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Self-Similar Tiling Systems, Topological Factors and Stretching Factors

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Abstract In this paper we prove that if two self-similar tiling systems, with respective stretching factors λ_1 and λ_2, have a common factor which is a nonperiodic tiling system, then λ_1 and λ_2 are multiplicatively dependent.

Keywords Tiling · Self-similar · Factor maps · Stretching factor · Cobham theorem

1 Introduction

Given a nonperiodic self-similar tiling $\mathcal{T}$ generated by some similarity S_1 with stretching factor λ_1, it is rather natural to ask if we could generate $\mathcal{T}$ using another similarity with a different stretching factor λ_2. This is of course possible taking a power of the similarity S_1, where λ_2 is in this case a power of λ_1. Holton, Radin, and Sadun show in [16] that the stretching factor of any other similarity which generates $\mathcal{T}$ is equal to a rational power of λ_1. More precisely, they prove that the stretching factors of conjugate tiling systems which are the orbit closure under Euclidean motions of some self-similar tilings are multiplicatively dependent. In this paper we look at tiling systems which are the orbit closure under translations of some self-similar tilings, in order to give a necessary condition to have nonperiodic common factors. The result we present in this paper is the following:

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Theorem 1 *Let $S_1(\mathcal{T}_1) = \mathcal{T}_1$ and $S_2(\mathcal{T}_2) = \mathcal{T}_2$ be two self-similar tilings satisfying the Finite Pattern Condition, where S_1 and S_2 are primitive substitutions. Let λ_1 and λ_2 be the Perron eigenvalues of the substitution matrices associated to S_1 and S_2, respectively. If there exist a nonperiodic tiling $\mathcal{T}$ and factor maps $\pi_i : \Omega_{\mathcal{T}_i} \to \Omega_{\mathcal{T}}$ for $i \in \{1, 2\}$, then λ_1 and λ_2 are multiplicatively dependent.*

The problem we are interested in has been considered a long time ago by Cobham in [5] and [6] for fixed points of substitutions of constant length. He showed that if $p, q > 1$ are two multiplicatively independent integers, then a sequence x on a finite alphabet is both p-substitutive and q-substitutive if and only if x is ultimately periodic, where p-substitutive means that x is the image by a letter-to-letter morphism of a fixed point of a substitution of constant length p. This theorem was the starting point of a lot of work in many different directions such as: numeration systems for $\mathbb{N}$, substitutive sequences and subshifts, automata theory and logic (for more details, see [1–4, 7–14, 17]). Later, Semenov [21] proved a "multidimensional" Cobham-type theorem, that is to say, a Cobham theorem for recognizable subsets of $\mathbb{N}^d$. This result can be stated in terms of self-similar tilings, and in the case these tilings are repetitive, our result is a generalization of Semenov Theorem.

This paper is organized as follows: in Sect. 2 we give some basic definitions relevant for the study of tiling systems and substitution tiling systems. In Sect. 3 we study the frequencies of the patches in self-similar tilings and in their factors. First we prove that the frequencies of the patches in a self-similar tiling $\mathcal{T}$ are included in a finite union of geometric progressions of rate λ, where λ is the stretching factor of $\mathcal{T}$ (in [15] the authors remarked this fact for minimal substitution subshifts). Next, we prove that the frequencies of the patches in a tiling $\mathcal{T}$, which is a factor of two self-similar tiling systems with stretching factors λ_1 and λ_2, respectively, are included in the intersection of two finite unions of geometric progressions, one of rate λ_1 and the other of rate λ_2. The proof of this result would be easier if the factor maps were given by a kind of "sliding block code" (as it can be the case for subshifts), because in this case the preimage of a patch would be a finite collection of patches. Nevertheless, this is no longer the case for the tiling systems we consider here (examples of factor maps, and even conjugacies, that are not given by a "sliding block codes" are given in [18] and [19]), but we overcome this problem selecting carefully some patches in the preimages we considered. Finally, in Sect. 4 we deduce the main theorem.

2 Definitions and Background

In this section we give the classical definitions concerning tilings. For more details, we refer to [22]. A *tiling* of $\mathbb{R}^d$ is a countable collection $\mathcal{T} = \{t_i : i \geq 0\}$ of closed subsets of $\mathbb{R}^d$ (which are known as *tiles*) whose union is the whole space and their interiors are pairwise disjoint. We assume that the tiles are homeomorphic to closed balls and that they belong, up to translations, to a finite collection of closed subsets of $\mathbb{R}^d$ whose elements are called *prototiles*. We say that two tiles are *equivalent* if they are equal up to translations. It is often useful to consider every prototile as a closed set endowed with a label. In this case, two tiles are equivalent if, in addition, their labels coincide.

The translation of the tiling $\mathcal{T}$ by a vector $v \in \mathbb{R}^d$ is the tiling $\mathcal{T} + v$ obtained after translating every tile of $\mathcal{T}$ by $-v$. The tiling $\mathcal{T}$ is said to be *aperiodic* (or *nonperiodic*) if $\mathcal{T} + v = \mathcal{T}$ implies $v = 0$.

The *support* of a tile t_i, denoted by $\mathrm{supp}(t_i)$, is the closed set that defines t_i. For every subset A of $\mathbb{R}^d$, we define, as usual, $\mathcal{T} \cap A$ to be the set $\{t_i \cap A : i \geq 0\}$. A *patch* P is a finite collection of tiles. The support of a patch P, denoted by $\mathrm{supp}(P)$, is the union of the supports of the tiles in P. The *diameter* of a patch P is the diameter of its support, we call it $\mathrm{diam}(P)$. We define $P + v$ as we defined $\mathcal{T} + v$.

The tiling $\mathcal{T}$ satisfies the *finite pattern condition* FPC (or equivalently, we say that it is *locally finite*) if for any $r > 0$, there are up to translation, only finitely many patches with diameter smaller than r. This condition is automatically satisfied in the case of a tiling whose tiles are polyhedra that meet face-to-face. A tiling $\mathcal{T}$ is *repetitive* if for any patch P in $\mathcal{T}$, there exists $r > 0$ such that for every open ball $B_r(v)$, the collection $\mathcal{T} \cap B_r(v)$ contains a patch P' equivalent to P (when it is clear from the context, we will say that P "appears" in $B_r(v)$). Nonperiodic repetitive tilings that satisfy FPC are called *perfect* tilings.

