On the Maximum Number of Translates in a Point Set

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Abstract Given a finite set $P \subseteq \mathbb{R}^d$, called a pattern, $t_P(n)$ denotes the maximum number of *translated* copies of P determined by n points in $\mathbb{R}^d$. We give the exact value of $t_P(n)$ when P is a rational simplex, that is, the points of P are rationally affinely independent. In this case, we prove that $t_P(n) = n - m_r(n)$, where r is the rational affine dimension of P, and $m_r(n)$ is the r-Kruskal–Macaulay function. We note that almost all patterns in $\mathbb{R}^d$ are rational simplices. The function $t_P(n)$ is also determined exactly when $|P| \leq 3$ or when P has rational affine dimension one and n is large enough. We establish the equivalence of finding $t_P(n)$ and the maximum number $s_R(n)$ of scaled copies of a suitable pattern $R \subseteq \mathbb{R}^+$ determined by n positive reals. As a consequence, we show that $s_{A_k}(n) = n - \Theta(n^{1-1/\pi(k)})$, where $A_k = \{1, 2, \ldots, k\}$ is an arithmetic progression of size k, and $\pi(k)$ is the number of primes less than or equal to k.

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1 Introduction

Some of the most natural and important problems in Combinatorial Geometry are those concerning *repeated configurations* and *pattern recognition*. A typical question in the area is the following:

> What is the maximum number of occurrences of a pattern among all subsets of an n-point set in the d-dimensional space?

Here an *occurrence* is determined by a class of equivalence under a fixed group of geometric transformations: translations, homotheties, congruences, similarities, and affinities. Many of these problems were first asked by Erdős and Purdy [10, 11]. For instance, when $d = 2$, the geometric group are congruences, and the pattern consists of the endpoints of a unit segment; we obtain Erdős' famous problem [9] (still wide open) about the maximum number of unit segments that can be determined by a set of n points in the plane.

In general, determining the exact maxima for these problems is hard. The only nontrivial instances where an exact formula is known are due to Brass [5] and van Wamelen [17], who worked with unit segments (congruences) in $\mathbb{R}^4$, and to Swanepoel [16], who very recently accomplished the impressive task of obtaining exact formulas for the maximum number of unit segments determined by n points in $\mathbb{R}^{2d}$, for $d \geq 3$ and for every n large enough. He also found structural properties for the optimal sets in odd dimensions d.

In spite of this, there are very few known asymptotic formulae. In the best case, one hopes to determine the order of magnitude for the corresponding maximum, and even this is only known for few particular patterns and geometric transformations. Elekes and Erdős [8] found the right order of magnitude for algebraic patterns in $\mathbb{R}^d$ under homotheties and in the plane under similarities. For arbitrary patterns, their upper bounds are almost matched by their constructions. Continuing with similarities in the plane, Laczkovich and Rusza [13] classified the patterns that can have a quadratic number of similar copies (the maximum possible), and Ábrego, Elekes, and Fernández-Merchant [1] found structural properties on those point-sets having a quadratic number of occurrences. The problem of similarities when $d \geq 3$ gets harder. Even for the simplest case of triangles in 3-space, we do not know the right order of magnitude.

It is worth mentioning that the algorithmic aspect of these combinatorial questions is motivated by real-life necessities: finding patterns among the huge amounts of data obtained by scanners, digital cameras, electron microscopes, telescopes, etc. The reader is encouraged to see [4, 6, 7], and [15] for further references and the current status of both the combinatorial and the algorithmic problems.

In this paper, we consider the problem of finding the maximum number of translated copies of a pattern that can occur among n points in d-dimensional space. More

$$n = 1716 = \binom{13}{6} + \binom{4}{5} + \binom{3}{4} + \binom{2}{3} + \binom{1}{2} + \binom{0}{1}$$

$$m_6(n) = 792 = \binom{12}{5} + \binom{3}{4} + \binom{2}{3} + \binom{1}{2} + \binom{0}{1} + \binom{-1}{0}$$
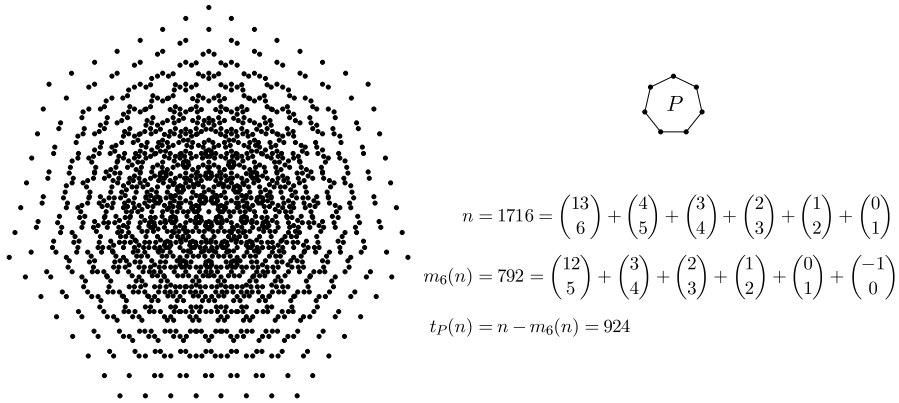
$$t_P(n) = n - m_6(n) = 924$$

Fig. 1 Optimal set with $n = 1716$ points and the maximum number of translates of P, the regular heptagon

precisely, let $P \subseteq \mathbb{R}^d$ be a fixed finite subset of the d-dimensional real space. We refer to P as a *pattern*. For any finite set $X \subseteq \mathbb{R}^d$, let $t_P(X)$ be the number of *translated copies*, or simply *translates*, of P contained in X. That is,

$$t_P(X) = \left| \left\{ P' \subseteq X : P' = v + P \text{ where } v \in \mathbb{R}^d \right\} \right|.$$

We want to determine the largest possible value of $t_P(X)$ among all n-point sets $X \subseteq \mathbb{R}^d$. We denote this by

$$t_P(n) = \max \left\{ t_P(X) : X \subseteq \mathbb{R}^d, |X| = n \right\}.$$

This problem was previously considered by Brass [4]. It was also presented as a prototype, by Pach and Sharir, for the many problems about repeated configurations and pattern recognition treated in [15]. Brass defined the *rational affine dimension*, $\dim_{\mathbb{Q}} P$, of a pattern $P \subseteq \mathbb{R}^d$ as the dimension of the affine space generated by P. This is precisely the *rational linear dimension* of the set generated by $P + (-P) = \{p_1 - p_2 : p_1, p_2 \in P\}$, i.e., the size of the smallest set B such that any element of $P + (-P)$ is a linear combination with rational coefficients of elements of B. Note that the rational linear dimension of a set may be larger than its rational affine dimension, but when P contains the origin, the two dimensions coincide. Brass found the asymptotic value of $t_P(n)$ in terms of the rational affine dimension r of P. He proved that

$$t_P(n) = n - \Theta\left(n^{1-1/r}\right). \tag{1}$$

Our main result gives the *exact value* of $t_P(n)$ when the pattern P is a *rational simplex*, that is, $|P| = \dim_{\mathbb{Q}} P + 1$. We remark that *almost all* patterns P in $\mathbb{R}^d$ *are* rational simplices. If we were to select a pattern at random (with a fixed number of points) from a compact subset of $\mathbb{R}^d$ with positive finite measure, then the pattern would be a rational simplex with probability one. Some interesting patterns are also rational simplices. For instance, a regular polygon with prime number of sides (Fig. 1), a regular pentagonal pyramid (Fig. 2), or the set $\{0, 1, \sqrt{2}, \sqrt{3}, e\}$.
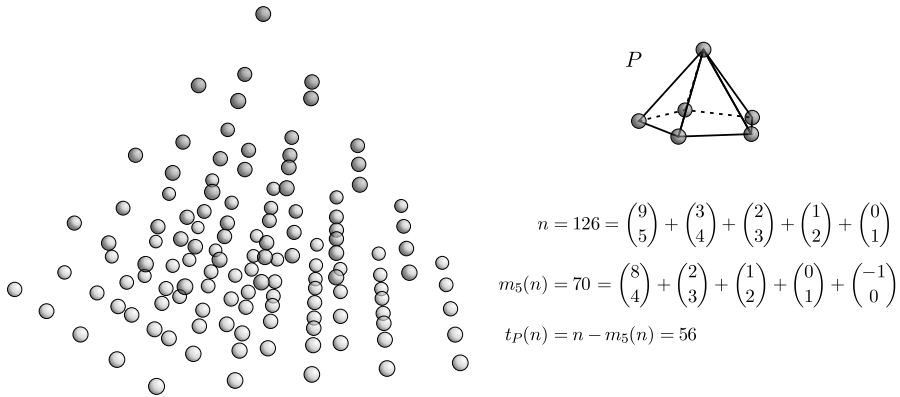
$$n = 126 = \binom{9}{5} + \binom{3}{4} + \binom{2}{3} + \binom{1}{2} + \binom{0}{1}$$

$$m_5(n) = 70 = \binom{8}{4} + \binom{2}{3} + \binom{1}{2} + \binom{0}{1} + \binom{-1}{0}$$

$$t_P(n) = n - m_5(n) = 56$$

Fig. 2 Optimal set with $n = 126$ points and the maximum number of translates of P, the pentagonal pyramid

