice path enumeration problem associated to the Delannoy array, while Sulanke [20] states that "the definition of Legendre polynomials does not appear to foster any combinatorial interpretation leading to enumeration". The present author has found a combinatorial interpretation in [11]; this interpretation involves, however, a variant of the Delannoy numbers whose table coincides on the main diagonal only.

2.3 Triangulation by Pulling the Vertices

Let $\mathcal{P}$ be a polytopal complex and L a set of points containing the set of vertices of $\mathcal{P}$. Let $<$ be a linear order on L. The pulling triangulation $\triangle_{L,<}(\mathcal{P})$ with respect to L is defined recursively as follows. We set $\triangle_{L,<}(\mathcal{P}) = \mathcal{P}$ if $\mathcal{P}$ consists of a single vertex. Otherwise, let v_1 be the least element of L with respect to $<$ and set

$$\triangle_{L,<}(\mathcal{P}) = \triangle(\mathcal{P} \setminus v_1) \cup \bigcup_F \bigl\{\operatorname{conv}\bigl(\{v_1\} \cup G\bigr) : G \in \triangle\bigl(\mathcal{P}(F)\bigr)\bigr\}.$$

Here the union runs over the facets F not containing v_1 of the maximal faces of $\mathcal{P}$ which contain v_1. The complex $\mathcal{P} \setminus v_1$ consists of all faces of $\mathcal{P}$ not containing v_1, $\mathcal{P}(F)$ consists of all faces of $\mathcal{P}$ contained in F. The triangulations $\triangle(\mathcal{P} \setminus v_1)$ and $\triangle(\mathcal{P}(F))$ are with respect to $L \setminus \{v_1\}$ and the restriction of $<$. This very general definition is given by Athanasiadis [1]. In most papers where this notion is used, L either equals the vertex set, or some additional assumptions are made. For example, Sullivant [21] makes a definition where $\mathcal{P}$ is the face complex of a lattice polytope P, and L is the set of lattice points in P. In the event when L equals the vertex set of $\mathcal{P}$, we will write $\triangle_<$ instead of $\triangle_{L,<}$ and refer to the triangulation as a pulling triangulation that uses only the vertices of $\mathcal{P}$.

The numbers of different dimensional faces in $\triangle_{L,<}(\mathcal{P})$ are hard to compute in general. In one important special case, however, Stanley [18] has shown how to calculate these numbers. Assume that $\mathcal{P}$ is the face complex of a polytope P whose vertices have integer coordinates and let $L = V(P)$. The order $<$ is *compressed* if all facets in $\triangle_<(P)$ have the same minimal volume. The face numbers of $\triangle_<(P)$ are the same for any compressed order $<$.

Stanley gave the following example of a compressed order [18, Example 2.4 (a)].

Proposition 2.1 (Stanley) *Suppose that one of the vertices of P is the origin and that the matrix whose rows are the vertices of P is totally unimodular. Let $<$ be any ordering on $V(P)$ such that the origin is the least vertex with respect to $V(P)$. Then $<$ is compressed.*

3 Non-Degenerate Central Sections of Centrally Symmetric Polytopes

Throughout this section we assume that $P \subset \mathbb{R}^n$ is a centrally symmetric polytope of dimension n, centered at the origin. Let H be a hyperplane containing the origin and not containing any vertex of P. We call the polytope $Q := P \cap H$ a *non-degenerate central section* of the polytope P.

Assume that H is given by the equation $\sum_{i=1}^n \lambda_i x_i = \langle \lambda \mid x \rangle = 0$ where the x_i's are the coordinate functions. The vertex set $V(P)$ of P is then the union of two disjoint sets $V_+(P) := \{(x_1, \ldots, x_n) \in V(P) : \langle \lambda \mid x \rangle > 0\}$ and $V_-(P) := \{(x_1, \ldots, x_n) \in V(P) : \langle \lambda \mid x \rangle < 0\} = -V_+(P)$. Each vertex of Q is of the form $H \cap [u, -v]$ where $[u, -v]$ is the line segment connecting $u \in V_+(P)$ and $-v \in V_-(P)$. Obviously, $[u, -v]$ must be an edge of P. We may thus represent the vertex set of $Q = P \cap H$ by a graph $G = G(P, H)$ on the vertex set $V(G) := V_+(P)$ and letting (u, v) be a directed edge in G exactly when $[u, -v] \cap H$ is a vertex of Q. Since P has no edge of he form $[u, -u]$, the graph $G(P, H)$ contains no loops.

Lemma 3.1 *Each vertex of Q is represented by a unique edge (u, v) in $G(P, H)$.*

In fact, if $[u_1, -v_1] \cap H = [u_2, -v_2] \cap H$ then the edges $[u_1, -v_1]$ and $[u_2, -v_2]$ intersect in an interior point and can not be edges of P.

Each face of Q is of the form $F \cap H$ where F is a face of P, and the set $V(F)$ of the vertices of F is a subset of $V(P)$. Introducing $V_+(F) := V_+(P) \cap V(F)$ and $V_-(F) := V_-(P) \cap V(F)$, each vertex of $F \cap H$ is of the form $[u, -v] \cap H$ where $u \in V_+(F)$ and $-v \in V_-(F)$. Since the set of vertices $F \cap H$ is a subset of the set of vertices of Q, each vertex $[u, -v] \cap H$ of $F \cap H$ must satisfy that (u, v) is an edge of $G(P, H)$. Conversely, if (u, v) is an edge in $G(P, H)$, $u \in V_+(F)$ and $-v \in V_-(F)$ then the segment $[u, -v]$ is a subset of F and $[u, -v] \cap H$ is a vertex in $F \cap H$.

Definition 3.2 Let G be a directed graph on the vertex set $V(G)$, edge set $E(G)$ with no multiple edges. Let S and T be disjoint subsets of V. The *directed restriction of G to (S, T)* is the digraph with vertex set $S \cup T$ with edge set $\{(s, t) \in E(G) : s \in S, t \in T\}$.