2.1 Tiling Systems

Let $\mathcal{A}$ be a finite collection of prototiles. We denote by $T(\mathcal{A})$ (*full tiling space*) the space of all tilings of $\mathbb{R}^d$ whose tiles are equivalent to some element in $\mathcal{A}$. We always suppose that $T(\mathcal{A})$ is nonempty. The group $\mathbb{R}^d$ acts on $T(\mathcal{A})$ by translations:

$$(v, \mathcal{T}) \to \mathcal{T} + v \quad \text{for } v \in \mathbb{R}^d \text{ and } \mathcal{T} \in T(\mathcal{A}).$$

Furthermore, this action is continuous with the topology induced by the following distance: take $\mathcal{T}, \mathcal{T}'$ in $T(\mathcal{A})$ and define A the set of $\varepsilon \in (0, 1)$ such that there exist v and v' in $B_\varepsilon(0)$ with

$$(\mathcal{T} + v) \cap B_{1/\varepsilon}(0) = \left(\mathcal{T}' + v'\right) \cap B_{1/\varepsilon}(0);$$

we set

$$d\left(\mathcal{T}, \mathcal{T}'\right) = \begin{cases} \inf A & \text{if } A \neq \emptyset, \\ 1 & \text{if } A = \emptyset. \end{cases}$$

Roughly speaking, two tilings are close if they have the same pattern in a large neighborhood of the origin, up to a small translation. A *tiling system* is a pair $(\Omega, \mathbb{R}^d)$ such that Ω is a translation invariant closed subset of some full tiling space. The orbit closure of a tiling $\mathcal{T}$ in $T(\mathcal{A})$ is the set $\Omega_{\mathcal{T}} = \overline{\{\mathcal{T} + v : v \in \mathbb{R}^d\}}$. When $\mathcal{T}$ satisfies the FPC, $\Omega_{\mathcal{T}}$ is compact (see [20]). If $\mathcal{T}$ is repetitive, then all the orbits are dense in $\Omega_{\mathcal{T}}$. In this case the tiling system $(\Omega_{\mathcal{T}}, \mathbb{R}^d)$ is said to be *minimal*.

A *factor map* between two tiling systems $(\Omega_1, \mathbb{R}^d)$ and $(\Omega_2, \mathbb{R}^d)$ is a continuous map $\pi : \Omega_1 \to \Omega_2$ such that $\pi(\mathcal{T} + v) = \pi(T) + v$ for all $\mathcal{T} \in \Omega_1$ and $v \in \mathbb{R}^d$.

In symbolic dynamics it is well known that topological factor maps between subshifts are always given by sliding-block-codes. There are examples which show that this result cannot be extended to tiling systems [18, 19]. The following lemma shows that factor maps between tiling systems are not far to be sliding-block-codes. A similar result can be found in [16].

Lemma 2 *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two tilings. Suppose that $\mathcal{T}_1$ verifies the FPC and $\pi : \Omega_{\mathcal{T}_1} \to \Omega_{\mathcal{T}_2}$ is a factor map. Then, there exists a constant $s_0 > 0$ such that to every $\varepsilon > 0$ it is possible to associate $R_\varepsilon > 0$ satisfying the following: Let $R \geq R_\varepsilon$. If $\mathcal{T}$ and $\mathcal{T}'$ in $\Omega_{\mathcal{T}_1}$ verify*

$$\mathcal{T} \cap B_{R+s_0}(0) = \mathcal{T}' \cap B_{R+s_0}(0),$$

then

$$\big(\pi(\mathcal{T}) + v\big) \cap B_R(0) = \pi\big(\mathcal{T}'\big) \cap B_R(0)$$

for some $v \in B_\varepsilon(0)$.

Proof The tiling $\mathcal{T}_2$ also satisfies the FPC because $\Omega_{\mathcal{T}_2}$ is compact. Since the tilings in $\Omega_{\mathcal{T}_2}$ have a finite number of tiles, up to translations, there exists $\delta_0' > 0$ such that if $y_1 \neq y_2 \in \mathbb{R}^d$ satisfy $(\mathcal{T} + y_1) \cap B_R(0) = (\mathcal{T} + y_2) \cap B_R(0)$ for some $\mathcal{T} \in \Omega_{\mathcal{T}_2}$ and some $R > \max\{\mathrm{diam}(p) : p \text{ prototile in } \mathcal{T}\}$, then $\|y_1 - y_2\| \geq \delta_0'$ (for the details, see [22]).

Let $0 < \delta_0 < \frac{\delta_0'}{2}$. Since π is uniformly continuous, there exists $s_0 > 1$ such that if $\mathcal{T}$ and $\mathcal{T}'$ in $\Omega_{\mathcal{T}_1}$ verify $\mathcal{T} \cap B_{s_0}(0) = \mathcal{T}' \cap B_{s_0}(0)$, then

$$\big(\pi(\mathcal{T}) + v\big) \cap B_{\frac{1}{\delta_0}}(0) = \pi\big(\mathcal{T}'\big) \cap B_{\frac{1}{\delta_0}}(0)$$

for some $v \in B_{\delta_0}(0)$.

Let $0 < \varepsilon < \delta_0$. By the uniform continuity of π there exists $0 < \delta < \frac{1}{s_0}$ such that if $\mathcal{T}$ and $\mathcal{T}'$ in $\Omega_{\mathcal{T}_1}$ verify $\mathcal{T} \cap B_{\frac{1}{\delta}}(0) = \mathcal{T}' \cap B_{\frac{1}{\delta}}(0)$, then

$$\big(\pi(\mathcal{T}) + v\big) \cap B_{\frac{1}{\varepsilon}}(0) = \pi\big(\mathcal{T}'\big) \cap B_{\frac{1}{\varepsilon}}(0) \tag{2.1}$$

for some $v \in B_\varepsilon(0)$.

Now fix $R \geq R_\varepsilon = \frac{1}{\delta} - s_0$ and $\mathcal{T}$ and $\mathcal{T}'$ two tilings in $\Omega_{\mathcal{T}_1}$ verifying

$$\mathcal{T} \cap B_{R+s_0}(0) = \mathcal{T}' \cap B_{R+s_0}(0). \tag{2.2}$$

Then, on one hand, the tilings $\mathcal{T}$ and $\mathcal{T}'$ satisfy (2.1), and, on the other hand, we obtain that $(\mathcal{T} + a) \cap B_{s_0}(0) = (\mathcal{T}' + a) \cap B_{s_0}(0)$ for every a in $B_R(0)$. By the choice of s_0 this implies that

$$\big(\pi(\mathcal{T}) + a + t_a\big) \cap B_{\frac{1}{\delta_0}}(0) = \big(\pi\big(\mathcal{T}'\big) + a\big) \cap B_{\frac{1}{\delta_0}}(0) \tag{2.3}$$

for some $t_a \in B_{\delta_0}(0)$.

Since $\delta_0 > \varepsilon$, from (2.1) we get

$$\big(\pi(\mathcal{T}) + v\big) \cap B_{\frac{1}{\delta_0}}(0) = \pi\big(\mathcal{T}'\big) \cap B_{\frac{1}{\delta_0}}(0). \tag{2.4}$$

We will show that $t_a = v$ for every a in $B_R(0)$. This property, together with (2.3) and (2.4), implies that

$$\big(\pi(\mathcal{T}) + v\big) \cap B_R(0) = \pi\big(\mathcal{T}'\big) \cap B_R(0).$$

For $a=0$, from (2.3) and (2.4) we have that $t_0=v$ or $\|v-t_0\|\geq\delta_0'$. Since $\|t_0-v\|\leq\delta_0+\varepsilon<2\delta_0<\delta_0'$, we conclude that $t_0=v$.

For $a\in B_R(0)$, consider $s>0$ such that for every $a'\in B_s(a)$, the patch

$$P=\big(\big(\pi(\mathcal{T}')+a\big)\cap B_{\frac{1}{\delta_0}}(0)\big)\cap\big(\big(\pi(\mathcal{T}')+a+\big(a'-a\big)\big)\cap B_{\frac{1}{\delta_0}}(0)\big)$$

contains a tile.