To present our explicit formula we need some definitions from finite set theory. Given positive integers r and n, there is a unique way of writing n as

$$n = \binom{n_r}{r} + \binom{n_{r-1}}{r-1} + \cdots + \binom{n_2}{2} + \binom{n_1}{1} \tag{2}$$

so that $0 \le n_1 < n_2 < \cdots < n_{r-1} < n_r$ are integers. Using this representation, called the *r-binomial representation* of n, the *r-Kruskal–Macaulay function,* is defined as

$$m_r(n) = \binom{n_r - 1}{r - 1} + \binom{n_{r-1} - 1}{r - 2} + \cdots + \binom{n_2 - 1}{1} + \binom{n_1 - 1}{0}.$$

(See [3, 12] for details.) Our main result, proved in Sect. 2, is the following:

Theorem 1 *If P is a rational simplex with $\dim_{\mathbb{Q}}(P) = r$, then*

$$t_P(n) = n - m_r(n).$$

Since any pattern P with rational affine dimension r contains an r-dimensional rational simplex, $t_P(n) \le n - m_r(n) \le n - \frac{r}{(r!)^{1/r}} n^{1-1/r} + O(n^{1-2/r})$. Brass also used the r-dimensional rational simplex to find an upper bound for $t_P(n)$. Implicit in his proof [4] is the bound $t_P(n) \le n - \sqrt{\pi}/(2\Gamma(r/2 + 1)^{1/r}) n^{1-1/r}$, where Γ is Euler's Gamma Function. Our result improves this inequality and gives a tight lower bound for the leading coefficient of the error term. As a way of comparing, as $r \to \infty$, Brass' coefficient goes to zero, whereas ours goes to e.

We also consider the following problem about maximizing scaled copies of a pattern. Let R be a fixed finite set of positive reals. Again, we refer to R as a (*real*) *pattern*. For any finite set $Y \subseteq \mathbb{R}^+$, let $s_R(Y)$ be the number of *scaled copies* of R contained in Y. That is,

$$s_R(Y) = \left| \left\{ R' \subseteq Y : R' = \lambda R \text{ for some } \lambda \in \mathbb{R}^+ \right\} \right|.$$

We want to determine the value of $s_R(n) = \max\{s_R(Y) : Y \subseteq \mathbb{R}^+, |Y| = n\}$.

The following result shows the equivalence of maximizing the number of scaled copies of a real pattern and maximizing the number of translates of an equivalent pattern. Its proof, included in Sect. 3, provides explicit ways of constructing a pattern P in $\mathbb{R}^d$ from a pattern R of positive reals, and vice versa. Before stating the theorem, we introduce some terminology from abstract algebra. If R is a finite set in $\mathbb{R}^+$, we denote by $\langle R \rangle$ the smallest multiplicative subgroup of $\mathbb{R}^+$ containing R. By construction $\langle R \rangle$ is a *finitely generated free abelian group*. That is, there is a finite subset of $\langle R \rangle$, called a *basis,* which satisfies that every element of the group can be written in one and only one way as a finite product of integer powers of the elements in the basis. In general, a free abelian group has many different bases, but all bases have the same cardinality, and this cardinality is called the *rank* of the group.

Theorem 2 *Let $n \in \mathbb{N}$. For every pattern $R \subseteq \mathbb{R}^+$, there exists a pattern $P \subseteq \mathbb{Z}^d$, where d is the rank of R, such that*

$$s_R(n) = t_P(n).$$

Reciprocally, for every dimension d and every pattern $P \subseteq \mathbb{R}^d$, there is a pattern $R \subseteq \mathbb{N}$ satisfying $s_R(n) = t_P(n)$. Moreover, $|R| = |P|$, and if $1 \in R$, or equivalently if $0 \in P$, then the rank of R and the rational affine dimension of P coincide.

Before proving this theorem, at the beginning of Sect. 3, we present some of its applications to the number of scaled copies of an arithmetic progression. Let $A_k = \{1, 2, 3, \ldots, k\}$; if $k \leq 4$, we determine precisely $s_{A_k}(n)$ for every n. We also show the following:

Theorem 3 *If $k \in \mathbb{N}$, then $s_{A_k}(n) = n - \Theta(n^{1-1/\pi(k)})$, where $\pi(k)$ is the number of primes not exceeding k.*

In Sect. 4, we concentrate on the 1-dimensional (rational affine) patterns. We essentially solve the problem. We give the exact value of $t_P(n)$ for n large enough and any pattern P with $|P| \geq 3$ by showing the following:

Theorem 4 *Let $P = \{0, a_1, a_2, \ldots, a_{k+1}\} \subseteq \mathbb{Z}$ with $k \geq 1$, $0 < a_1 < a_2 < \cdots < a_{k+1} = M$, and such that $\gcd\{a_i\}_{i=1}^{k+1} = 1$. Then*

$$t_P(n) = n - M \quad \text{for every } n \geq M(M - k).$$

We give the exact values of $t_P(n)$ for all n when $|P| \leq 3$: $t_P(n) = n$ for $|P| = 1$ is trivial, $t_P(n) = n - 1$ if $|P| = 2$ is a consequence of Theorem 1, and for $|P| = 3$, we show the following:

Theorem 5 *Let $P = \{0, a, b\} \subseteq \mathbb{Z}$ with $0 < a < b$ and $\gcd(a, b) = 1$. Then*

$$t_P(n) = \begin{cases} n - \left\lceil \sqrt{2n + \frac{1}{4}} - \frac{1}{2} \right\rceil & \text{if } n \leq \binom{b}{2}, \\ n - b & \text{if } n > \binom{b}{2}. \end{cases}$$

Finally, in Sect. 5, we study 2-dimensional (rational affine) patterns with four points. We determine $t_\square(n)$ and $t_L(n)$, where $\square$ represents the pattern in $\mathbb{R}^2$ consisting of the four vertices of a square, and $L = \{(0, 0), (1, 0), (2, 0), (0, 1)\}$. We also present some conjectures about the exact value of $t_P(n)$ for every set P with $|P| = 4$.

In this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{N}^* = \mathbb{N} \cup \{0\}$ the set of nonnegative integers.

2 Proof of Theorem 1: Translates of a Rational Simplex

2.1 Preliminaries and Discretization

Let P be a finite set in $\mathbb{R}^d$. By translating P if necessary, we can assume that $0 \in P$. Let $\langle P \rangle_+$ be the additive subgroup of $\mathbb{R}^d$ generated by P and $\{v_1, v_2, \ldots, v_r\}$ a basis of $\langle P \rangle_+$. Then $\langle P \rangle_+ = \{k_1 v_1 + k_2 v_2 + \cdots + k_r v_r : k_i \in \mathbb{Z}\}$, and, since $0 \in P$, r is the rational affine dimension of P. Note that any translate $v + P$ of P is contained in a single equivalence class $v + \langle P \rangle_+ \in \mathbb{R}^d / \langle P \rangle_+$. The following lemma restricts the maximum for the function $t_P(n)$ to $\langle P \rangle_+$.

Lemma 1 *For any finite set $P \subseteq \mathbb{R}^d$,*

$$t_P(n) = \max\{t_P(X) : X \subseteq \langle P \rangle_+, |X| = n\}.$$

Proof Take $X \subseteq \mathbb{R}^d$ and assume that $X \supseteq X_1 \cup X_2$, where X_1 and X_2 are contained in different classes $u_1 + \langle P \rangle_+$ and $u_2 + \langle P \rangle_+$, respectively. For $k \in \mathbb{Z}$ large enough, the set $X_2' = (u_1 - u_2 + kv_1) + X_2 \subseteq u_1 + \langle P \rangle_+$ has no elements in common with X. Also X_2' has as many translates of P as X_2. Replacing X_2 by X_2' in X, we obtain a set intersecting a fewer number of equivalence classes in $\mathbb{R}^d / \langle P \rangle_+$ and with at least as many translates of P as X. Now, if X is contained in a single class $u_1 + \langle P \rangle_+$, consider the set $(-u_1) + X \subseteq \langle P \rangle_+$. Since this set contains as many translated copies of P as X, the identity is satisfied. $\square$