We obtained the following description of the face structure of $Q = P \cap H$.

Proposition 3.3 *Assume the vertices of $Q = P \cap H$ are represented by the edges of the graph $G = G(P, H)$. Given a face F of P, the vertices contained in the face $F \cap H$ of Q are represented by the edges in the directed restriction of $G(P, H)$ to $(V_+(F), -V_-(F))$.*

Assume from now on that P is simplicial. Then, for each face $F \subset P$, $[u, -v]$ is an edge if $u \in V_+(F)$ and $-v \in V_-(F)$. Thus, the directed restriction of $G(P, H)$ to $(V_+(F), -V_-(F))$ is a complete bipartite graph with each edge directed towards its endpoint in $V_-(F)$. The following lemma generalizes a result of Kapranov, Postnikov, and Zelevinski (see the first half of Lemma 12.5 in [16]).

Lemma 3.4 *Let $F \subset P$ be a face of P. A subset S of the edges of the directed restriction of $G(P, H)$ to $(V_+(F), -V_-(F))$ represents a simplex if and only if, disregarding the orientation of the edges, the set S contains no cycle.*

Proof Observe first that for any $v_i \in V_+(F)$ and $-v_j \in V_-(F)$, the intersection $v_{i,j}$ of H with $[v_i, -v_j]$ is given by

$$v_{i,j} = \frac{\langle \lambda \mid v_j \rangle}{\langle \lambda \mid v_i + v_j \rangle} v_i - \frac{\langle \lambda \mid v_i \rangle}{\langle \lambda \mid v_i + v_j \rangle} v_j .$$

The set $\{v_{i,j} : (v_i, v_j) \in S\}$ is affinely dependent if and only if there exist coefficients $\alpha_{i,j}$, not all zero, such that

$$\sum_{(v_i,v_j)\in S} \alpha_{i,j} v_{i,j} = 0 \quad \text{and} \quad \sum_{(v_i,v_j)\in S} \alpha_{i,j} = 0$$

hold. Since F is a simplex, the set of vectors $V_+(F) \cup V_-(F)$ is linearly independent. Thus, the condition on affine dependence is equivalent to stating

$$\sum_j \alpha_{i,j} \frac{\langle \lambda \mid v_j \rangle}{\langle \lambda \mid v_i + v_j \rangle} = 0 \quad \text{for all } v_i, \tag{5}$$

$$\sum_i \alpha_{i,j} \frac{\langle \lambda \mid v_i \rangle}{\langle \lambda \mid v_i + v_j \rangle} = 0 \quad \text{for all } -v_j, \quad \text{and} \tag{6}$$

$$\sum_{i,j} \alpha_{i,j} = 0. \tag{7}$$

Assume first that S contains no cycle, yet it represents an affinely dependent set. Then there is either a $v_i \in V_+(F)$ or a $v_j \in -V_-(F)$ belonging to a unique edge $(v_i, v_j) \in S$. For such an i (resp., j), Condition (5) (resp., (6)) may be only fulfilled by setting $\alpha_{i,j} = 0$ which allows to remove (v_i, v_j) from S. The set $S' := S \setminus \{(v_i, v_j)\}$ still contains no cycles, and the restriction of the $\alpha_{i,j}$'s to S' exhibits that S' still represents an affinely dependent set. Repeating this argument finitely many times we reach a contradiction. Therefore, cycle-free sets of edges represent affinely independent sets.

Assume now that the set S contains a cycle. Ignoring the orientation of the edges yields a bipartite graph, thus our cycle has even length $2m$ for some m. Label the vertices $v_1, \dots, v_{2m}$ along the cycle in such a way that the odd indexed vertices belong to $V_+(F)$, the rest belongs then to $-V_-(F)$. The set of directed edges covered by our cycle is $\{(v_1, v_2), (v_3, v_2), (v_3, v_4), (v_4, v_3), \dots, (v_{2m-1}, v_{2m}), (v_1, v_{2m})\}$. Define the coefficients $\alpha_{i,j}$ by setting

$$\alpha_{2i-1,2i} := \frac{\langle \lambda \mid v_{2i-1} + v_{2i}\rangle}{\langle \lambda \mid v_{2i-1}\rangle\langle \lambda \mid v_{2i}\rangle} \quad \text{for } i = 1, \dots, m,$$
$$\alpha_{2i+1,2i} := -\frac{\langle \lambda \mid v_{2i+1} + v_{2i}\rangle}{\langle \lambda \mid v_{2i+1}\rangle\langle \lambda \mid v_{2i}\rangle} \quad \text{for } i = 1, \dots, m-1, \quad \text{and}$$
$$\alpha_{1,2m} := -\frac{\langle \lambda \mid v_1 + v_{2m}\rangle}{\langle \lambda \mid v_1\rangle\langle \lambda \mid v_{2m}\rangle}.$$

Set all other $\alpha_{i,j}$'s to zero. It is easy to verify that the resulting set coefficients satisfies (5), (6) and (7). In fact, introducing $v_0 := v_{2m}$ and writing our indices "modulo $2m$" we get

$$\alpha_{2i-1,2i}\frac{\langle \lambda \mid v_{2i}\rangle}{\langle \lambda \mid v_{2i-1} + v_{2i}\rangle} + \alpha_{2i-1,2i-2}\frac{\langle \lambda \mid v_{2i-2}\rangle}{\langle \lambda \mid v_{2i-1} + v_{2i-2}\rangle}$$
$$= \frac{1}{\langle \lambda \mid v_{2i-1}\rangle} - \frac{1}{\langle \lambda \mid v_{2i-1}\rangle} = 0;$$

thus, (5) is satisfied for all v_{2i-1} where $i = 1, 2, \dots, m$. The verification of (6) is similar. To verify (7) it is worth observing that $\alpha_{2i-1,2i}$ may be rewritten as $1/\langle \lambda \mid v_{2i-1}\rangle + 1/\langle \lambda \mid v_{2i}\rangle$ thus the sum of all $\alpha_{2i-1,2i}$'s is $\sum_{j=1}^{2m} \langle \lambda \mid v_j\rangle$, whereas the sum of all $\alpha_{2i+1,2i}$'s is completely similarly $-\sum_{j=1}^{2m} \langle \lambda \mid v_j\rangle$. Therefore, if S contains a cycle, the vertices of F represented by the edges of this cycle form an affinely dependent set. □