From (2.3) we get $\pi(\mathcal{T})+a+t_a+(a-a')\cap\operatorname{supp}(P)=P$. Replacing a by a' in (2.3), we obtain $\pi(\mathcal{T})+a+t_a'+(a'-a)\cap\operatorname{supp}(P)=P$. This implies that the norm of t_a-t_a' is equal to 0 or greater than δ_0'. Since $\|t_a-t_a'\|\leq 2\delta_0<\delta_0'$, we get $t_a=t_a'$. Thus we conclude that the function that associates t_a to a is constant, which implies that $t_a=t_0=v$ for every a in $B_R(0)$. □

2.2 Linearly Recurrent Tilings

A tiling $\mathcal{T}$ is *linearly recurrent* (or strongly repetitive, or linearly repetitive) if there exists a constant $L>0$ such that for every patch P in $\mathcal{T}$, any ball of radius $L\operatorname{diam}(P)$ contains a translate of P. Every tiling in the orbit closure of a linearly recurrent tiling is linearly recurrent with the same constant. When $\mathcal{T}$ is linearly recurrent, we call $(\Omega_{\mathcal{T}},\mathbb{R}^d)$ a *linearly recurrent* tiling system.

Lemma 3 *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two tilings verifying the FPC. If $\pi:\Omega_{\mathcal{T}_1}\to\Omega_{\mathcal{T}_2}$ is a factor map and $\mathcal{T}_1$ is linearly repetitive, then $(\Omega_{\mathcal{T}_2},\mathbb{R}^d)$ is linearly recurrent.*

Proof Let $\mathcal{T}\in\Omega_{\mathcal{T}_1}$. Consider $\varepsilon>0$ and $R>0$ the positive number of Lemma 2 associated to ε. Since $\mathcal{T}$ is linearly repetitive with some constant L, for any $y\in\mathbb{R}^d$, there exists $v\in B_{L(R+s_0)}(y)$ such that $B_{R+s_0}(v)\subseteq B_{L(R+s_0)}(y)$ and $(\mathcal{T}+v)\cap B_{R+s_0}(0)=\mathcal{T}\cap B_{R+s_0}(0)$. By Lemma 2 there exists $t\in B_\varepsilon(0)$ such that $(\pi(\mathcal{T})+v+t)\cap B_R(0)=\pi(\mathcal{T})\cap B_R(0)$. This implies that any ball of radius $L(R+s_0)+2\varepsilon$ in $\pi(\mathcal{T})$ contains a copy of $\pi(\mathcal{T})\cap B_R(0)$. Since $Ls_0+2\varepsilon$ is smaller than some constant, it follows that $\pi(\mathcal{T})$ is linearly recurrent. □

2.3 Substitution Tiling Systems

Let M be a linear map on $\mathbb{R}^d$. It is called *expansive* if there exists $\lambda>1$ such that

$$\|Mv\|\geq\lambda\|v\|\quad\text{for all } v\in\mathbb{R}^d.$$

The map M is a *similarity* if $\|Mv\|=\lambda\|v\|$ for all $v\in\mathbb{R}^d$.

Let α be an eigenvalue of the expansive (resp. similar) linear map M, and let $v\neq 0$ be an eigenvector associated to α. We have $\|Mv\|=|\alpha|\|v\|$, which implies that $|\alpha|\geq\lambda$ (resp. $|\alpha|=\lambda$) and then $|\det(M)|\geq\lambda^d$ (resp. $|\det(M)|=\lambda^d$). Thus, if Θ is a Borel set in $\mathbb{R}^d$, we obtain

$$\operatorname{vol}(M\Theta)=\big|\det(M)\big|\operatorname{vol}(\Theta)\geq\lambda^d\operatorname{vol}(\Theta)\quad\text{if } M \text{ is expansive,}$$

$$\operatorname{vol}(M\Theta)=\big|\det(M)\big|\operatorname{vol}(\Theta)=\lambda^d\operatorname{vol}(\Theta)\quad\text{if } M \text{ is a similarity.}$$

Let $\mathcal{A}$ be a finite collection of prototiles, and let M be an expansive linear map on $\mathbb{R}^d$. A *substitution* is a function S on the set of prototiles $\mathcal{A}$ that associates to each p in P a patch $S(p)$ such that

- the support of $S(p)$ is $M\,\mathrm{supp}(p)$,
- for every $q \in \mathcal{A}$, there exist $n_{p,q} \geq 0$ and, for each $1 \leq k \leq n_{p,q}$, $v_{p,q,k} \in \mathbb{R}^d$ such that

$$S(p) = \{q + v_{p,q,k} : 1 \leq k \leq n_{p,q},\ q \in \mathcal{A}\}.$$

The *substitution matrix* of S is the matrix $A \in \mathcal{M}_{\mathcal{A}\times\mathcal{A}}(\mathbb{Z}^+)$ which contains, in the coordinate (p, q), the number of different tiles in $S(p)$ which are equivalent to q. That is, $A_{p,q} = n_{p,q}$ for all $p, q \in \mathcal{A}$.

The substitution S can be defined on $T(\mathcal{A})$ in the following way: if t is a tile in $\mathcal{T} \in T(\mathcal{A})$ such that t is equivalent to the prototile $p \in \mathcal{A}$, we define

$$S(t) = S(p) + Mv,$$

where $v \in \mathbb{R}^d$ is such that $\mathrm{supp}(t) = \mathrm{supp}(p) + v$. Then, we define

$$S(\mathcal{T}) = \bigcup_{t\in\mathcal{T}} S(t) \in T(\mathcal{A}).$$

The substitution is *primitive* if A is primitive, that is, there exists $k > 0$ such that $A^k > 0$. In this case, the Perron eigenvalue of A is $|\det(M)|$ [22].

In this paper, we always suppose that S is primitive.

The *substitution tiling system* associated to S is the tiling system $(X_S, \mathbb{R}^d)$, where X_S is the space of all tilings $\mathcal{T}$ in $T(\mathcal{A})$ such that for every patch P of $\mathcal{T}$, there exist a prototile $p \in \mathcal{A}$ and $k > 0$ satisfying $P \subseteq S^k(p)$. The action of $\mathbb{R}^d$ on X_S is the translation. Because S is primitive, there always exist a tiling $\mathcal{T}_0 \in T(\mathcal{A})$ and $k_0 > 0$ such that $S^{k_0}(\mathcal{T}_0) = \mathcal{T}_0$. It is classical (in the primitive case) that $\Omega_{\mathcal{T}_0} = X_S = X_{S^k}$ for every $k > 0$. So, without loss of generality we can suppose that $S(\mathcal{T}_0) = \mathcal{T}_0$. In addition, we will always suppose that the fixed point of S satisfies the FPC. In this case, X_S is a compact metric space, and $(X_S, \mathbb{R}^d)$ is minimal.

A tiling $\mathcal{T}$ in $T(\mathcal{A})$ which satisfies the FPC is *self-affine* if it is the fixed point of a substitution. The tiling $\mathcal{T}$ is said to be *self-similar* if it is the fixed point of a substitution S which is defined by a similarity M with constant λ (for more details, see [22]). We say that λ is the *stretching factor* of S or $\mathcal{T}$.