2.2 Reduction to the Standard Simplex and Construction of $B_r(n)$

Let $P_r \subseteq \mathbb{R}^r$ be the *standard r-simplex* consisting of 0 and the r standard basis vectors $e_1, e_2, \ldots, e_r$. If P is a rational simplex with $\dim_{\mathbb{Q}}(P) = r$, we can assume that $P = \{0, v_1, v_2, \ldots, v_r\}$ by using a suitable translation. Under the linear transformation (from $\langle P \rangle_+$ to $\mathbb{Z}^r$) that sends v_i to e_i, translates of P correspond to translates of P_r. Thus, by Lemma 1, we have that

$$
\begin{aligned}
t_P(n) &= \max\{t_P(X) : X \subseteq \langle P \rangle_+, |X| = n\} \\
&= \max\{t_{P_r}(X) : X \subseteq \mathbb{Z}^r, |X| = n\}. \tag{3}
\end{aligned}
$$

We first construct an n-set $B_r(n) \subseteq \mathbb{Z}^r$ having at least $n - m_r(n)$ translates of P_r. Then by (3), $t_P(n) \geq n - m_r(n)$. We construct $B_r(n)$ recursively as follows. For every $n \in \mathbb{N}$, set $B_1(n) = \{1, 2, \ldots, n\}$. For $r \geq 2$, if $n = \binom{n_r}{r}$ for some $n_r \in \mathbb{Z}$, let

$$B_r\left(\binom{n_r}{r}\right) = \{(x_1, x_2, \ldots, x_r) \in \mathbb{Z}^r : x_i \geq 0, x_r \geq 1,$$
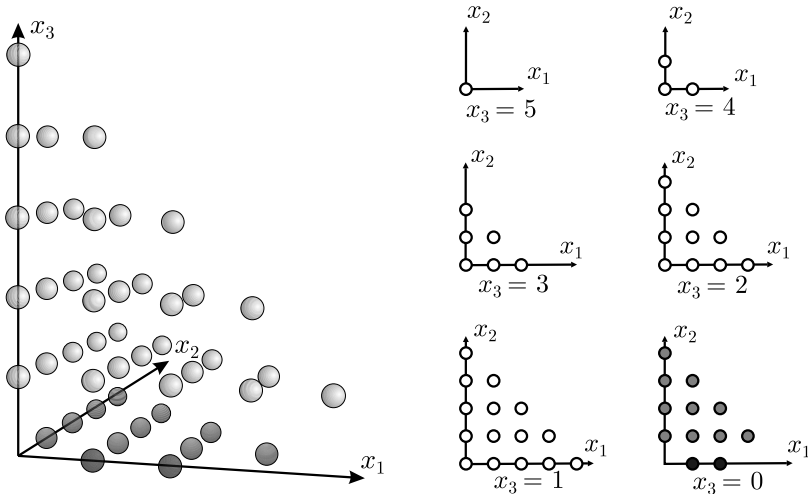$$\text{and } x_1 + x_2 + \cdots + x_r \leq n_r - r + 1\}.$$

Fig. 3 Construction of $B_3(n)$ with $n = \binom{7}{3} + \binom{5}{2} + \binom{2}{1} = 47$ points and $n - m_3(n) = 27$ translates of P_3

Note that this set has exactly $\binom{n_r}{r}$ points. For all other values of n, consider the r-binomial representation $n = \binom{n_r}{r} + \binom{n_{r-1}}{r-1} + \cdots + \binom{n_2}{2} + \binom{n_1}{1}$ and let $n' = n - \binom{n_r}{r} > 0$. Define

$$B_r(n) = B_r\left(\binom{n_r}{r}\right) \cup \left(B_{r-1}(n') * \{0\}\right),$$

where $B_{r-1}(n') * \{0\}$ consists of the points in $B_{r-1}(n')$ with an extra coordinate zero appended to them (see Fig. 3). Observe that $B_r(n)$ has exactly n points and, by construction, all points $x = (x_1, x_2, \ldots, x_r)$ satisfy that $x_1 + x_2 + \cdots + x_r \leq n_r - r + 1$. (This is clear for $x \in B_r(\binom{n_r}{r})$, otherwise $x \in B_{r-1}(n') * \{0\}$ and thus $(x_1 + x_2 + \cdots + x_{r-1}) + x_r \leq n_{r-1} - (r-1) + 1 + 0 \leq n_r - r + 1$ since $n_{r-1} < n_r$.)

2.3 The Number of Translates of the Standard Simplex in $B_r(n)$

We prove that $t_{P_r}(B_r(n)) \geq n - m_r(n)$ by induction on r. If $r = 1$, then $B_r(n) = \{1, 2, \ldots, n\}$, which clearly contains $n - 1$ copies of $P_1 = \{0, 1\}$, that is, $t_{P_1}(B_r(n)) = n - 1 = n - m_1(n)$. Let $r \geq 2$ and assume that the result is true for $r - 1$. We bound the number of points x in $B_r(n)$ such that $x + P_r \subseteq B_r(n)$, observing that different points generate different translates of P_r. We do this first for points in $B_r(\binom{n_r}{r})$ and then for points in $B_{r-1}(n') * \{0\}$.

If $x \in B_r(\binom{n_r}{r})$ and $x_1 + x_2 + \cdots + x_r \leq n_r - r$, then $x + e_i \in B_r(\binom{n_r}{r})$ for $1 \leq i \leq r$. Therefore, $x + P_r \subseteq B_r(\binom{n_r}{r})$, and thus $t_{P_r}(B_r(\binom{n_r}{r})) \geq \binom{n_r}{r} - \binom{n_r - 1}{r - 1}$.

Now, suppose that $x' + P_{r-1}$ is a translate of P_{r-1} in $B_{r-1}(n')$ with $x' = (x_1, x_2, \ldots, x_{r-1})$. Then the point $x' * \{0\} \in B_{r-1}(n') * \{0\} \subseteq B_r(n)$, and we claim that $x' * \{0\} + P_r \subseteq B_r(n)$. Indeed, for $1 \leq i < r$, we have that $x' + e_i \in B_{r-1}(n')$ and then $x' * \{0\} + e_i = (x' + e_i) * \{0\} \in B_{r-1}(n') * \{0\} \subseteq B_r(n)$. In particular, $x_1 + x_2 + \cdots + x_{r-1} + 1 \leq n_r - r + 1$, and thus $x' * \{0\} + e_r = (x_1, x_2, \ldots, x_{r-1}, 1) \in$

$B_r\left(\binom{n_r}{r}\right) \subseteq B_r(n)$. All this gives at least $t_{P_{r-1}}(n')$ translates $x + P_r$ in $B_r(n)$ with $x \in B_{r-1}(n') * \{0\}$. Finally, by induction $t_{P_{r-1}}(n') \geq n' - m_{r-1}(n')$, and thus,

$$t_{P_r}\big(B_r(n)\big) \geq t_{P_r}\big(B_r\big(\tbinom{n_r}{r}\big)\big) + t_{P_{r-1}}(n')$$

$$\geq \binom{n_r}{r} - \binom{n_r - 1}{r - 1} + n' - m_{r-1}(n') = n - m_r(n).$$

2.4 Bounding $t_P(n)$ Above

To prove the reverse inequality, $t_P(n) \leq n - m_r(n)$, we use the following result from [2]. This inequality is tight for every r and infinitely many values of n. In [2] the authors use this inequality to give a short proof of Macaulay's theorem (see [12, 14]).

Lemma 2 *If $r \geq 2$ and $0 \leq a < m_r(n)$, then*

$$m_{r-1}(a) + m_r(n - a) \geq m_r(n). \tag{4}$$

Moreover, if $n = \binom{N}{r}$ for some $N \geq r$, then equality in (4) *occurs only when $a = 0$.*

Let X be an n-point set in $\mathbb{Z}^r$. A point $x \in X$ is called a *deficit point with respect to P_r* if $x + P_r \nsubseteq X$. Define $d_r(X)$ as the number of deficit points with respect to P_r in the set X. Clearly $t_{P_r}(X) = n - d_r(X)$. We prove by induction, on r first and then on n, that $d_r(X) \geq m_r(n)$ for all n-point sets $X \subseteq \mathbb{Z}^r$. This, together with (3), gives $t_P(n) = n - \min\{d_r(X) : X \subseteq \mathbb{Z}^r, |X| = n\} \leq n - m_r(n)$.

If $r = 1$, then clearly the largest point in $X \subseteq \mathbb{Z}$ is a deficit point, thus $d_1(X) \geq 1 = m_1(n)$. Suppose that $r \geq 2$ and that the inequality holds for $r - 1$. If $n \leq r$, then the result is trivially true since $d_r(X) = n = m_r(n)$. Assume that $n \geq r + 1$ and that the result holds for values smaller than n. By using an appropriate translation, we can assume that the r-coordinate of every point in X is greater than zero and that there are some points with r-coordinate equal to one. For every $i \geq 1$, let L_i be the points in X with r-coordinate equal to i. Let $l_i = |L_i|$ and note that $l_1 \geq 1$.

First assume that $l_1 \geq m_r(n)$. Observe that every non-deficit point x in L_i is bijectively associated to the point $x + e_r \in L_{i+1}$. Then there are at least $l_i - l_{i+1}$ deficit points in L_i. Assuming that u is the largest subindex such that $l_u \neq 0$, we get $d_r(X) \geq \sum_{i=1}^{u}(l_i - l_{i+1}) = l_1 \geq m_r(n)$.