4 Legendre Polytopes, Root Polytopes $P_{A_n^+}$, and Their Pulling Triangulations

The *standard cross-polytope* $\mathcal{O}_{n+1} \subset \mathbb{R}^{n+1}$ is the convex hull of $\{e_0, -e_0, \dots, e_n, -e_n\}$, where e_i is the ith standard basis vector. We start with index 0 for technical reasons. For integrality reasons, we will work with $2\mathcal{O}_{n+1}$, the convex hull of $\{2e_0, -2e_0, \dots, 2e_n, -2e_n\}$.

Definition 4.1 We define the *Legendre polytope* $\mathcal{L}_n$ as the non-degenerate central section of $2\mathcal{O}_{n+1}$ with the hyperplane

$$H_n := \left\{(x_0, \dots, x_n) : \sum_{i=0}^{n} x_i = 0\right\}.$$

It is easy to see that the vertices of $\mathcal{L}_n$ are all points of the form $e_i - e_j$ where $i \neq j$. Using the observations of Sect. 3, we may encode each vertex of $\mathcal{L}_n$ by an edge $(2e_i, 2e_j)$. The graph $G(\mathcal{L}_n, H_n)$ is then the complete digraph on the vertex set $\{2e_0, \dots, 2e_n\}$, containing all directed edges $(2e_i, 2e_j)$ where $i \neq j$. Each proper face of $\mathcal{L}_n$ is of the form $F \cap H_n$ where F is a face of $2\mathcal{O}_{n+1}$. A subset of

$\{2e_0, -2e_0, \dots, 2e_n, -2e_n\}$ is the vertex set of a face F of $2\mathcal{O}_{n+1}$ exactly when it does not contain both $2e_i$ and $-2e_i$ for some i. In particular, each facet of $2\mathcal{O}_{n+1}$ is of the form $\{\varepsilon_0 2e_0, \dots, \varepsilon_n 2e_n\}$ where $\varepsilon_0, \dots, \varepsilon_n \in \{1, -1\}$. Thus, we may observe the following.

Lemma 4.2 *A set $S \subset \{(2e_i, 2e_j) : i \neq j\}$ of edges represents all vertices in a facet of the boundary $\partial\mathcal{L}_n$ of $\mathcal{L}_n$ if and only if there is a proper subset A of $A \subset \{2e_0, \dots, 2e_n\}$ such that S consists of all edges starting in A and ending in $\{2e_0, \dots, 2e_n\} \setminus A$.*

This lemma may be rephrased in terms of *admissibility*, originally introduced by Gelfand, Graev and Postnikov [7] for root polytopes.

Definition 4.3 We call a set S of edges in a directed graph G on the vertex set *admissible* if there is no vertex v such that both $(u, v) \in G$ $(v, w) \in G$ hold for some vertices u and w.

Lemma 4.2 is equivalent to the following.

Lemma 4.4 *A set $S \subset \{(2e_i, 2e_j) : i \neq j\}$ of edges represents a subset of the vertex set of a face in $\partial\mathcal{L}_n$ if and only if it is admissible.*

Proof Clearly, facets of $\partial\mathcal{L}_n$ are represented by admissible sets. Conversely, given an admissible set S, let A be the set of all vectors $2e_i$ such that $(2e_i, 2e_j) \in S$ for some j. Then S is easily seen to be a subset of the facet whose vertex set is represented by all edges starting in A and ending in $\{2e_0, \dots, 2e_n\} \setminus A$. □

The convex hull of the origin $\mathbf{0}$ and the vertex set $\{e_i - e_j : i < j\}$ is the *root polytope* $P_{A_n^+}$, first studied by Gelfand, Graev and Postnikov [7]. Results on these polytopes were extended to more general root polytopes by Postnikov [16] and (to the root systems B_n, C_n, and D_n) by Wungkum Fong [6]. Since all vertices of $\mathcal{L}_n$ not belonging to $P_{A_n^+}$ form $V(-P_{A_n^+}) \setminus \{\mathbf{0}\}$, we may think of $\mathcal{L}_n$ as the convex hull of $P_{A_n^+}$ and $-P_{A_n^+}$. The set of facets of $\partial\mathcal{L}_n$ may be partitioned into three classes:

(1) The facets of $P_{A_n^+}$ not containing the origin.
(2) The facets of $-P_{A_n^+}$ not containing the origin.
(3) Facets which contain at least one vertex of $P_{A_n^+}$ and one vertex of $-P_{A_n^+}$.

In our notation, the elements of $V(P_{A_n^+}) \setminus \{\mathbf{0}\}$ are denoted by directed edges $(2e_i, 2e_j)$ where $i < j$. All edges are directed from the lower index edge towards the higher index, thus one may omit indicating the orientation, and use (i, j) as a shorthand for $(2e_i, 2e_j)$. This is the notation used in [7]. All facets of $P_{A_n^+}$ contain $\mathbf{0}$ since $P_{A_n^+}$ is a cone over $\mathbf{0}$. The faces of $P_{A_n^+}$ that do not contain $\mathbf{0}$ are subsets of $\partial\mathcal{L}_n$, their union is $P_{A_n^+} \cap \partial\mathcal{L}_n$. By Lemma 4.4, a set of vertices contained in a face of $\mathcal{L}_n$ must be represented by an admissible set of edges, and here our notion of admissibility specializes to the definition given in [7].