Let $\mathcal{T}_0$ be a self-similar tiling which is the fixed point of a primitive substitution S satisfying the FPC. The following two results are included in [23].

Lemma 4 *$\mathcal{T}_0$ is linearly recurrent.*

Lemma 5 *There exists $N > 0$ such that if P is a patch in $\mathcal{T}_0$ whose support contains a ball of radius R, then whenever $P + v$ is a patch of $\mathcal{T}_0$ with $v > 0$, $\|v\| > \frac{R}{N}$.*

These two lemmata mean that the minimal distance between two equivalent patches in a self-similar tiling is neither too large nor too small compared to their sizes.

3 Frequencies

Consider a tiling $\mathcal{T}$ of $\mathbb{R}^d$. For a set $F \subseteq \mathbb{R}^d$, we write

$$\mathcal{T}[\![F]\!] = \{t \in \mathcal{T} : t \cap F \neq \emptyset\}.$$

A $\mathcal{T}$-*corona* is a patch $\mathcal{T}[\![\text{supp}(t)]\!]$, where t is a tile in $\mathcal{T}$. Remark that for some $\epsilon \in \mathbb{R}^d$, we could have $\mathcal{T}[\![F + \epsilon]\!] = \mathcal{T}[\![F]\!]$. To avoid this situation we define, for $v \in \mathbb{R}^d$, $\mathcal{T}[F, v] = \mathcal{T}[\![F]\!] - v$. When F is a ball $B_R(v)$, we write $\mathcal{T}[B_R(v)]$ instead of $\mathcal{T}[B_R(v), v]$.

In the sequel we suppose that $\mathcal{T}_0$ is a self-similar tiling which is the fixed point of a primitive substitution S, with stretching factor λ, satisfying the FPC.

3.1 Van Hove Sequences

In order to define the notion of frequency of a patch, we need the concept of Van Hove sequences.

Let P be a patch in $\mathcal{T}_0$, and let $\Theta \subset \mathbb{R}^d$. Denote by $L_P(\Theta)$ the number of patches included in $\mathcal{T}_0 \cap \Theta$ which are equivalent to P [22].

A sequence $(\Theta_n)_{n\geq 0}$ of subsets of $\mathbb{R}^d$ is a *Van Hove* sequence if for any $r > 0$,

$$\lim_{n\to\infty} \frac{\text{vol}((\partial\Theta_n)^{+r})}{\text{vol}(\Theta_n)} = 0,$$

where

$$\Theta^{+r} = \{x \in \mathbb{R}^d : \text{dist}(x, \Theta) \leq r\},$$

and $\partial\Theta$ is the border of Θ.

In [22], it was shown that for any patch P in $\mathcal{T}_0$, there is a number $\text{freq}(P) > 0$ such that for any Van Hove sequence $(\Theta_n)_{n\geq 0}$,

$$\lim_{n\to\infty} \frac{L_P(\Theta_n)}{\text{vol}(\Theta_n)} = \text{freq}(P).$$

Suppose that P and Q are two patches in $\mathcal{T}_0$. In order to simplify the notation, we will write $L_P(Q)$, $\text{vol}(P)$ and $(\partial P)^{+r}$ instead of $L_P(\text{supp}(Q))$, $\text{vol}(\text{supp}(P))$ and $(\partial\,\text{supp}(P))^{+r}$ respectively.

It is easy to show that $(M^n\Theta)_{n\geq 0}$ is a Van Hove sequence when $M : \mathbb{R}^d \to \mathbb{R}^d$ is an expansive linear map and Θ is a compact subset of $\mathbb{R}^d$ with nonempty interior and such that $\text{vol}(\partial\Theta) = 0$. Consequently, to compute $\text{freq}(P)$ we will use the limit

$$\text{freq}(P) = \lim_{k\to\infty} \frac{L_P(S^k(p))}{\text{vol}(S^k(p))}$$

for any prototile p in $\mathcal{A}$.

3.2 Patch Frequencies of a Self-Similar Tiling

The next proposition extends a result of C. Holton and L. Zamboni [15] obtained for minimal substitution subshifts. But before we will need the following technical lemma:

Lemma 6 *Suppose that $\mathcal{T}$ satisfies the FPC. Then there exists a constant $\eta > 0$ such that for every $y \in \mathbb{R}^d$, the ball $B_\eta(y)$ is contained in the support of a corona in $\mathcal{T}$.*

Proof Let t be a tile in $\mathcal{T}$. The number

$$\eta_t = \operatorname{dist}\bigl(\partial t, \partial \mathcal{T}[\![\operatorname{supp}(t)]\!]\bigr)$$

is positive for every tile t. The FPC implies that there is a finite number of coronas up translations. Hence we get

$$\eta = \min\{\eta_t : t \in \mathcal{T}\} > 0.$$

Notice that the set

$$\bigl\{x \in \mathbb{R}^d : \operatorname{dist}(x, t) \leq \eta\bigr\}$$

is contained in the support of $\mathcal{T}[\![\operatorname{supp}(t)]\!]$ for every tile t in $\mathcal{T}$. Thus if y is a point in $\mathbb{R}^d$ belonging to the tile $t \in \mathcal{T}$, then the ball $B_\eta(y)$ is contained in the support of $\mathcal{T}[\![\operatorname{supp}(t)]\!]$. □

Proposition 7 *There exists a finite set $F \subset \mathbb{R}$ such that for every patch P in $\mathcal{T}_0$ satisfying $P = \mathcal{T}_0[B_R(y)]$, for some $R > 0$ and $y \in \mathbb{R}^d$,*

$$\operatorname{freq}(P) = \frac{f}{\lambda^{dk}},$$

where $f \in F$ and $k > 0$ is such that

$$\lambda^{k-1}\eta \leq \operatorname{diam}(P) < \lambda^k \eta$$

with η the constant of Lemma 6.

Proof Let $\mathcal{A}$ be the prototile set associated to $\mathcal{T}_0$. We define

$$\bar{l} = \max\bigl\{\operatorname{diam}(p) : p \in \mathcal{A}\bigr\}.$$

Let P be a patch in $\mathcal{T}_0$ such that

$$P = \mathcal{T}_0\bigl[\![B_R(y)]\!\bigr] \quad \text{for some } R > 0 \text{ and } y \in \mathbb{R}^d.$$

This implies that

$$\operatorname{diam}(P) \leq 2\bigl(R + \bar{l}\bigr). \tag{3.1}$$

Let $k \geq 0$ be such that

$$\lambda^{k-1}\eta \leq \operatorname{diam}(P) < \lambda^k \eta. \tag{3.2}$$

By Lemma 6, there exists a corona B the support of which contains the ball $B_\eta(M^{-k}y)$. Because the support of $S^k(B)$ contains the ball $B_{\lambda^k\eta}(y)$, by (3.2) we deduce that $S^k(B)$ contains the patch P. From Lemma 5 we have

$$L_P\big(S^k(B)\big) \leq \frac{\operatorname{vol}(S^k(B))}{\operatorname{vol}(B_{\frac{R}{N}}(0))} = \frac{\lambda^{kd}}{\frac{R^d}{N^d}} \frac{\operatorname{vol}(B)}{\operatorname{vol}(B_1(0))}. \tag{3.3}$$