On the other hand, if $l_1 < m_r(n)$, then, by Lemma 2, inequality (4) holds with $a = l_1$. This time note that every deficit point with respect to P_{r-1} in L_i is also a deficit point with respect to P_r in X, which means that $d_r(L_i) \geq d_{r-1}(L_i)$. Thus $d_r(X) \geq d_{r-1}(L_1) + d_r(X \backslash L_1)$. By induction on r, $d_{r-1}(L_1) \geq m_{r-1}(l_1)$, and by induction on n, $d_r(X \backslash L_1) \geq m_r(n - l_1)$. These, together with (4), give $d_r(X) \geq m_r(n)$.

Remark 1 The problem of classifying the n-sets X for which $t_{P_r}(X) = t_{P_r}(n)$ is open. The first difficulty would be to determine the pairs (a, n) for which we get identity in (4). However, for n of a particular form, we can determine uniqueness for the optimal sets constructed in Sect. 2.2 (in particular the sets shown in Figs. 1 and 2 are unique).

Proposition 1 *If* $n = \binom{N}{r}$, *then the set* $X = B_r(n)$ *is the unique* n-*set in* $\mathbb{Z}^r$ *(modulo translations) for which* $t_{P_r}(X) = t_{P_r}(n)$.

Proof According to the proof of Theorem 1, $d_r(X) = m_r(n)$. If $l_1 < m_r(n) = \binom{N-1}{r-1}$, then $d_r(X) \geq d_{r-1}(L_1) + d_r(X \backslash L_1) \geq m_{r-1}(l_1) + m_r(n - l_1)$. Because $l_1 > 0$, it follows, by Lemma 2, that (4) is strict. Thus,

$$m_r(n) = d_r(X) \geq m_{r-1}(l_1) + m_r(n - l_1) > m_r(n),$$

which is impossible. For the other case, we must have $l_1 \geq m_r(n)$. Then $m_r(n) = d_r(X) \geq l_1 \geq m_r(n)$, i.e., $l_1 = m_r(n) = \binom{N-1}{r-1}$. Further,

$$m_r(n) = d_r(X) \geq d_{r-1}(L_1) + d_r(X \backslash L_1) \geq m_{r-1}(l_1) + m_r(n - l_1)$$

$$\geq m_{r-1}\left(\binom{N-1}{r-1}\right) + m_r\left(\binom{N-1}{r}\right) = \binom{N-2}{r-2} + \binom{N-2}{r-1} = \binom{N-1}{r-1} = m_r(n).$$

Thus equality must occur everywhere. Then L_1 is an optimal set for P_{r-1} with $\binom{N-1}{r-1}$ points, and $X \backslash L_1$ is an optimal set for P_r with $\binom{N-1}{r}$ points. By induction, L_1 and $X \backslash L_1$ are translates of $B_{r-1}(l_1)$ and $B_r(n - l_1)$, the optimal sets constructed in Sect. 2.2. Moreover, to have equality in $d_r(X) \geq d_{r-1}(L_1) + d_r(X \backslash L_1)$, we need that every non-deficit point in L_1 is exactly below, in the x_r coordinate, a point from $X \backslash L_1$. This can only happen if L_1 is the bottom layer (in the x_r coordinate) of $B_r(n)$. □

3 Equivalence Between Translates and Scaled Copies

3.1 Applications of the Equivalence

Before we present the proof of Theorem 2, we show some of its applications. In particular, we bound the maximum number of scaled copies of an arithmetic progression. Recall that $A_k = \{1, 2, 3, \ldots, k\}$. Clearly $s_{A_1}(n) = n$. Under the equivalence given by Theorem 2, the real patterns A_2 and A_3 correspond to the patterns $P_1 = \{0, 1\}$ and $P_2 = \{(0, 0), (1, 0), (0, 1)\}$, respectively. Thus, by Theorem 1, $s_{A_2}(n) = t_{P_1}(n) = n - m_1(n) = n - 1$ and $s_{A_3}(n) = t_{P_2}(n) = n - m_2(n) = n - \lceil \sqrt{2n + \frac{1}{4}} - \frac{1}{2} \rceil$ (the last identity is an elementary calculation). Similarly, A_4 corresponds to the pattern $L = \{(0, 0), (1, 0), (2, 0), (0, 1)\}$, and thus $s_{A_4}(n) = t_L(n) = n - \lceil 2\sqrt{n} - 1 \rceil$ by Theorem 6 (see Sect. 5). For larger arithmetic progressions, we do not have exact values of $s_{A_k}(n)$. Yet, we know its asymptotic behavior in terms of the rank of $\langle A_k \rangle$ as a multiplicative subgroup of $\mathbb{R}^+$. For example, if $k = 5$, then A_5 has rank 3 (it actually corresponds to the pattern $\{(0, 0, 0), (1, 0, 0), (2, 0, 0), (0, 1, 0), (0, 0, 1)\}$). Then by (1), $s_{A_5}(n) = n - \Theta(n^{2/3})$. In general, using the Fundamental Theorem of Arithmetic, any element of A_k can be multiplicatively generated by the primes within A_k, and all these primes are necessary to generate A_k. That is, the rank of A_k is equal to $\pi(k)$, the number of primes that are less than or equal to k. Thus by (1), $s_{A_k}(n) = n - \Theta(n^{1-1/\pi(k)})$, proving Theorem 3.

The values of $s_R(n)$ can be determined for other patterns. For example, if $R = \{1, 2, 3, 6\}$, then R corresponds, under Theorem 2, to $\square = \{(0, 0), (1, 0), (1, 1), (0, 1)\}$. By Theorem 6, $s_R(n) = t_\square(n) = n - \lceil 2\sqrt{n} - 1 \rceil$. Other patterns like $R = \{1, \sqrt{2}, \pi\}$ correspond to the same pattern P_2 as A_3 (above), and thus $s_R(n) = n - \lceil \sqrt{2n + \frac{1}{4}} - \frac{1}{2} \rceil$.

3.2 Proof of Theorem 2

Let R be a finite set in $\mathbb{R}^+$. By dilating R, we can assume that $1 \in R$. Recall that $\langle R \rangle$ is the multiplicative subgroup of $\mathbb{R}^+$ generated by R. Let $\{x_1, x_2, \ldots, x_q\}$ be a basis of $\langle R \rangle$. Then q is the rank of $\langle R \rangle$, and $\langle R \rangle = \{x_1^{k_1} x_2^{k_2} \cdots x_q^{k_q} : k_i \in \mathbb{Z}\}$. Since any scaled copy λR of R is contained in a single equivalence class $\lambda \langle R \rangle \in \mathbb{R}^+/\langle R \rangle$, we can restrict the maximum for the function $s_R(n)$ to $\langle R \rangle$. The following lemma is the multiplicative analogue of Lemma 1.

Lemma 3 *For any finite set $R \subseteq \mathbb{R}^+$,*

$$s_R(n) = \max\{s_R(Y) : Y \subseteq \langle R \rangle, |Y| = n\}.$$

Proof Take $Y \subseteq \mathbb{R}^+$ and assume that $Y \supseteq Y_1 \cup Y_2$, where Y_1 and Y_2 are contained in the different classes $\lambda_1 \langle R \rangle$ and $\lambda_2 \langle R \rangle$, respectively. For $k \in \mathbb{Z}$ large enough, the set $Y_2' = (\lambda_1 \lambda_2^{-1} x_1^k) Y_2 \subseteq \lambda_1 \langle R \rangle$ has no elements in common with Y, and Y_2' has as many scaled copies of R as Y_2. Replacing Y_2 by Y_2' in Y gives a set intersecting a fewer number of equivalence classes in $\mathbb{R}^+/\langle R \rangle$ and with at least as many scaled copies of R as Y. Now, if Y is contained in a single class $\lambda_1 \langle R \rangle \in \mathbb{R}^+/\langle R \rangle$, the set $(\lambda_1^{-1}) Y \subseteq \langle R \rangle$ contains as many scaled copies of R as Y, proving the identity. $\square$

The function $G : \langle R \rangle \to \mathbb{Z}^q$ given by $G(x_1^{k_1} x_2^{k_2} \cdots x_q^{k_q}) = (k_1, k_2, \ldots, k_q)$ is a group isomorphism. Thus $G(\lambda R) = G(\lambda) + G(R)$ for any $\lambda \in \langle R \rangle$, and $v + G(R) = G(G^{-1}(v)R)$ for any $v \in \mathbb{Z}^q$. That is, scaled copies of R in $\langle R \rangle \subseteq \mathbb{R}^+$ uniquely correspond to translates of $G(R)$ in $\mathbb{Z}^q \subseteq \mathbb{R}^q$ under G. Hence,

$$Y \subseteq \langle R \rangle \quad \text{implies} \quad |Y| = \big|G(Y)\big| \quad \text{and} \quad s_R(Y) = t_{G(R)}\big(G(Y)\big). \tag{5}$$