As seen in Lemma 3.4 (or Lemma 12.5 in [16]), a set of vertices contained in a face of $P_{A_n^+} \cap \partial\mathcal{L}_n$ is affinely independent if and only if the associated edges form no cycle. In particular, facets in any triangulation of $P_{A_n^+}$ are associated with admissible trees on the vertex set $\{2e_0, \dots, 2e_n\}$. The vertex sets represented by these trees are facets in triangulations of $P_{A_n^+} \cap \partial\mathcal{L}_n$, coning over $\mathbf{0}$ yields all triangulations of $P_{A_n^+}$. There are two triangulations of $P_{A_n^+} \cap \partial\mathcal{L}_n$ explicitly given in [7]: the *standard* and the *anti-standard* triangulation. The standard triangulation is associated to the set of all admissible trees having *no intersections*: the edges $(2e_i, 2e_j)$ and $(2e_k, 2e_l)$ are intersecting if $i < k < j < l$ holds. The anti-standard triangulation is associated to the set of all admissible trees having *no enclosed edges*: $(2e_i, 2e_j)$ and $(2e_k, 2e_l)$ are enclosed if $i < k < l < j$ holds.

Both the standard and anti-standard triangulations are pulling triangulations of $P_{A_n^+}$.

Definition 4.5 The *revlex order* on $V(P_{A_n^+}) \setminus \{\mathbf{0}\}$ is defined by setting $(2e_i, 2e_j) < (2e_k, 2e_l)$ if $j < l$ or $j = l$ and $i > k$. The *lexicographic order* on $V(P_{A_n^+}) \setminus \{\mathbf{0}\}$ is defined by setting $(2e_i, 2e_j) < (2e_k, 2e_l)$ if $i < k$ or $i = k$ and $j < l$.

Lemma 4.6 *The standard triangulation of $P_{A_n^+}$ is the pulling triangulation with $L = V(P_{A_n^+})$ and the revlex order on $V(P_{A_n^+}) \setminus \{\mathbf{0}\}$ extended to $V(P_{A_n^+})$ by making $\mathbf{0}$ the least element.*

Proof Since $\mathbf{0}$ is the least element, the pulling triangulation is a cone over it, the same holds also for the standard triangulation. We are left to show that the family of facets of $P_{A_n^+} \cap \partial\mathcal{L}_n$ whose vertex set is represented by standard trees is exactly the family of facets in the revlex pulling triangulation of $P_{A_n^+} \cap \partial\mathcal{L}_n$. Consider a facet F of the revlex pulling triangulation. The vertices of F are represented by an admissible tree T on $\{2e_0, \dots, 2e_n\}$. Assume, by way of contradiction that $(2e_i, 2e_j)$ and $(2e_k, 2e_l)$ are crossing edges in T, satisfying $i < k < j < l$. The edge $(2e_i, 2e_j)$ represents the vertex $e_i - e_j$, the edge $(2e_k, 2e_l)$ represents the vertex $e_k - e_l$ in $\mathcal{L}_n$. These two vertices, together with $e_i - e_l$ and $e_k - e_j$ form a square on the boundary of $\mathcal{L}_n$. In fact, the four points belong to H_n and the convex hull of $\{2e_i, -2e_j, 2e_k, -2e_l\}$, a 2-face of $2\mathcal{O}_n$. No other $e_u - e_v$ belongs to this intersection. This square is also a face of $P_{A_n^+}$. The least vertex in the revlex order is the face represented by $(2e_k, 2e_j)$, when we pull it, an edge between $e_i - e_l$ and $e_k - e_j$ arises. This makes the adding of the other diagonal between $e_i - e_j$ and $e_k - e_l$ at a later pulling stage impossible.

We have shown that every facet in the revlex pulling triangulation is also a facet in the standard triangulation. The converse must also be true since both triangulations triangulate the same polytope, using the same vertices. □

Lemma 4.7 *The anti-standard triangulation of $P_{A_n^+}$ is the pulling triangulation with $L = V(P_{A_n^+})$ and the lex order on $V(P_{A_n^+}) \setminus \{\mathbf{0}\}$ extended to $V(P_{A_n^+})$ by making $\mathbf{0}$ the least element.*

The proof of this lemma is similar to the previous one and omitted.

It was observed in [7] that the facets in the standard and anti-standard triangulations have all the same minimal volume. In fact, Lemma 5.3 in [7] states that every simplex associated to an admissible tree has the same minimal volume. This may be rephrased as follows:

Proposition 4.8 (Gelfand–Graev–Postnikov) *Any pulling triangulation $\triangle_{L,<}(P_{A_n^+})$ of $P_{A_n^+}$ satisfying $L = V(P_{A_n^+})$ and that $\mathbf{0}$ is the least element in the order, is compressed.*

Proposition 4.8 is also an easy consequence of Proposition 2.1 and the following well-known result.

Theorem 4.9 (Heller) *The incidence matrix of a directed graph is totally unimodular.*

See [9] and [10]. In fact, the rows of the matrix whose rows are the vertices of $V(P_{A_n^+})$ are the rows of incidence matrix of the directed graph consisting of all edges $(2e_i, 2e_j)$ where $i < j$, plus there is an extra row, consisting of zeros only and not changing the total unimodularity property.

The proof of Proposition 4.8 that relies on Proposition 2.1 and Theorem 4.9 extends easily to a similar statement on the Legendre polytope.