From (3.1) and (3.2) we obtain

$$\frac{1}{2(R+\bar{l})} \leq \frac{1}{\operatorname{diam}(P)} \leq \frac{1}{\lambda^{k-1}\eta},$$

which implies that there exists C, independent of k, such that

$$\frac{\lambda^{kd}}{R^d} \leq \left(\frac{2\lambda}{\eta - \frac{2\bar{l}}{\lambda^{k-1}}}\right)^d \leq C. \tag{3.4}$$

From (3.3) and (3.4) we conclude that there exists a constant K, independent on P, k and B, such that

$$L_P\big(S^k(B)\big) \leq K.$$

Let P' be any patch in $\mathcal{T}_0$, and let D be the set of all $\mathcal{T}_0$-coronas, up to translation. We have

$$L_P\big(S^k\big(P'\big)\big) = \sum_{B \in D} L_B\big(P'\big) N\big(P', P, B\big),$$

where $N(P', P, B)$ is some integer in $\{0, \ldots, L_P(S^k(B))\} \subseteq \{0, \ldots, K\}$. Thus, for $p \in \mathcal{A}$ and $n > k$,

$$\begin{aligned}
\frac{L_P(S^n(p))}{\operatorname{vol}(S^n(p))} &= \frac{L_P(S^k(S^{n-k}(p)))}{\operatorname{vol}(S^n(p))} \\
&= \sum_{B \in D} \frac{L_B(S^{n-k}(p)) N(S^{k-n}(p), P, B)}{\operatorname{vol}(S^n(p))} \\
&= \sum_{B \in D} \frac{L_B(S^{n-k}(p))}{\operatorname{vol}(S^{n-k}(p))} \frac{\operatorname{vol}(S^{n-k}(p))}{\operatorname{vol}(S^n(p))} N\big(S^{k-n}(p), P, B\big) \\
&= \frac{1}{\lambda^{kd}} \sum_{B \in D} \frac{L_B(S^{n-k}(p))}{\operatorname{vol}(S^{n-k}(p))} N\big(S^{k-n}(p), P, B\big).
\end{aligned}$$

Because $N(S^{k-n}(p), P, B)$ is in $\{1, \ldots, K\}$ for every $n > k$, we can take a convergent subsequence to obtain

$$\begin{aligned}\mathrm{freq}(P) &= \frac{1}{\lambda^{kd}} \lim_{n\to\infty} \sum_{B\in D} \frac{L_B(S^{n-k}(p))}{\mathrm{vol}(S^{n-k}(p))} N\big(S^{k-n}(p), P, B\big)\\ &= \frac{1}{\lambda^{kd}} \sum_{B\in D} \mathrm{freq}(B) N(P, B),\end{aligned}$$

where $N(P, B)$ is some integer in $\{0, \ldots, K\}$ for every $B \in D$. Because D is finite, to conclude it suffices to take

$$F = \left\{ \sum_{B\in D} \mathrm{freq}(B) N_B : N_B \in \{0, \ldots, K\} \right\}. \qquad \square$$

Remark 8 From [22] we know that $(\Omega_{\mathcal{T}_0}, \mathbb{R}^d)$ is uniquely ergodic. Hence, the frequency of a patch P does not depend on the tiling. That is, $\mathrm{freq}(P)$ is the same for every $\mathcal{T}$ in $\Omega_{\mathcal{T}_0}$.

3.3 Patch Frequency in the Factor

The next result extends Proposition 7 to tiling factors of self-similar tiling systems. The main problem we have to overcome is that the factor map is not necessarily given by a sliding block code. Hence the first part of the next proof consists in selecting carefully the preimages of a given patch P by means of a finite induction procedure. Then, we show that the frequency of the patch P is the sum of the frequencies of the selected patches.

Proposition 9 *Let $\mathcal{T}$ be a nonperiodic tiling. If there exists a factor map $\pi : \Omega_{\mathcal{T}_0} \to \Omega_{\mathcal{T}}$, then there exists a finite set $F \subseteq \mathbb{R}$ such that for every patch P in $\mathcal{T}$ satisfying $P = \mathcal{T}[B_R(y)]$, for some $R > 0$ and $y \in \mathbb{R}^d$,*

$$\mathrm{freq}(P) = \frac{f}{\lambda^{dk}},$$

where $f \in F$ and $k > 0$ is such that

$$\eta\lambda^{k-3} \le \mathrm{diam}(P) < \eta\lambda^{k-1}$$

if R is large enough.

Proof Let $\mathcal{T}_2 \in \Omega_{\mathcal{T}}$, and let $\mathcal{T}_1 \in \Omega_{\mathcal{T}_0}$ be such that $\pi(\mathcal{T}_1) = \mathcal{T}_2$. Let $s_0 > 0$ be the constant of Lemma 2.

The linear recurrence of $\mathcal{T}_1$ implies that the tiling $\mathcal{T}_2$ is also linearly recurrent. Let L be the constant of linear recurrence of $\mathcal{T}_1$, and let M and N be the constants of Lemma 5 associated to $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. We set

$$K = \max\big\{(8LN)^d, (8LM)^d\big\}$$

and

$$\eta_i = \max\{\operatorname{diam}(t) : t \text{ is a tile in } \mathcal{T}_i\} \quad \text{for } i \in \{1, 2\}.$$

Let $\varepsilon > 0$. Let $R_\varepsilon > 0$ be the positive number associated to ε as in Lemma 2. Notice that R_ε can be chosen large enough in order that

$$R_\varepsilon \geq \max \begin{cases} s_0 + \eta_1 + \eta_2 + \varepsilon, \\ 4N(2K+1)\varepsilon, \\ 2M\varepsilon - s_0, \\ 2(\eta_1 + \varepsilon) - (s_0 + \eta_2), \\ \eta\lambda^{\lceil \log_\lambda \frac{2\eta_1}{\eta(\lambda-1)} \rceil}, \\ \eta\lambda^{\lceil \log_\lambda \frac{2(s_0+\eta_1+\eta_2+2\varepsilon)}{\eta(\lambda-1)} \rceil + 2}, \\ \eta/2. \end{cases} \tag{3.5}$$

Let $R \geq R_\varepsilon$, and let $P = \mathcal{T}_2[B_R(y)]$, $y \in \mathbb{R}^d$.

Suppose that $v_1, \dots, v_l$ are all the points in $B_{2L(R+s_0+\varepsilon+\eta_1+\eta_2)}(0)$ such that

$$\mathcal{T}_2[B_R(v_i)] = P.$$

If $v_i \neq v_j$, we have $\|v_i - v_j\| > \frac{R}{N}$. This implies that in a ball of radius $\frac{R}{2N}$, there is at most one point v such that $\mathcal{T}_2[B_R(v)] = P$. Using (3.5), it follows that in $B_{2L(R+s_0+\varepsilon+\eta_1+\eta_2)}(0)$ there are at most

$$\frac{\operatorname{vol}(B_{2L(R+s_0+\varepsilon+\eta_1+\eta_2)}(0))}{\operatorname{vol}(B_{\frac{R}{2N}}(0))} \leq (8LN)^d \leq K$$

points v such that $\mathcal{T}_2[B_R(v)] = P$. This implies that for any patch P, we have $l \leq K$.