Let P be a finite set in $\mathbb{R}^d$ containing 0 and with rational affine dimension r. Assume that $\{v_1, v_2, \ldots, v_r\}$ is a basis of $\langle P \rangle_+$ and consider any r distinct primes $p_1, p_2, \ldots, p_r$. Then $\langle p_1, p_2, \ldots, p_r \rangle = \{p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} : k_i \in \mathbb{Z}\}$ is a multiplicative subgroup of $\mathbb{R}^+$, and the function $F : \langle P \rangle_+ \to \langle p_1, p_2, \ldots, p_r \rangle$ given by $F(k_1 v_1 + k_2 v_2 + \cdots + k_r v_r) = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ is a group isomorphism. Thus $F(v + P) = F(v)F(P)$ for any $v \in \langle P \rangle_+$, and $\lambda F(P) = F(F^{-1}(\lambda) + P)$ for any $\lambda \in \langle p_1, p_2, \ldots, p_r \rangle$. That is, translates of P in $\langle P \rangle_+ \subseteq \mathbb{R}^d$ uniquely correspond to scaled copies of $F(P)$ in $\langle p_1, p_2, \ldots, p_r \rangle \subseteq \mathbb{R}^+$. Hence,

$$X \subseteq \langle P \rangle_+ \quad \text{implies} \quad |X| = \big|F(X)\big| \quad \text{and} \quad t_P(X) = s_{F(P)}\big(F(X)\big). \tag{6}$$

By Lemma 3, (5), and Lemma 1 we have

$$
\begin{aligned}
s_R(n) &= \max\{s_R(Y) : Y \subseteq \langle R \rangle, |Y| = n\} \\
&= \max\{t_{G(R)}(G(Y)) : Y \subseteq \langle R \rangle, |Y| = n\} \\
&= \max\{t_{G(R)}(X) : X \subseteq G(\langle R \rangle), |X| = n\} = t_{G(R)}(n).
\end{aligned}
$$

To obtain the last identity, observe that $\langle G(R) \rangle_+ \subseteq G(\langle R \rangle) \subseteq \mathbb{R}^q$. Lemma 1 guarantees that the maximum of $t_{G(R)}(X)$ over all X in a subset of $\mathbb{R}^q$ containing $\langle G(R) \rangle_+$ is exactly $t_{G(R)}(n)$. Moreover, since G is an isomorphism and $1 \in R$, $G(1) = 0 \in G(R)$ and $q = \dim_\mathbb{Q} G(R)$.

Similarly, by Lemma 1, (6), and Lemma 3, together with $\langle F(P) \rangle \subseteq F(\langle P \rangle_+) \subseteq \mathbb{R}^+$, we have that

$$
\begin{aligned}
t_P(n) &= \max\{t_P(X) : X \subseteq \langle P \rangle_+, |X| = n\} \\
&= \max\{s_{F(P)}(F(X)) : X \subseteq \langle P \rangle_+, |X| = n\} \\
&= \max\{s_{F(P)}(Y) : Y \subseteq F(\langle P \rangle_+), |Y| = n\} = s_{F(P)}(n).
\end{aligned}
$$

Since F is an isomorphism and $0 \in P$, we have that $F(0) = 1 \in F(P)$ and r is the rank of P. Finally, observe that for a suitable $\lambda > 0$, we can guarantee that $\lambda F(P) \subseteq \mathbb{N}$, so we may assume that $F(P) \subseteq \mathbb{N}$.

4 One-Dimensional Translated Patterns

Let $P \subseteq \mathbb{R}^d$ be a pattern with $\dim_\mathbb{Q}(P) = 1$. Then all members of P are rational multiples of a vector $z \in \mathbb{R}^d$. Thus, using a suitable affine transformation, we can assume that $P = \{0, a_1, a_2, \ldots, a_k, a_{k+1}\} \subseteq \mathbb{Z}$ with $0 < a_1 < a_2 < \cdots < a_k < a_{k+1} = M$ and $\gcd\{a_i\}_{i=1}^{k+1} = 1$. Clearly, if $|P| = 1$, then $t_P(n) = n$. If $|P| = 2$, then we can assume that $P = \{0, 1\}$. By Theorem 1, $t_P(n) = n - m_1(n) = n - 1$. Theorem 4 shows that $t_P(n) = n - M$ for $|P| \geq 3$ and n large enough, namely $n \geq M(M - k)$.

4.1 Proof of Theorem 4

Let $A_n = \{1, 2, 3, \ldots, n\}$. If $n \geq M$, clearly $x + P \subseteq A_n$ if and only if $1 \leq x \leq n - M$. Thus, $t_P(n) \geq t_P(A_n) = n - M$.

Now we prove that $t_P(n) \leq n - M$ for any n-set X and $n \geq M(M - k)$. We study the subset of X consisting of the first points of translates of P. Specifically, let $X_0 = \{x \in X : x + P \subseteq X\}$. Clearly $|X_0| = t_P(X)$. We consider the residue classes of X_0 modulo M. For $1 \leq i \leq M$, let $B_i = \{x \in X_0 : x + i \equiv 0 (\mathrm{mod}\, M)\}$, $C_i = \{x \in X \setminus X_0 : x + i \equiv 0 (\mathrm{mod}\, M)\}$, and set $x_i = |B_i|$, $y_i = |C_i|$. Of course,

$$
\sum_{i=1}^{M} (x_i + y_i) = n \quad \text{and} \quad \sum_{i=1}^{M} x_i = t_P(X). \tag{7}
$$

Now observe that if $B_i \neq \varnothing$ and $m_i = \max B_i$, then $m_i + M \in C_i \neq \varnothing$, i.e.,

$$y_i = 0 \quad \text{implies} \quad x_i = 0. \tag{8}$$

By definition, if $x \in B_i \subseteq X_0$, then $x + a_j + (i - a_j) \equiv x + i \equiv 0 \pmod{M}$ and $x + a_j \in X$. Thus, $B_i + a_j \subseteq B_{i-a_j} \cup C_{i-a_j}$, that is, $x_i \leq x_{i-a_j} + y_{i-a_j}$. Replacing i by $i + a_j$, we get

$$x_{i+a_j} - x_i \leq y_i \quad \text{for } 0 \leq i \leq M - 1 \text{ and } 1 \leq j \leq k, \tag{9}$$

where the indices are taken $\mathrm{mod}\, M$.

Let $I = \{i : y_i = 0\}$. If $I = \varnothing$, then $t_P(X) = \sum_{i=1}^{M} x_i = n - \sum_{i=1}^{M} y_i \leq n - M$ by (7). If $I \neq \varnothing$, we may assume, using a suitable translation of X, that $y_M = 0$. Then by (8) $x_M = x_0 = 0$ and by (9) $x_{a_j} = 0$ for all j. Let $A = (a_1, a_2, \ldots, a_k)$; for every $N = (n_1, n_2, \ldots, n_k) \in (\mathbb{N}^*)^k$, we define the dot product $N \cdot A = \sum_{j=1}^{k} n_j a_j$ and the norm $|N| = \sum_{j=1}^{k} n_j$. We also define a partial order on the vectors in $(\mathbb{N}^*)^k$: If $N, N' \in (\mathbb{N}^*)^k$, we say that $N' \prec N$ if $n'_i \leq n_i$ for $1 \leq i \leq k$ and $N' \neq N$. We claim the following:

Claim *For every $1 \leq u < M$, there is a vector $N = (n_1, n_2, \ldots, n_k) \in (\mathbb{N}^*)^k$ such that $N \cdot A \equiv u \pmod{M}$ and $N_1 \cdot A \not\equiv N_2 \cdot A \pmod{M}$ for every $N_1 \prec N_2 \preceq N$.*

Proof First, since $\gcd\{a_i\}_{i=1}^{k+1} = 1$, the set of vectors $N \in (\mathbb{N}^*)^k$ such that $N \cdot A \equiv u \pmod{M}$ is nonempty. Suppose that $N \in (\mathbb{N}^*)^k$ satisfies $N \cdot A \equiv u \pmod{M}$ and minimizes $|N|$. Clearly, if $N_1 \prec N$, then $|N_1| < |N|$ and consequently $N_1 \cdot A \not\equiv u \pmod{M}$. This proves the result when $N_2 = N$. Similarly, if $0 \prec N' \prec N$, then $N' \cdot A \not\equiv 0 \pmod{M}$. Otherwise $|N - N'| < |N|$ and $(N - N') \cdot A \equiv u \pmod{M}$. Suppose that $N_1 \prec N_2 \preceq N$ and $N_1 \cdot A \equiv N_2 \cdot A \pmod{M}$. Then $0 \prec N_2 - N_1 \prec N$ and $(N_2 - N_1) \cdot A \equiv 0 \pmod{M}$, which contradicts the observation above. This proves the claim. $\qquad \square$