Theorem 4.10 *Any pulling triangulation $\triangle_{L,<}(\mathcal{L}_n)$ of $\mathcal{L}_n$ satisfying $L = V(\mathcal{L}_n) \cup \{\mathbf{0}\}$ and that $\mathbf{0}$ is the least element in the order, is compressed.*

Proof Since $\mathbf{0}$ is the least point in the order, each facet in the pulling triangulation $\triangle_{L,<}(\mathcal{L}_n)$ arises as the join of $\mathbf{0}$ and a facet in a pulling triangulation of a facet of $\partial\mathcal{L}_n$. Thus, it is sufficient to prove that for any facet F of $\partial\mathcal{L}_n$, any pulling triangulation $\triangle_{L,<}(\mathrm{conv}(F \cup \{\mathbf{0}\}))$ of $\mathrm{conv}(F \cup \{\mathbf{0}\})$, satisfying $L = V(\mathrm{conv}(F \cup \{\mathbf{0}\}))$ and that $\mathbf{0}$ is the least element in the order, is compressed. This auxiliary statement is an immediate consequence of Proposition 2.1, Theorem 4.9, and the fact that the row matrix associated to the vertices of F is the incidence matrix of a digraph. □

Since pulling the interior point $\mathbf{0}$ results in coning over a pulling triangulation of $\partial\mathcal{L}_n$, an immediate consequence of Theorem 4.10 is the following.

Corollary 4.11 *All pulling triangulations of $\partial\mathcal{L}_n$ (using only the vertices) have the same face numbers.*

5 The F-Polynomials of the Pulling Triangulations

In this section, we compute the face numbers in any pulling triangulation $\triangle_<(\partial\mathcal{L}_n)$ of $\partial\mathcal{L}_n$ that uses only the vertices of $\mathcal{L}_n$. By Corollary 4.11, we only need to find these numbers for one linear order on the vertices. The easiest to use seems the *lexicographic order*: we set $(2e_i, 2e_j) < (2e_k, 2e_l)$ if $i < k$ or $i = k$ and $j < l$. In the previous section, this order was considered for $P_{A_n^+}$ only, now we extend it to all vertices of $\mathcal{L}_n$.

Lemma 5.1 *A set of vertices $\{(2e_{s_1}, 2e_{t_1}), (2e_{s_2}, 2e_{t_2}), \ldots, (2e_{s_j}, 2e_{t_j})\}$, satisfying $(2e_{s_1}, 2e_{t_1}) < \cdots < (2e_{s_j}, 2e_{t_j})$ in the lexicographic order, is a face of $\triangle_<(\partial\mathcal{L}_n)$ if and only if the following holds:*

(i) *The sets* $\{s_1, \dots, s_j\}$ *and* $\{t_1, \dots, t_j\}$ *are disjoint.*
(ii) *We have* $t_1 \le \cdots \le t_j$.

Proof By Lemma 4.4, Condition (i) is equivalent to stating that $\{(2e_{s_1}, 2e_{t_1}), (2e_{s_2}, 2e_{t_2}), \dots, (2e_{s_j}, 2e_{t_j})\}$ is contained in some face of $\partial\mathcal{L}_n$. By $(2e_{s_1}, 2e_{t_1}) < \cdots < (2e_{s_j}, 2e_{t_j})$ we must have $s_1 \le \cdots \le s_j$. Condition (ii) is sufficient since it guarantees that $e_{s_i} - e_{t_i}$ is the lexicographically least vertex in

$$H_n \cap \operatorname{conv}\bigl(\{2e_{s_i}, 2e_{s_{i+1}}, \dots, 2e_{s_j}, -2e_{t_i}, -2e_{t_{i+1}}, \dots, -2e_{t_j}\}\bigr) \quad \text{for } i = 1, \dots, j,$$

the smallest face of $\partial\mathcal{L}_n$ containing $\{e_{s_i} - e_{t_i}, e_{s_{i+1}} - e_{t_{i+1}}, \dots, e_{s_j} - e_{t_j}\}$. Using this observation, sufficiency may be shown by an easy induction. Condition (ii) is also necessary. Assume, by way of contradiction, that we have $t_i > t_{i+1}$ for some i. By $(2e_{s_i}, 2e_{t_i}) < (2e_{s_{i+1}}, 2e_{t_{i+1}})$ we must have $s_i < s_{i+1}$. Just like in the proof of Lemma 4.6, the vertices $e_{s_i} - e_{t_i}$, $e_{s_i} - e_{t_{i+1}}$, $e_{s_{i+1}} - e_{t_i}$ and $e_{s_{i+1}} - e_{t_{i+1}}$ form a square on the boundary of $\mathcal{L}_n$. The least vertex is $e_{s_i} - e_{t_{i+1}}$, when we pull it, and edge connecting it to $e_{s_{i+1}} - e_{t_i}$ arises. Thus, there cannot be an edge connecting $e_{s_i} - e_{t_i}$ and $e_{s_{i+1}} - e_{t_{i+1}}$ in the pulling triangulation since this other diagonal would cross the previous one. □

Theorem 5.2 *The number of* $(j-1)$*-dimensional faces in any pulling triangulation of* $\partial\mathcal{L}_n$ *that uses only the vertices of* $\mathcal{L}_n$ *is*

$$f_{j-1}\bigl(\triangle_<(\partial\mathcal{L}_n)\bigr) = \binom{n+j}{j}\binom{n}{j}.$$

Proof As noted above, it is sufficient to consider the pulling triangulation with respect to the lexicographic order. By Lemma 5.1, selecting a $(j-1)$-face in that triangulation is equivalent to performing the following two steps:

(1) Select two disjoint nonempty subsets $\{s_1, \dots, s_j\}$ and $\{t_1, \dots, t_j\}$ of $\{0, \dots, n\}$.
(2) Select the values of $s_1 \le \cdots \le s_j$ and $t_1 \le \cdots \le t_j$. Here, for each $i < j$ either $s_i = s_{i+1}$ or $t_i = t_{i+1}$ is allowed, but both cannot occur simultaneously.