For every $1 \leq i \leq l$, we set

$$P_i = \mathcal{T}_1[B_{R+s_0+\eta_2}(v_i)].$$

Now, for every $1 \leq i \leq l$, we will define, by induction on i, k_i different patches as follows (see Fig. 1).

For $i = 1$, we take all the patches P' in $\mathcal{T}_1$ satisfying the following two conditions:

$$P' = \mathcal{T}_1[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}(v)] \quad \text{for some } v \in \mathbb{R}^d, \tag{3.6}$$

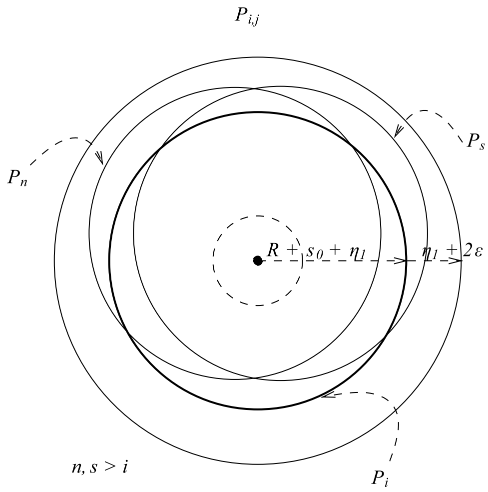
$$P_1 = \mathcal{T}_1[B_{R+s_0+\eta_2}(v)]. \tag{3.7}$$

Because $\mathcal{T}_1$ satisfies the FPC, there exists a finite number k_1 of different patches satisfying the previous condition. We call these patches $P_{1,1}, \dots, P_{1,k_1}$. Moreover, k_1 is bounded by K. Indeed, if v and v' are two different points in $\mathbb{R}^d$ such that

$$P_{1,j} = \mathcal{T}_1[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}(v)],$$

$$P_{1,i} = \mathcal{T}_1[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}(v')],$$

Fig. 1 Situation during the selection of the $P_{i,j}$ when $i \geq 2$



for some $1 \leq i,\ j \leq k_1$, then

$$P_1 = \mathcal{T}_1\big[B_{R+s_0+\eta_2}(v)\big] = \mathcal{T}_1\big[B_{R+s_0+\eta_2}\big(v'\big)\big].$$

By Lemma 5, this implies that

$$\big\|v - v'\big\| > \frac{R + s_0 + \eta_2}{M}.$$

It follows that in a ball of radius $\frac{R+s_0+\eta_2}{2M}$ there is at most one point w which is the center of some $P_{1,j}$. Since $\mathcal{T}_1$ is linearly recurrent with constant L and for every $1 \leq j \leq k_1$,

$$\operatorname{diam}(P_{1,j}) \leq 2(R + s_0 + \eta_1 + \eta_2 + 2\varepsilon) + 2\eta_1,$$

all the patches $P_{1,j}$ appear in the ball $B_{2L(R+s_0+2\eta_1+\eta_2+2\varepsilon)}(0)$ in $\mathcal{T}_1$. Using (3.5), this implies

$$k_1 \leq \frac{\operatorname{vol}(B_{2L(R+s_0+2\eta_1+\eta_2+2\varepsilon)}(0))}{\operatorname{vol}(B_{\frac{R+s_0+\eta_2}{2M}}(0))} \leq (8LM)^d \leq K.$$

For $1 < i \leq l$, we take all the patches P' in $\mathcal{T}_1$ satisfying the following three conditions:

$$P' = \mathcal{T}_1\big[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}(v)\big] \quad \text{for some } v \in \mathbb{R}^d, \tag{3.8}$$

$$P_i = \mathcal{T}_1\big[B_{R+s_0+\eta_2}(v)\big], \quad \text{if} \tag{3.9}$$

$$\mathcal{T}_1\big[B_{R+s_0+\eta_2}(v + t)\big] = P_j \quad \text{for some } t \in B_{2\varepsilon}(0),\ \text{then } j \geq i. \tag{3.10}$$

As for the case $i = 1$, we remark there is a finite number k_i of different patches satisfying the previous conditions and that k_i is smaller than K. We call these patches $P_{i,1}, \ldots, P_{i,k_i}$.

Remark 10 The linear recurrence of $\mathcal{T}_1$ and (3.5) imply that if $v \in \mathbb{R}^d$ satisfies

$$\mathcal{T}_1\big[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}(v)\big] = P_{i,j}$$

for some $1 \le i \le l$ and $1 \le j \le k_i$, then $\mathcal{T}_1[B_{R+s_0+\eta_2}(v+t)] \neq P_i$ for every $t \in B_{2\varepsilon}(0) \setminus \{0\}$.

Remark 11 By Remark 10 and (3.10), if $v \in \mathbb{R}^d$ satisfies

$$\mathcal{T}_1\big[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}(v)\big] = P_{i,j}$$

for some $1 \le i \le l$ and $1 \le j \le k_i$, then $\mathcal{T}_1[B_{R+s_0+\eta_2}(v+t)] \neq P_s$ for every $1 \le s \le i$ and $t \in B_{2\varepsilon}(0) \setminus \{0\}$.

Remark 12 By the construction of the patches $P_{i,j}$, if $v \in \mathbb{R}^d$ satisfies

$$\mathcal{T}_1\big[B_{R+s_0+\eta_2}(v)\big] = P_i$$

for some $1 \le i \le l$ and $j > i$ whenever $\mathcal{T}_1[B_{R+s_0+\eta_2}(v+t)] = P_j$ for some $t \in B_{2\varepsilon}(0) \setminus \{0\}$, then

$$\mathcal{T}_1\big[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}(v)\big] = P_{i,k}$$

for some $1 \le k \le k_i$.

In the sequel we will show that

$$\mathrm{freq}(P) = \sum_{i=1}^{l}\sum_{j=1}^{k_i}\mathrm{freq}(P_{i,j}).$$

Lemma 13 *Let $v \in \mathbb{R}^d$ be such that*

$$\mathcal{T}_1\big[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}(v)\big] = P_{i,j}$$

for some $1 \le i \le l$ and $1 \le j \le k_i$. Then there exists a point $w(v) \in B_\epsilon(v)$ verifying $\mathcal{T}_2[B_R(w(v))] = P$ Moreover, if $v' \neq v$, then $w(v') \neq w(v)$ and

$$\sum_{i=1}^{l}\sum_{j=1}^{k_i}\mathrm{freq}(P_{i,j}) \le \mathrm{freq}(P). \tag{3.11}$$

Proof Consider $v \in \mathbb{R}^d$ such that

$$\mathcal{T}_1\big[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}(v)\big] = P_{i,j}$$

for some $1 \le i \le l$ and $1 \le j \le k_i$. Since $\mathcal{T}_1[B_{R+s_0+\eta_2}(v)] = P_i$, we have

$$(\mathcal{T}_1 + v) \cap B_{R+s_0+\eta_2}(0) = (\mathcal{T}_1 + v_i) \cap B_{R+s_0+\eta_2}(0).$$