Let N be a vector given by the claim. For every $1 \leq t \leq |N|$, we define $v(t) \in (\mathbb{N}^*)^k$ as

$$v(t) = (n_1, n_2, \ldots, n_{q_t}, r_t, 0, \ldots, 0),$$

where $0 \leq q_t \leq k - 1$ and $1 \leq r_t \leq n_{q_t+1}$ are the unique integers such that $|v(t)| = n_1 + \cdots + n_{q_t} + r_t = t$. By construction $0 \prec v(1) \prec v(2) \prec \cdots \prec v(|N|) = N$, and by the claim all residues $\mathrm{mod}\, M$ of $v(t) \cdot A$, $1 \leq t \leq |N|$ are pairwise different. By (9) we have for $1 \leq t \leq |N| - 1$,

$$x_{v(t+1) \cdot A} - x_{v(t) \cdot A} \leq y_{v(t) \cdot A}.$$

Adding all these inequalities, since $x_{a_1} = 0$, we obtain

$$x_u = x_u - x_{a_1} = x_{N \cdot A} - x_{v(1) \cdot A} \leq \sum_{t=1}^{|N|-1} y_{v(t) \cdot A} \leq \sum_{j=0}^{M-1} y_j = n - t_P(X).$$

The last inequality follows from the fact that all the indices $v(t) \cdot A$, $1 \leq t \leq |N| - 1$ are pairwise different mod M. If we now add all these inequalities for every u except $u = 0, a_1, \ldots, a_k$ (x_u is equal to zero for these values), using (7), we get

$$t_P(X) = \sum_{u=0}^{M-1} x_u \leq (M - k - 1)(n - t_P(X)).$$

Finally, since $n \geq M(M - k)$, we have $t_P(X) \leq n - \frac{n}{M-k} \leq n - M$.

4.2 Proof of Theorem 5

Theorem 4 only works for n large enough. Theorem 5 gives the exact value of $t_P(n)$ for all n when $|P| = 3$. In fact, Theorem 4 is implied (and improved) by Theorem 5 when M is relatively prime to some a_i, $1 \leq i \leq k$.

Let $P = \{0, a, b\} \subseteq \mathbb{Z}$ with $0 < a < b$ and $\gcd(a, b) = 1$. For $n \leq \binom{b}{2}$, let m and p be the unique positive integers such that $n = \binom{m}{2} + p$ with $1 \leq p \leq m$. A simple calculation shows that $m = \lceil \sqrt{2n + \frac{1}{4}} - \frac{1}{2} \rceil$. Note that $1 \leq m \leq b - 1$. We want to prove

$$t_P(n) = \begin{cases} n - m & \text{if } n \leq \binom{b}{2}, \\ n - b & \text{if } n > \binom{b}{2}. \end{cases} \tag{10}$$

If $n \geq \binom{b}{2} + 1$, then $t_P(n) \geq t_P(A_n) \geq n - b$, as shown in the proof of Theorem 4. If $n \leq \binom{b}{2}$, we construct the n-set

$$X_n = \bigcup_{j=0}^{m-1} \left\{ ai + bj : 0 \leq i \leq m - j - 2 + \varepsilon_j \text{ and } \varepsilon_j = \begin{cases} 1 \text{ if } j \leq p - 1 \\ 0 \text{ if } j \geq p \end{cases} \right\}$$

such that $t_P(X_n) = n - m$.

To check that X_n has in fact n elements, we show that the sets in the union above are pairwise disjoint, and thus $|X_n| = \sum_{j=0}^{m-1} (m - j - 1) + \sum_{j=0}^{p-1} 1 = \binom{m}{2} + p = n$. Indeed, if $ai_1 + bj_1 = ai_2 + bj_2$, then $a(i_1 - i_2) = b(j_1 - j_2)$, and since $\gcd(a, b) = 1$, $i_1 \equiv i_2 \pmod{b}$. But $0 \leq i_1, i_2 < m \leq b - 1$, and thus $i_1 = i_2$ and $j_1 = j_2$.

To count the number of translates in X_n, define $X_n' = \{x \in X_n : x + P \subseteq X_n\}$. By definition, $t_P(X_n) = |X_n'|$. We claim that

$$X_n' \supseteq \bigcup_{j=0}^{m-2} \{ai + bj : 0 \leq i \leq m - j - 3 + \varepsilon_{j+1}\}.$$

Indeed, suppose that $x = ai + bj$ with $0 \leq j \leq m - 2$ and $0 \leq i \leq m - j - 3 + \varepsilon_{j+1}$. Note that $x \in X_n$. Since $\varepsilon_{j+1} = 1$ implies $\varepsilon_j = 1$, we have $x + a = a(i + 1) + bj \in X_n$ and thus $1 \leq i + 1 \leq m - j - 2 + \varepsilon_{j+1} \leq m - j - 2 + \varepsilon_j$. Similarly, $x + b = ai + b(j + 1) \in X_n$, since $0 \leq i \leq m - (j + 1) - 2 + \varepsilon_{j+1}$. Finally,

$$t_P(X_n) = |X_n'| \geq \sum_{j=0}^{m-2} (m - j - 2) + \sum_{j=0}^{p-2} 1 = \binom{m-1}{2} + p - 1$$

$$= \binom{m-1}{2} + n - \binom{m}{2} - 1 = n - m.$$

Now, we argue that the right-hand side of (10) bounds $t_P(n)$ above. We follow the proof and notation of Theorem 4 with $k = 1$, $a_1 = a$, and $a_2 = M = b$. But B_i and C_i are somewhat different: $B_i = \{x \in X_0 : x + ia \equiv 0 (\text{mod } b)\}$ and $C_i = \{x \in X \backslash X_0 : x + ia \equiv 0 (\text{mod } b)\}$ for $0 \leq i \leq b - 1$. Again, let $x_i = |B_i|$, $y_i = |C_i|$, and $I = \{i : y_i = 0\}$. In this case, (7) and (8) still hold, and (9) becomes

$$x_i - x_{i-1} \leq y_{i-1}, \tag{11}$$

where all indices of x and y are taken mod b. Again, if $I = \varnothing$, then $t_P(X) \leq n - b \leq n - m$. The same occurs if $s = \sum_{i=1}^{b} y_i \geq b$. From this point on, the proofs of Theorems 4 and 5 diverge.

Suppose now that $I \neq \varnothing$, $s < b$, and $\{x_i\}$, $\{y_i\}$ are nonnegative integers maximizing $\sum_{i=1}^{b} x_i$ subject to the restrictions in (7), (8), and (11). If $y_i = 0$, then by (8) $x_i = 0$ and then by (11) $x_{i+1} = 0$. Suppose that $1 \leq z(1) < z(2) < \cdots < z(k)$ are the members of I. By a suitable translation of X, we may assume that $z(1) = 1$. Note that $1 \leq k < b \leq s$. Based on our previous observation and (11), we have for $1 \leq j \leq k$,

$$x_{z(j)} = x_{z(j)+1} = 0 = y_{z(j)},$$

$$x_{z(j)+2} \leq y_{z(j)+1},$$

$$x_{z(j)+3} \leq y_{z(j)+1} + y_{z(j)+2},$$

$$\vdots$$

$$x_{z(j+1)-1} \leq y_{z(j)+1} + y_{z(j)+2} + \cdots + y_{z(j+1)-2}.$$

Thus,

$$\sum_{i=0}^{b-1} x_i \leq \sum_{j=1}^{k} \sum_{u=0}^{z(j+1)-z(j)-1} y_{z(j)+u} \big(z(j+1) - z(j) - 1 - u\big). \tag{12}$$

(For convenience, we added zero-terms to the second sum when $u = 0$ or $u = z(j+1) - z(j) + 1$. Also $z(k+1) = b + 1$ by convention.) The following lemma allows us to bound the right-hand side of (12). The proof of the lemma consists of solving a simple linear programming problem. We defer the proof of this lemma to Sect. 4.3 and continue with the proof of the theorem.

Lemma 4 *Let $b, k, s \in \mathbb{Z}^+$ be such that $1 \leq k < b \leq s$. Consider the function*

$$G(y_1, y_2, \ldots, y_b) = \sum_{j=1}^{k} \sum_{u=0}^{z(j+1)-z(j)-1} y_{z(j)+u} \big(z(j+1) - z(j) - 1 - u\big), \tag{13}$$

where $y_i \in \mathbb{N}^*$, $I = \{i : y_i = 0\} = \{z(1), z(2), \ldots, z(k)\}$ with $1 = z(1) < z(2) < \cdots < z(k)$ ($z(k+1) = b + 1$ by convention), and $s = \sum_{i=1}^{b} y_i$.

G is maximized (uniquely except for symmetries mod b) by $y_1 = y_2 = \cdots = y_k = 0$, $y_{k+1} = s - (b - k - 1)$, and $y_{k+2} = y_{k+3} = \cdots = y_b = 1$.