If we choose $\{s_1, \dots, s_j\}$ to have u elements and $\{t_1, \dots, t_j\}$ to have v elements, then there are $\binom{n+1}{u, v, n+1-u-v}$ ways to perform the first step. Given $\{s_1, \dots, s_j\}$ and $\{t_1, \dots, t_j\}$, performing the second step is equivalent to selecting a lattice path of length $j-1$ from (s_1, t_1) to (s_j, t_j) in $\{s_1, \dots, s_j\} \times \{t_1, \dots, t_j\}$ such that in each step we increase either the s-value, or the t value, or both, to the next available element. We will have $j-u$ steps when we increase only the s-coordinate, $j-v$ steps when we only increase the right coordinate and $u+v-j-1$ steps when we increase both coordinates. There are $\binom{j-1}{j-u, j-v, u+v-j-1}$ ways to perform this step. Thus, we obtain

$$f_{j-1}\bigl(\triangle_<(\partial\mathcal{L}_n)\bigr) = \sum_{u,v} \binom{n+1}{u, v, n+1-u-v}\binom{j-1}{j-u, j-v, u+v-j-1}. \tag{8}$$

The sum on the right hand side is also the number of ways to select a pair (U, V) of disjoint subsets in an $(n+1)$-element set (here $|U| = u$ and $|V| = v$) and to complement each set to a j element set such that they remain disjoint, and the added elements come from a new $(j-1)$-element set. Equivalently, one may choose two disjoint j-element subsets in an $n+j$-element set, which may be done

$$\binom{n+j}{j, j, n-j} = \binom{n+j}{j}\binom{n}{j} \quad \text{ways.}$$

□

Theorem 5.2 is the main reason behind the name "Legendre polytope". To justify it, let us introduce the *F-polynomial* of a simplicial complex $\triangle$ by the formula

$$F_{\triangle}(x) := \sum_{j=0}^{d} f_{j-1}\left(\frac{x-1}{2}\right)^{j}.$$

Here $(d-1)$ is the dimension of the simplicial complex and f_{j-1} is the number of $(j-1)$-dimensional faces. This polynomial was shown to be related to certain orthogonal polynomials for the order complexes of some spherical posets in [12] and for a triangulation in [13].

Corollary 5.3 *The F-polynomial of any pulling triangulation of $\partial\mathcal{L}_n$ that uses only the vertices of $\mathcal{L}_n$ is $P_n(x)$, the nth Legendre polynomial.*

We conclude this section by computing the face numbers in any pulling triangulation of $P_{A_n^+} \cap \partial\mathcal{L}_n$ that uses only the vertices. The pulling triangulations of the root polytope $P_{A_n^+}$ whose least element is $\mathbf{0}$ are obtained from the pulling triangulations of $P_{A_n^+} \cap \partial\mathcal{L}_n$ via coning over $\mathbf{0}$. At the level of the F-polynomials, coning over a single vertex induces multiplication by a factor of $(x+1)/2$.

Theorem 5.4 *The number of $(j-1)$-dimensional faces in any pulling triangulation of $P_{A_n^+} \cap \partial\mathcal{L}_n$ that uses only the vertices is*

$$f_{j-1}\big(\triangle_{<}(P_{A_n^+} \cap \partial\mathcal{L}_n)\big) = \frac{1}{j+1}\binom{n+j}{j}\binom{n}{j}.$$

Proof By Proposition 4.8, we may restrict ourselves to the standard triangulation, induced by the revlex order. Introducing $f(n, j)$ as a shorthand for $f_{j-1}(\triangle_{<}(P_{A_n^+} \cap \partial\mathcal{L}_n))$, we have

$$f(n, j) = \sum_{0\le u<v\le n} \sum_{k=0}^{j-1} f(v-u-1, k) \cdot f(n-v, j-k-1). \tag{9}$$

In fact, $f(n, j)$ is the number of ways to select a system of j edges $(2e_s, 2e_t)$ on the vertex set $\{2e_0, \ldots, 2e_n\}$ such that $s < t$ is satisfied for each edge and the edges are pairwise non-crossing. Let $(2e_u, 2e_v)$ represent the least vertex of a face in the revlex order. Then $2e_u$ is the leftmost left end in our system of edges, and $2e_v$ is the rightmost right end of an edge starting in $2e_u$. By the non-crossing property, the remaining edges may be partitioned into two classes:

(1) Edges $(2e_s, 2e_t)$ satisfying $u < s < v$: for these we must have $s < t \le v$. These form a system of k non-crossing edges on $\{2e_u, 2e_{u+1}, \dots, 2e_v\}$ for some $0 \le k \le j-1$, so there are $f(v-u-1, k)$ ways to select them.
(2) Edges $(2e_s, 2e_t)$ satisfying $s = u$ or $s \ge v+1$. (By admissibility, $s = v$ is not allowed.) By the choice of v, $s = u$ implies $t > v$. Let us replace each edge $(2e_u, 2e_t)$ in this class with $(2e_v, 2e_t)$. (This cannot create duplicate edges since originally no edge started at e_v.) We obtain a system of $j-k-1$ non-crossing edges on $\{2e_v, 2e_{v+1}, \dots, 2e_n\}$. Conversely, given any system of $j-k-1$ non-crossing edges on $\{2e_v, 2e_{v+1}, \dots, 2e_n\}$, we may uniquely recover a system in our second class by replacing each $(2e_v, 2e_t)$ by $(2e_u, 2e_t)$. Thus, there are $f(n-v, j-k-1)$ ways to select the edges in this class.