Thus from Lemma 2 we obtain that there exists $t \in B_\varepsilon(0)$ verifying

$$(\mathcal{T}_2 + v + t) \cap B_{R+\eta_2}(0) = (\mathcal{T}_2 + v_i) \cap B_{R+\eta_2}(0),$$

which implies that $\mathcal{T}_2[B_R(v+t)] = P$. Now, if $v' \in \mathbb{R}^d$ is another point such that

$$\mathcal{T}_1\big[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}\big(v'\big)\big] = P_{i',j'}$$

for some $1 \le i' \le l$ and $1 \le j' \le k'_i$, in a similar way we get that there exists $t' \in B_\varepsilon(0)$ satisfying $\mathcal{T}_2[B_R(v'+t')] = P$. Suppose that $v + t = v' + t'$. This implies that $\|v - v'\| < 2\varepsilon$, i.e., $v - v' \in B_{2\varepsilon}(0)$. But since

$$\begin{aligned} P_{i,j} &= \mathcal{T}_1\big[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}(v)\big], \\ P_i &= \mathcal{T}_1\big[B_{R+s_0+\eta_2}(v)\big], \\ P_{i'} &= \mathcal{T}_1\big[B_{R+s_0+\eta_2}\big(v + \big(v' - v\big)\big)\big], \end{aligned}$$

the condition (3.10) implies that $i' \ge i$. In the same way we obtain that $i' \le i$, which implies $i = i'$. Since $2\varepsilon < \frac{R+s_0}{M}$, we get that $v' - v = 0$. Hence we deduce that it is possible to associate to each v in $\mathbb{R}^d$ which satisfies

$$\mathcal{T}_1\big[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}(v)\big] = P_{i,j}$$

for some $1 \le i \le l$ and $1 \le j \le k_i$, a point $w(v) \in \mathbb{R}^d$ verifying

$$\mathcal{T}_2\big[B_R\big(w(v)\big)\big] = P$$

and such that $w(v) \ne w(v')$ if $v \ne v'$. Thus we deduce that

$$\sum_{i=1}^{l} \sum_{j=1}^{k_i} \mathrm{freq}(P_{i,j}) \le \mathrm{freq}(P).$$

□

Lemma 14 *Let $v \in \mathbb{R}^d$ be such that $\mathcal{T}_2[B_R(v)] = P$. Then there exists a point $p(v) \in B_{(2l+1)\epsilon}(v)$ verifying*

$$\mathcal{T}_1\big[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}\big(p(v)\big)\big] = P_{i,j}$$

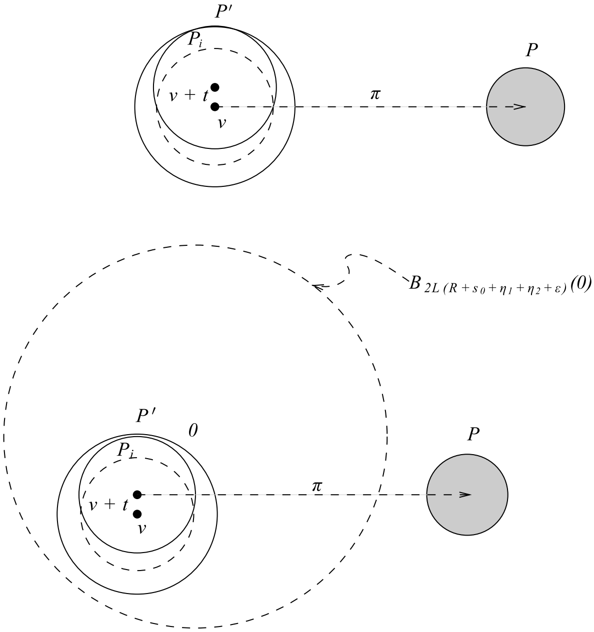
for some $1 \le i \le l$ and $1 \le j \le k_i$. Moreover, if $v' \ne v$, then $p(v') \ne p(v)$ and

$$\sum_{i=1}^{l} \sum_{j=1}^{k_i} \mathrm{freq}(P_{i,j}) \ge \mathrm{freq}(P). \tag{3.12}$$

Proof Let $v \in \mathbb{R}^d$ be such that

$$\mathcal{T}_2\big[B_R(v)\big] = P$$

Fig. 2 Situation before Step 0



and consider

$$P' = \mathcal{T}_1\big[B_{R+s_0+\eta_2+\varepsilon}(v)\big].$$

Since L is the constant of linear recurrence of $\mathcal{T}_1$ and

$$\operatorname{diam}\big(P'\big) \le 2(R+s_0+\eta_2+\varepsilon)+2\eta_1,$$

there exists a translate of P' the support of which is included in the ball

$$B_{2L(R+s_0+\eta_1+\eta_2+\varepsilon)}(0).$$

In other words, there exists $v' \in B_{2L(R+s_0+\eta_1+\eta_2+\varepsilon)}(0)$ such that the support of the patch $\mathcal{T}_1[\![B_{R+s_0+\eta_2+\varepsilon}(v')]\!]$ is contained in the ball $B_{2L(R+s_0+\eta_1+\eta_2+\varepsilon)}(0)$ and satisfies

$$\begin{aligned} P' &= \mathcal{T}_1\big[B_{R+s_0+\eta_2+\varepsilon}\big(v'\big)\big] \\ &= \mathcal{T}_1\big[B_{R+s_0+\eta_2+\varepsilon}(v)\big]. \end{aligned}$$

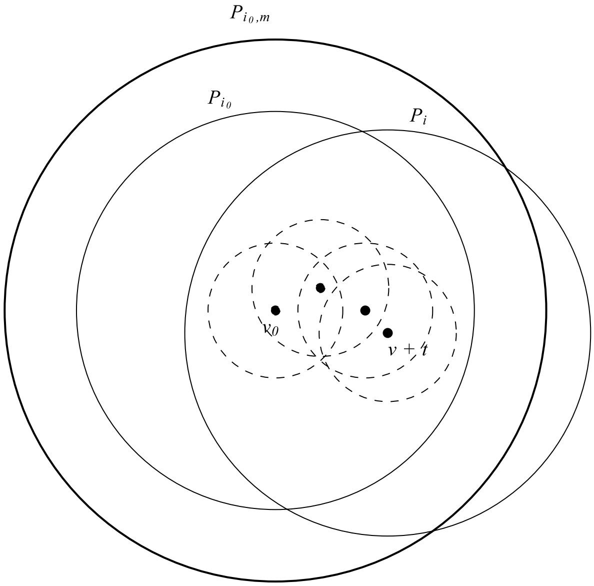
This implies that

$$(\mathcal{T}_1+v)\cap B_{R+s_0+\eta_2}(0) = \big(\mathcal{T}_1+v'\big)\cap B_{R+s_0+\eta_2}(0).$$

So, by Lemma 2 there exists $t \in B_\varepsilon(0)$ verifying

$$\big(\mathcal{T}_2+v'+t\big)\cap B_{R+\eta_2}(0) = (\mathcal{T}_2+v)\cap B_{R+\eta_2}(0).$$