By Lemma 4, we can assume that $y_1 = y_2 = \cdots = y_k = 0$, $y_{k+1} = s - (b - k - 1)$, and $y_{k+2} = y_{k+3} = \cdots = y_b = 1$. Then,

$$t_P(X) = \sum_{i=0}^{b-1} x_i \leq \sum_{u=1}^{b-k-1} y_{k+u}(b - k - u)$$

$$\leq \left(s - (b - k - 1)\right)(b - k - 1) + \binom{b-k-1}{2} = s(b - k - 1) - \binom{b-k}{2}.$$

Now recall that $s = n - t_P(X)$. Thus,

$$t_P(X) \leq n - \left(\frac{n}{b - k} + \frac{b - k - 1}{2}\right).$$

Let $l = m$ if $n \leq \binom{b}{2}$ and $l = b$ otherwise. Note that $n > \binom{l}{2}$. Since $k \in \mathbb{Z}$, we have

$$(b - k)(l - 1) - \binom{b-k}{2} = \frac{1}{2}\left(l - \frac{1}{2}\right)^2 - \frac{1}{2}\left(b - k - \left(l - \frac{1}{2}\right)\right)^2$$

$$\leq \frac{1}{2}\left(\left(l - \frac{1}{2}\right)^2 - \frac{1}{4}\right) = \binom{l}{2}.$$

Divide by $b - k$ to get $l - 1 \leq \binom{l}{2}/(b - k) + (b - k - 1)/2$, and thus

$$t_P(X) < n - \left(\frac{\binom{l}{2}}{b - k} + \frac{b - k - 1}{2}\right) \leq n - (l - 1),$$

that is, $t_P(X) \leq n - l$, and this completes the proof of the theorem.

4.3 Proof of Lemma 4

We first establish the following:

Claim *Let $l \geq 1$ be an integer and $z > l$ a real number. The expression $x_2 + 2x_3 + 3x_4 + \cdots + lx_{l+1}$ subject to the conditions $x_i \geq 1$ and $x_1 + x_2 + \cdots + x_{l+1} = z$ is maximized uniquely by $x_1 = x_2 = \cdots = x_l = 1$ and $x_{l+1} = z - l$.*

Proof If $x_i > 1$ for some $i \leq l$, then let $x'_{l+1} = x_{l+1} + 1$, $x'_i = x_i - 1$, and $x'_j = x_j$ for all other indices. Then $(x'_2 + 2x'_3 + 3x'_4 + \cdots + lx'_{l+1}) - (x_2 + 2x_3 + 3x_4 + \cdots + lx_{l+1}) = l(x_{l+1} + 1) + (i - 1)(x_i - 1) - lx_{l+1} - (i - 1)x_i = l - i + 1 \geq 1$, which proves the claim. ∎

Suppose y maximizes $G(y)$. By using a suitable rotation $\bmod\, b$ we can assume that $y_b \neq 0$. Let $s_j = \sum_{i=z(j)}^{z(j+1)} y_i$. If $z(j + 1) - z(j) \geq 2$, then, by the last claim, the jth sum in (13) is maximized uniquely when $y_{z(j)+1} = s_j - (z(j + 1) - z(j) - 2)$ and $y_{z(j)+2} = y_{z(j)+3} = \cdots = y_{z(j+1)-1} = 1$. For the sake of brevity, let $d_j = z(j + 1) - z(j)$. Then the jth sum in (13) equals

$$(s_j - d_j + 2)(d_j - 2) + \binom{d_j-2}{2} = \left(s_j(d_j - 2) - \binom{d_j-1}{2}\right).$$

If $z(j+1) - z(j) = 1$, then the jth sum in (13) is zero. Note that this agrees with the right-hand side of the previous equality. Thus we may assume that

$$G(y) = \sum_{j=1}^{k} \left(s_j(d_j - 2) - \binom{d_j - 1}{2} \right).$$

Now suppose that $s_j > 0$ for some $j < k$. Note that $s_j \geq z(j+1) - z(j) - 1 = d_j - 1 \geq 1$. We will show that this is not the case where y maximizes $G(y)$ by "moving" the $(j+1)$th zero next to the jth zero. Let $y'_{z(j)+1} = 0$, $y'_{z(j)+2} = s(j) + s(j+1) - (z(j+2) - z(j) - 3)$ and $y'_{z(j)+3} = y'_{z(j)+4} \cdots = y'_{z(j+2)-1} = 1$. All other coordinates of y' are the same as those of y. Then $s'_j = 0$, $s'_{j+1} = s_j + s_{j+1}$, and $s'_i = s_i$ for all $i \neq j, j+1$. Also $z'(j+1) = z(j) + 1$ and $z'(i) = z(i)$ for all $i \neq j+1$, i.e., $d'_j = 1$, $d'_{j+1} = d_{j+1} + d_j - 1$, and $d'_i = d_i$ for all $i \neq j, j+1$. Thus,

$$
\begin{aligned}
G(y') - G(y) &= s'_{j+1}(d'_{j+1} - 2) - \binom{d'_{j+1} - 1}{2} \\
&\quad - \left(s_j(d_j - 2) - \binom{d_j - 1}{2} + s_{j+1}(d_{j+1} - 2) - \binom{d_{j+1} - 1}{2} \right) \\
&= (s_j + s_{j+1})(d_{j+1} + d_j - 3) - \binom{d_{j+1} + d_j - 2}{2} \\
&\quad + \binom{d_{j+1} - 1}{2} + \binom{d_j - 1}{2} - s_j(d_j - 2) - s_{j+1}(d_{j+1} - 2) \\
&= (d_{j+1} - 1)\left(s_j - (d_j - 1) \right) + s_{j+1}(d_j - 1) \\
&\geq s_{j+1}(d_j - 1) \geq s_{j+1} \geq 0.
\end{aligned}
$$

Moreover, unless all the k zeros are the first k integers, repeating this operation will eventually yield a strict inequality (one with $s_{j+1} > 0$) because $y_b \neq 0$. This completes the proof of the lemma.

5 Four-Point Sets with Rational Dimension Two

If P is a 3-point pattern, then an exact formula for $t_P(n)$ follows from Theorem 1 if $\dim_{\mathbb{Q}} P = 2$ or from Theorem 5 if $\dim_{\mathbb{Q}} P = 1$. For sets P with 4 points, we also have exact formulas for $t_P(n)$ when P has rational dimension 3 or 1 (at least for n large enough). The only remaining case is when $\dim_{\mathbb{Q}} P = 2$. In what follows, we show formulas for two particular patterns and present some conjectures for the general situation.

5.1 The Square and the L

First we present exact results for two important patterns. In particular the pattern L below is equivalent, under Theorem 2, to the four-term arithmetic progression A_4. Thus, $s_{A_4}(n) = t_L(n)$.
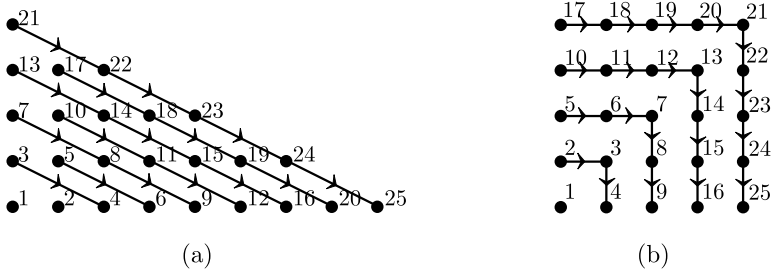
Fig. 4 Constructions of optimal sets X_{25} and Y_{25}

Theorem 6 *Let* $L = \{(0,0), (1,0), (2,0), (0,1)\}$ *and* $\square = \{(0,0), (1,0), (0,1), (1,1)\}$. *Then for every* $n \in \mathbb{N}$, $t_L(n) = t_\square(n) = n - \lceil 2\sqrt{n} - 1 \rceil$.

Proof For the constructive lower bounds, see Fig. 4(a) for the set L and Fig. 4(b) for the set $\square$. For both L and $\square$, the respective optimal sets X_n and Y_n consist of the points with labels 1 to n. The labels are assigned by following the pattern shown in the figure. It can be verified that $t_L(X_n) = t_\square(Y_n) = n - \lceil 2\sqrt{n} - 1 \rceil$. We omit the details.

We now prove both upper bounds. The proof follows the ideas and notation of Theorem 1. Let $P \in \{L, \square\}$ and suppose that X is an n-set with $t_P(X) = t_P(n)$. As in the proof of Theorem 1, we can assume that $X \subseteq \mathbb{Z}^2$, all y-coordinates of X are positive, and some are equal to one. We also use the notation $d_P(X)$ to denote the number of deficit points of X with respect to P, L_i to denote the points in X with y-coordinate equal to i, and $|L_i| = l_i$. Note that $l_1 \geq 1$.