The initial condition on the numbers $f(n, j)$ is $f(n, 0) = 1$, easy substitution into (9) gives $f(n, 1) = \binom{n+1}{2}$. We may use (9) to prove by induction on j that $f(n, j)$ is of the form $C_j\binom{n+j}{2j}$ where the number C_j does not depend on n. In fact, the statement is true for $j = 0$ and $j = 1$ and the induction step is the following:

$$\begin{aligned} f(n,j) &= \sum_{0\le u<v\le n}\sum_{k=0}^{j-1} C_k\binom{v-u-1+k}{2k}\cdot C_{j-k-1}\binom{n-v+j-k-1}{2j-2k-2} \\ &= \sum_{k=0}^{j-1} C_k C_{j-k-1}\sum_{0\le u<v\le n}\binom{v-u-1+k}{2k}\binom{n-v+j-k-1}{2j-2k-2} \\ &= \sum_{k=0}^{j-1} C_k C_{j-k-1}\binom{n+j}{2j}. \end{aligned}$$

We have also shown that the numbers C_j satisfy $C_j = \sum_{k=0}^{j-1} C_k C_{j-k-1}$. This, together with $C_0 = C_1 = 1$, implies that C_j is the jth Catalan number, and we have

$$f(n,j) = \frac{1}{j+1}\binom{2j}{j}\binom{n+j}{2j} = \frac{1}{j+1}\binom{n+j}{j}\binom{n}{j},$$

as stated. □

We should note that Theorem 5.2 also has a proof, analogous to the proof of Theorem 5.4, that relies on the following extension of the revlex order to the vertex set of $\mathcal{L}_n$: we set $(2e_i, 2e_j) < (2e_k, 2e_l)$ if one of the following holds:

(i) $\max(i, j) < \max(k, l)$.
(ii) $\max(i, j) = \max(k, l)$ and $\min(i, j) > \min(k, l)$.
(iii) $\{i, j\} = \{k, l\}$ and $i < j$ and $k > l$.

In other words, we take the revlex order on $\{(2e_i, 2e_j) : 0 \le i < j \le n\}$ and insert each $(2e_j, 2e_i)$ satisfying $j > i$ right above $(2e_i, 2e_j)$. It is easy to see that the resulting triangulation consists of all faces whose representation as a graph does not contain any pair $(2e_i, 2e_j)$, $(2e_k, 2e_l)$ with any of the following properties: $i < k < j < l$, $l < j < k < i$, $i < l < j < k$, $l < i < k < j$, $i < l < k < j$, $j < k < l < i$. The first four forbidden patterns here form the list of all ways two edges can cross, the last two forbids exclude the possibility of two nested edges to have opposite directions. Introducing $g(n, j)$ as a shorthand for $f_{j-1}(\partial\mathcal{L}_n)$, an argument similar to the one in the proof of Theorem 5.4 yields

$$g(n, j) = 2 \sum_{0\le u<v\le n} \sum_{k=0}^{j-1} f(v-u-1, k) g(n-v, j-k-1).$$

Here $f(n, j) = \frac{1}{j+1}\binom{n+j}{j}\binom{n}{j}$, by Theorem 5.4. In analogy to the proof seen there, we may show by induction that the numbers $g(n, j)$ are of the form $B_j\binom{n+j}{2j}$, where the numbers B_j are independent of n, satisfy the initial condition $B_0 = 1$, $B_1 = 2$, and the recursion formula $B_j = \sum_{k=0}^{j-1} B_j B_{j-k-1}$. Thus, B_j is the central binomial coefficient $\binom{2j}{j}$ and $g(n, j) = \binom{2j}{j}\binom{n+j}{2j}$.

Remark 5.5 The number of facets in any pulling triangulation of $P_{A_n^+} \cap \partial\mathcal{L}_n$ is the Catalan number C_n, as it was already stated in [7]. As a consequence of Theorem 5.2, the number of facets in any pulling triangulation of $\partial\mathcal{L}_n$ is the central binomial coefficient $\binom{2n}{n}$. The relation between the root polytope $P_{A_n^+}$ and the Legendre polytope $\mathcal{L}_n$ is thus a "geometric enhancement" of the relation between the Catalan numbers and central binomial coefficients, and this undercurrent seems especially highlighted by the use of the revlex order. It is unknown to the present author, whether the lex order could also be used efficiently to establish such a connection.

In analogy to Corollary 5.3 we have the following consequence of Theorem 5.4.

Corollary 5.6 *The F-polynomial of any pulling triangulation of $P_{A_n^+} \cap \partial\mathcal{L}_n$ that uses only the vertices is $P_n^{(1,-1)}(x)/(n+1)$. Here $P_n^{(1,-1)}(x)$ is a Jacobi polynomial.*

In fact, by Theorem 5.4 we obtain the F-polynomial

$$\sum_{j=0}^{n} \frac{1}{j+1}\binom{n+j}{j}\binom{n}{j}\left(\frac{x-1}{2}\right)^j = \frac{1}{n+1}\sum_{j=0}^{n}\binom{n+j}{j}\binom{n+1}{j+1}\left(\frac{x-1}{2}\right)^j.$$

The right hand side is $P_n^{(1,-1)}(x)/(n+1)$, by (2).

6 Delannoy Numbers and Generalized Orthants in the Legendre Polytope

Since the Delannoy number $d_{n,n-i}$ enumerates the number of lattice paths from $(0, 0)$ to $(n, n-i)$, using only steps $(1, 0)$, $(0, 1)$ and $(1, 1)$, the following equality is obvious:

$$d_{n,n-i} = \sum_{k=0}^{n-i} \binom{n+k}{n-i-k, k+i, k} = \sum_{k=0}^{n-i} \binom{n+k}{k+i}\binom{n-i}{k}. \tag{10}$$

Here k is the number of $(0, 1)$-steps, $k + i$ is the number of $(1, 0)$ steps and $n - i - k$ is the number of $(1, 1)$ steps. Inspired by (8), let us rewrite $\binom{n+k}{k+i}\binom{n-i}{k}$ as follows:

$$\begin{aligned}
&\binom{n+k}{k+i}\binom{n-i}{k}\\
&\quad=\sum_{u=1}^{n-i+1}\binom{n-i+1}{u}\binom{k+i-1}{k+i-u}\sum_{v}\binom{n-i+1-u}{v}\binom{u-1}{k-v}\\
&\quad=\sum_{u,v}\binom{n-i+1}{u, v, n-i+1-u-v}\binom{k+i-1}{k+i-u, k-v, u+v-k-1}.
\end{aligned}$$