Fig. 3 Situation when the algorithm finished



It follows that $\mathcal{T}_2[B_R(v'+t)] = P$, and, since $v'+t$ is in $B_{L(R+s_0+\eta_1+\eta_2+\varepsilon)}(0)$, we deduce that $v'+t = v_i$ for some $1 \le i \le l$. Because $\mathcal{T}_1[B_{R+s_0+\eta_2}(v'+t)] = P_i$ is included in $\mathcal{T}_1[B_{R+s_0+\eta_2+\varepsilon}(v')] = P'$, we obtain that

$$\mathcal{T}_1\big[B_{R+s_0+\eta_2}(v+t)\big] = P_i.$$

Now, we will show that in the ball $B_{(2l+1)\varepsilon}(v)$ there is a point $p(v)$ such that

$$\mathcal{T}_1\big[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}\big(p(v)\big)\big] = P_{m,j}$$

for some $1 \le m \le l$ and $1 \le j \le k_m$. For that, consider the following algorithm (see Fig. 3):

Step 0: We put $v_0 = v+t$ and $i_0 = i$.

Step 1: We have $\mathcal{T}_1[B_{R+s_0+\eta_2}(v_0)] = P_{i_0}$.

If $\mathcal{T}_1[B_{R+s_0+\eta_2}(v_0+s)] = P_j$ for some $s \in B_{2\varepsilon}(0)$ implies $j \ge i_0$, then from the definition of the patches $P_{i,k}$ we obtain that

$$\mathcal{T}_1\big[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}(v_0)\big] = P_{i_0,m}$$

for some m in $\{1,\dots,k_{i_0}\}$.

Step 2: If there exists $s \in B_{2\varepsilon}(0)$ such that $\mathcal{T}_1[B_{R+s_0+\eta_2}(v_0+s)] = P_j$ with $j < i_0$, then we put

$$i_0 = \min\big\{j : \exists s \in B_{2\varepsilon}(0) \text{ such that } \mathcal{T}_1\big[B_{R+s_0+\eta_2}(v_0+s)\big] = P_j\big\}.$$

If $s \in B_{2\varepsilon}(0)$ is such that $\mathcal{T}_1[B_{R+s_0+\eta_2}(v_0+s)] = P_{i_0}$, then we put $v_0 = v_0+s$. With these new values of v_0 and i_0, we go to Step 1.

This algorithm finishes in at most l steps. The result is a point $p(v)=v_0$ the distance of which to v is at most $(2l+1)\varepsilon$ and such that

$$\mathcal{T}_1\big[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}(v_0)\big]=P_{i_0,m}$$

for some m in $\{1,\dots,k_{i_0}\}$.

If $w\in\mathbb{R}^d$ is another point satisfying $\mathcal{T}_2[B_R(w)]=P$, we have

$$\begin{aligned}\frac{R}{N}&\le\|v-w\|\\&\le\big\|p(v)-v\big\|+\big\|p(v)-p(w)\big\|+\big\|p(w)-w\big\|\\&\le 2(2l+1)\varepsilon+\big\|p(v)-p(w)\big\|.\end{aligned}$$

Thus we get

$$0<\frac{R}{2N}<\frac{R}{N}-2(2l+1)\varepsilon\le\big\|p(v)-p(w)\big\|.$$

This implies that it is possible to associate to each v in $\mathbb{R}^d$ which satisfies $\mathcal{T}_2[B_R(v)]=P$ a point $p(v)\in\mathbb{R}^d$ verifying

$$\mathcal{T}_1\big[B_{R+s_0+\eta_1+\eta_2+2\varepsilon}\big(p(v)\big)\big]=P_{i,j}$$

for some $1\le i\le l$ and $1\le j\le k_i$ and such that $p(v)\ne p(w)$ if $v\ne w$. Hence we deduce that

$$\operatorname{freq}(P)\le\sum_{i=1}^{l}\sum_{j=1}^{k_i}\operatorname{freq}(P_{i,j}).$$

□

From (3.11) and (3.12) we get

$$\operatorname{freq}(P)=\sum_{i=1}^{l}\sum_{j=1}^{k_i}\operatorname{freq}(P_{i,j}).\tag{3.13}$$

As $R>\eta/2$, there exists $k>0$ such that

$$\eta\lambda^{k-2}\le 2(R+s_0+\eta_1+\eta_2+2\varepsilon)<\eta\lambda^{k-1}.\tag{3.14}$$

Since

$$2(R+s_0+\eta_1+\eta_2+2\varepsilon)\le\operatorname{diam}(P_{i,j})\le 2(R+s_0+\eta_1+\eta_2+2\varepsilon)+2\eta_1$$

and $R\ge\eta\lambda^{\lceil\log_\lambda\frac{2\eta_1}{\eta(\lambda-1)}\rceil}$, we have

$$\eta\lambda^{k-2}\le\operatorname{diam}(P_{i,j})<\eta\lambda^{k}.$$

Hence, by Proposition 7, we get

$$\operatorname{freq}(P_{i,j})\in\left\{\frac{f}{\lambda^{dk}},\frac{f}{\lambda^{d(k-1)}}:f\in F\right\},$$

where F is the finite set of Proposition 7. Thus we obtain

$$\mathrm{freq}(P) = \frac{f}{\lambda^{dk}},$$

where f is an element in

$$F' = \left\{ \sum_{i=1}^{K} f_i \ : f_i \in F \cup \lambda^d F, \forall 1 \le i \le K \right\},$$

which is a finite subset of $\mathbb{R}^d$.

Notice that

$$2R \le \mathrm{diam}(P) \le 2(R + \eta_2).$$

Thus from (3.14) we have

$$\eta\lambda^{k-2} - 2(s_0 + \eta_1 + \eta_2 + 2\varepsilon) \le \mathrm{diam}(P) < \eta\lambda^{k-1},$$

and by the choice of R in (3.5), we obtain

$$\eta\lambda^{k-3} \le \mathrm{diam}(P) < \eta\lambda^{k-1}.$$

□

4 Proof of Theorem 1

By Proposition 9, there exist two finite sets F_1 and F_2 such that for $R > 0$ and $P = \mathcal{T}[B_R(0)]$, there exist k_1 and k_2 such that

$$\mathrm{freq}(P) = \frac{f_1}{\lambda_1^{k_1}} = \frac{f_2}{\lambda_2^{k_2}}$$

for some $f_1 \in F_1$ and $f_2 \in F_2$.

Because F_1 and F_2 are finite, we can find $a \in F_1$, $b \in F_2$, $n_2 > n_1$, $m_2 > m_1$ and patches P_1 and P_2 in $\mathcal{T}$ such that

$$\mathrm{freq}(P_1) = \frac{a}{\lambda_1^{n_1}} = \frac{b}{\lambda_2^{m_1}},$$

$$\mathrm{freq}(P_2) = \frac{a}{\lambda_1^{n_2}} = \frac{b}{\lambda_2^{m_2}}.$$

This implies that

$$\lambda_1^{n_2-n_1} = \lambda_2^{m_2-m_1},$$

which means that λ_1 and λ_2 are multiplicatively dependent.

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