We first consider $P = L$. Assume that $l_1 \geq 2\sqrt{n} - 1$. Every non-deficit point $x \in L_i$ is bijectively associated to $x + (0,1) \in L_{i+1}$. Thus there are at least $l_i - l_{i+1}$ deficit points in L_i and then $d_L(X) \geq l_1 \geq 2\sqrt{n} - 1$. Now assume that $l_1 < 2\sqrt{n} - 1$. If $l_1 = 1$, then the point in l_1 is clearly a deficit point, so $d_L(X) \geq 1 + d_L(X \backslash L_1) \geq 1 + d_L(n-1)$. By induction, $1 + d_L(n-1) \geq 2\sqrt{n} - 1$, which is at least $2\sqrt{n} - 1$ for $n \geq 2$. If $l_1 \geq 2$, then the last two points in L_1 are deficit points. In that case, $d_L(X) \geq 2 + d_L(X \backslash L_1) \geq 2 + d_L(n - l_1)$. By induction, $2 + d_L(n - l_1) \geq 2 + 2\sqrt{n - l_1} - 1$, which is greater than $2\sqrt{n} - 1$ as long as $l_1 < 2\sqrt{n} - 1$. In all cases, $t_L(n) = n - d_L(n) \leq n - \lceil 2\sqrt{n} - 1 \rceil$.

Let $P = \square$ and proceed by induction on n. Suppose that u is the largest number for which $l_u \geq 1$. If $l_j = 0$ for some $1 < j < u$, then the result follows by induction applied to $L_1 \cup \cdots \cup L_{j-1}$ and $L_{j+1} \cup \cdots \cup L_u$ (the function $2\sqrt{n} - 1$ is concave down). Assume that $l_j \geq 1$ for all $1 \leq j \leq u$. We have

$$d_\square(X) \geq \sum_{i=1}^{u-1} \max(1, l_i - l_{i+1} + 1) + l_u.$$

Indeed, every last point in L_i is a deficit point with respect to $\square$. Also, if $i < u$, there are at most $l_{i+1} - 1$ points x in L_{i+1} which are *not* deficit points with respect to $\{(0,0), (1,0)\}$. Then there are at least $l_i - (l_{i+1} - 1) = l_i - l_{i+1} + 1$ points in L_i

which are *not* exactly below any of these points x, and thus they are deficit points with respect to $\square$. Finally, all points in L_u are clearly deficit points.

Since $\max(1, l_i - l_{i+1} + 1) = (2 + l_i - l_{i+1} + |l_i - l_{i+1}|)/2$, using the conventions $l_0 = l_{u+1} = 0$, we have

$$d_\square(X) \geq \frac{1}{2} \sum_{i=1}^{u-1} (2 + l_i - l_{i+1} + |l_i - l_{i+1}|) + l_u$$

$$= u - 1 + \frac{l_1}{2} + \frac{l_u}{2} + \frac{1}{2} \sum_{i=1}^{u-1} |l_i - l_{i+1}|$$

$$\geq u - 1 + \frac{1}{2} \sum_{i=0}^{u} |l_i - l_{i+1}|$$

$$\geq u - 1 + \max_{1 \leq i \leq u} l_i \quad \text{(by the triangle inequality)}$$

$$\geq u - 1 + \frac{1}{u} \sum_{i=1}^{u} l_i = u + \frac{n}{u} - 1 \geq 2\sqrt{n} - 1.$$

Thus, $t_\square(n) = n - d_\square(n) \leq n - \lceil 2\sqrt{n} - 1 \rceil$. $\qquad\square$

5.2 Conjectures for Arbitrary 4-Point Sets P

Now we discuss a plan and some conjectures to completely determine the function $t_P(n)$ when P is a four-point-set of affine rational dimension two. First note that, under an affine transformation T, translates of P correspond bijectively to translates of $T(P)$. Since P has rational dimension two, we can assume that $P = \{(0,0), (1,0), (0,1), (x,y)\}$, where $x, y \in \mathbb{Q}$. Moreover, by choosing appropriately the point in P corresponding to (x, y), we can assume that (x, y) is not in the interior or the boundary of the triangle $(0,0), (1,0), (0,1)$. That is, we can assume that $x < 0$ or $y < 0$ or $x + y \geq 1$.

Let $z = (x, y) \in \mathbb{Q}^2$. We can uniquely write $x = p_x/q$ and $y = p_y/q$, where p_x, p_y, q are integers, $q > 0$, and $\gcd(p_x, p_y, q) = 1$. Define $F : \mathbb{Q}^2 \to \mathbb{N}$ as

$$F(z) = F(x, y) = \begin{cases} p_x + p_y & \text{if } p_x, p_y \geq 0, \\ \max(p_x, -p_y + q) & \text{if } p_y < 0 \leq p_x, \\ \max(p_y, -p_x + q) & \text{if } p_x < 0 \leq p_y, \\ -p_x - p_y + q & \text{if } p_x, p_y < 0. \end{cases}$$

Let $\Omega = \{(x, y) \in \mathbb{Q}^2 : x < 0 \text{ or } y < 0 \text{ or } x + y \geq 1\}$. For every $z = (x, y) \in \Omega$, let $P(z) = \{(0,0), (1,0), (0,1), (x,y)\}$. We conjecture that:

Conjecture 1 *If $z_1, z_2 \in \Omega$ and $F(z_1) = F(z_2)$, then $t_{P(z_1)}(n) = t_{P(z_2)}(n)$ for every $n \in \mathbb{N}$.*
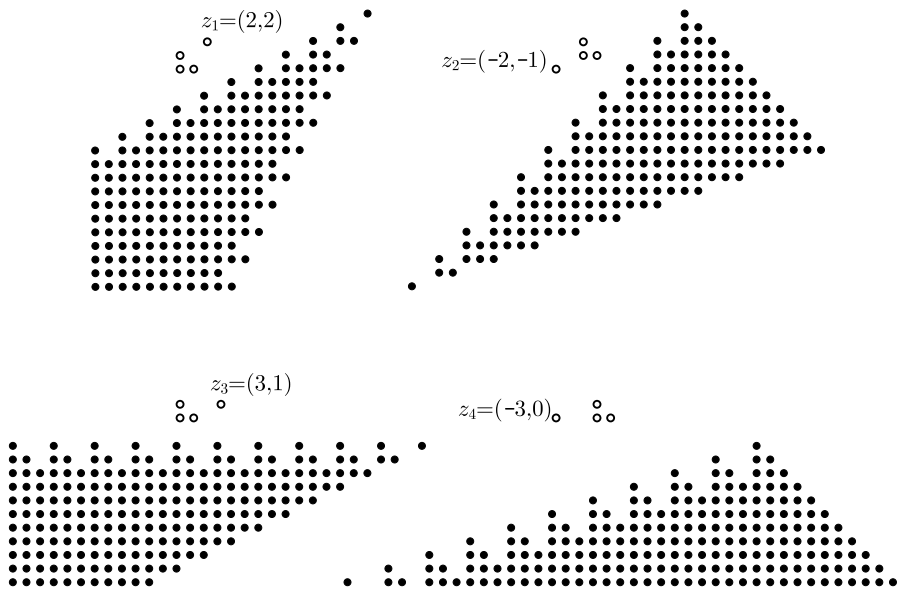
Fig. 5 Conjectured optimal constructions for the functions $t_{P(z_i)}(n)$, where $n = 102$

The previous section shows how this conjecture is true when $F(z) = 2$. Figure 5 shows four sets $P(z_i)$, $1 \le i \le 4$, for which $F(z_i) = 4$ and the conjectured optimal constructions for $n = 102$. In fact, every other $z \in \Omega$ with $F(z) = 4$ satisfies that $P(z)$ is affinely equivalent to one of the sets in Fig. 5. If Conjecture 1 is true in general, then it would be enough to determine $t_{P(z)}(n)$ for z of the form $z = (a, 0)$ with $a \in \mathbb{N}$. By considering these particular sets, we further believe that the correct coefficient for the $n^{1/2}$ term is $\sqrt{2F(z)}$ and that the rest of the error term is constant. More precisely:

Conjecture 2 *For every $z \in \Omega$,*

$$t_{P(z)}(n) = n - \sqrt{2F(z)}n^{1/2} + O(1).$$

For every set $P(z)$ with $z \in \Omega$, we can construct n-element sets X_n with $n - \sqrt{2F(z)}n^{1/2} + O(1)$ translates of $P(z)$. However we were able to prove a matching upper bound only for sets of the form $z = (a, 0)$. For instance, we can prove $t_{P(3,0)}(n) = n - \lceil \sqrt{6n - 15/4} - 3/2 \rceil$ for $n \ge 4$ and, in general, $t_{P(a,0)}(n) \le n - \lceil \sqrt{2an - a^2(a^2 - 4)/12} - a/2 \rceil$ for $a \ge 2$ and $n \ge \binom{a+1}{3}$. Already for $a = 4$ we cannot match, for every n, the $O(1)$ terms from our construction and from the previous upper bound. It is likely that an exact formula will depend on some kind of binomial-type representation, much like Theorem 1.

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