Using the above formula, we may observe that $\binom{n+k}{k+i}\binom{n-i}{k}$ is the number of ways to perform the following procedure:

(1) For some u and v, select an u-element subset U and a v-element subset V of $\{i, i + 1, \dots, n\}$ such that U and V are disjoint ($\binom{n-i+1}{u,v,n-i+1-u-v}$ ways).
(2) Add $\{0, 1, \dots, i - 1\}$ to V, and select $s_1 \le \dots \le s_{k+i}$ and $t_1 \le \dots \le t_{k+i}$ such that $\{s_1, \dots, s_{k+i}\} = U$, $\{t_1, \dots, t_{k+i}\} = V \cup \{0, 1, \dots, i - 1\}$ and $(2e_{s_1}, 2e_{t_1}) < \dots < (2e_{s_{k+i}}, 2e_{t_{k+i}})$ is a strictly increasing chain in the lexicographic order.

The second phase may be performed in $\binom{k+i-1}{k+i-u,k-v,u+v-k-1}$ ways since it will happen $k - v$ times that we increase only the first coordinate, $k + i - u$ times that we only increase the second coordinate, and $u + v - k - 1$ times that we increase both coordinates. In analogy to Theorem 5.2, we obtained the following result.

Theorem 6.1 *The number of those $(k + i - 1)$-dimensional faces in the lexicographic pulling triangulation of $\partial\mathcal{L}_n$ (using only the vertices of $\mathcal{L}_n$) which contain at least one vertex of the form $e_s - e_t$ for each $t \in \{0, 1, \dots, i - 1\}$ is $\binom{n+k}{k+i}\binom{n-i}{k}$.*

Note that this theorem includes Theorem 5.2 as the special case $i = 0$. However, we have this generalization for the lexicographic pulling triangulation only. As a consequence of (10) and Theorem 6.1 we have the following result.

Corollary 6.2 *The Delannoy number $d_{n,n-i}$ is the number of all those faces in the lexicographic pulling triangulation of $\partial\mathcal{L}_n$ which contain at least one vertex of the form $e_s - e_t$ for each $t \in \{0, 1, \dots, i - 1\}$.*

In particular, $d_{n,n}$ is the number of all faces in the lexicographic pulling triangulation of $\partial\mathcal{L}_n$ and thus equal to $P_n(3)$, by Corollary 5.3.

Corollary 6.2 has the following geometric interpretation.

Theorem 6.3 *For $i > 0$, the Delannoy number $d_{n,n-i}$ is the number of all faces F in the lexicographic pulling triangulation of $\partial\mathcal{L}_n$ that contain at least one point in the generalized orthant $\{(x_0, \dots, x_n) : x_0 < 0, x_1 < 0, \dots, x_i < 0\}$.*

Proof A face in $\triangle_<(\partial\mathcal{L}_n)$ contains at least one vertex of the form $e_s - e_t$ for each $t \in \{0, 1, \dots, i - 1\}$ if and only if it contains a point $(x_0, \dots, x_n)$ satisfying $x_0 < 0$,

$x_1 < 0, \dots, x_{i-1} < 0$. In fact, if there exist vertices of the form $e_{s_0} - e_0, e_{s_1} - e_1, \dots, e_{s_{i-1}} - e_{i-1}$ then the coordinates $(x_0, \dots, x_n)$ of the point $\frac{1}{i}\sum_{j=0}^{i-1}(e_{s_j} - e_j)$ satisfy $x_0 < 0, x_1 < 0, \dots, x_{i-1} < 0$. Conversely, if there is a point $(x_0, \dots, x_n)$ satisfying $x_0 < 0, x_1 < 0, \dots, x_{i-1} < 0$ in the convex hull, then for each $j \le i-1$ the inequality $x_j < 0$ forces the existence of a vertex $(e_{s_j} - e_j)$ in the triangulation. □

Theorem 6.3 motivates calling the generalized orthants $\{(x_0, \dots, x_n) : x_0 < 0, x_1 < 0, \dots, x_i < 0\} \cap \mathcal{L}_n$ *Delannoy orthants* of the Legendre polytope.

7 Concluding Remarks

Fundamentally, there are two ways to define a polytope: as a convex hull of vertices, or as an intersection of half-spaces and hyperplanes. It appears that the root polytopes introduced by Gelfand, Graev, and Postnikov [7] are most easily generalized in terms of the first approach: one replaces the set of vertices with a set that has a more complicated geometry, but still many symmetries. Legendre polytopes, however, are very naturally generalized in terms of the second approach, in fact, we obtained the definition of a Legendre polytopes by specializing the general approach outlined in Sect. 3. It seems reasonable to expect that non-degenerate central sections of other centrally symmetric polytopes will have interesting geometric and combinatorial properties. Lemma 3.4 indicates that the digraph generalization of the graphical approach that can be found in [7] is suitable to visualize the face structure of any non-degenerate central section of any simplicial centrally symmetric polytope.

In our work it was convenient to know that face enumeration in a pulling triangulation does not depend on the order of vertices because of Stanley's example [18] and a well-known unimodularity result. The question naturally arises: under which conditions can we guarantee that a non-degenerate central section of a centrally symmetric polytope is compressed? Here the term "compressed" seems to be most naturally defined in terms of considering all pulling triangulations that use the vertices, the origin, and make the origin the least point in the order. In particular, is it true that, subject to assumptions, a non-degenerate central section of a compressed centrally symmetric polytope is compressed?

Our main result on the Delannoy numbers counting certain faces in the Legendre polytope applies to the lexicographic pulling triangulation only. It seems worth exploring whether an analogous result exists for the revlex pulling triangulation, or whether there is a result that can be stated independently of the order on the vertices.

Acknowledgements I wish to thank Alexander Postnikov and Richard Stanley, for valuable background information on root polytopes. This work was supported by the NSA grant # H98230-07-1-0073.

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