Discrete Comput Geom (2009) 42: 640–653
DOI 10.1007/s00454-008-9090-x

More on an Erdős–Szekeres-Type Problem for Interior Points

Xianglin Wei · Ren Ding

Received: 7 December 2007 / Revised: 22 April 2008 / Accepted: 25 May 2008 /
Published online: 20 June 2008

Abstract An interior point of a finite planar point set is a point of the set that is not on the boundary of the convex hull of the set. For any integer $k \geq 1$, let $g(k)$ be the smallest integer such that every planar point set in general position with at least $g(k)$ interior points has a convex subset of points with exactly k interior points of P. In this article, we prove that $g(3) = 9$.

Keywords Interior point · General position · Empty convex polygon · Splitter · Deficient point set

1 Introduction

In this article, we consider only point sets in the plane. All points are assumed to be in *general position*, that is, no three of the points are collinear. The most famous and attractive problem concerning finite point sets is the following Erdős–Szekeres Problem posed in 1935 (see [5–7, 13]): *For any integer $n \geq 3$, determine the smallest positive integer $N(n)$ such that any set of at least $N(n)$ points contains n points that are the vertices of a convex n-gon.* In their first joint paper [6], Erdős and Szekeres proved the existence of the number $N(n)$ by two different methods. In [7] they obtained the lower bound $N(n) \geq 2^{n-2} + 1$ for all $n \geq 3$. $N(3) = 3$ is trivial. The only known nontrivial exact values of $N(n)$ are $N(4) = 5$ (see [4]) and $N(5) = 9$

This research was supported by National Natural Science Foundation of China (10571042, 10701033), NSF of Hebei (A2005000144, A2007000226, A2007000002) and the SF of Hebei Normal University (L2004202).

X. Wei
College of Science, Hebei University of Science and Technology, Shijiazhuang 050018, China
e-mail: wxlhebtu@126.com

X. Wei · R. Ding (✉)
College of Mathematics, Hebei Normal University, Shijiazhuang 050016, China
e-mail: rending@hebtu.edu.cn

(see [11]). For generalizations of Erdős–Szekeres Problem to families of convex sets, see [14] and [15].

In 2001, Avis, Hosono, and Urabe [2] investigated the following question: *For any integer $k \geq 1$, determine the smallest integer $g(k)$ such that every set of points containing at least $g(k)$ interior points has a subset containing exactly k interior points.*

In [2] the authors showed that $g(1) = 1, g(2) = 4$, and $g(3) \geq 8$ and also that every point set with at least k interior points contains a subset with between k and $\lfloor \frac{3k}{2} \rfloor$ interior points, and every point set with at least 3 interior points contains a subset with either 3 or 4 interior points. In 2005, Hosono [9] presented the following result: if a point set has at least 8 interior points and no empty convex hexagons, then it contains a subset with precisely 3 interior points. The progress in the investigation of the lower bound of $g(k)$ for integer k is as follows: In [2] it is proved that $g(k) \geq k+2$ for $k \geq 4$. In [1] it is proved that $g(k) \geq \lceil (3k+3)/2 \rceil$ for $k \geq 6$. Fevens [8] showed that $g(k) \geq 3k - 1$ for $k \geq 3$. Recently, we improved the lower bound by proving that $g(k) \geq 3k$ for $k \geq 3$ (see [17]).

The existence or finiteness of $g(k)$ for any nonnegative integer k is still an open problem. Let $g_{\triangle}(k)$ be the smallest integer such that every set of points whose convex hull is a triangle and which has at least $g_{\triangle}(k)$ interior points also contains a subset with exactly k interior points. By using some results from [12] and [16], Hosono et al. [10] and Bisztriczky et al. [3] proved that if $g_{\triangle}(k)$ is finite, $g(k)$ is also finite for every nonnegative integer k.

In this paper, we discuss the existence of a point subset with exactly 3 interior points of P and obtain that $g(3) = 9$.

2 Definitions and Notation

Let P be a finite planar point set in general position. Let $CH(P)$ denote the convex hull of P, and $V(P) = \{v_1, v_2, \ldots, v_m\}$ be the vertex set of $CH(P)$, or briefly, of P. Here the vertices $v_1, v_2, \ldots, v_m$ are always in counter-clockwise order. An interior point of P is a point of P that is not on the boundary of $CH(P)$. Let $I(P)$ denote the interior point set of P. Let $Q \subset P$ be a subset of P; we say that Q contains an interior point p of P if $p \in I(P) \cap intCH(Q)$, where $intCH(Q)$ is the interior of the convex hull of Q, and we shorten $I(P) \cap intCH(Q)$ to $I^*(Q)$. It is easy to give examples showing that $I(Q) \neq I^*(Q)$. Denote by $i^*(Q) = |I^*(Q)|$ the number of interior points of P contained in Q. When $i^*(Q) = k$, we say that Q is a "*subset with exactly k interior points of P*," or a "*subset with exactly k interior points*" for brevity. If a point set R (a connected region or a finite point set in the plane) contains no interior point of P, we say that R is *empty*, which is denoted by $R \approx \emptyset$.

Using the above notation, $g(k)$ can be defined as follows:

$$g(k) = \min\bigl\{s : |I(P)| \geq s \Rightarrow \exists\, Q \subset P \text{ such that } i^*(Q) = |I(P) \cap intCH(Q)| = k\bigr\}.$$

A point set T is called a triangle and denoted by $T = \triangle v_1 v_2 v_3$ if $V(T) = \{v_1, v_2, v_3\}$. An interior point of a triangle T is called an (x, y, z)-splitter, or a splitter of type (x, y, z) of T if it partitions T into three triangles with $x \geq y \geq z$ interior points, respectively. In the case $z=0$, we abbreviate it to (x, y)-splitter. Let $H(ab; c)$ denote the *half plane* bounded by the straight line ab with c in the interior of the half plane, and $C(a; b, c)$ a *convex cone*, where a is the apex of the cone and the two rays ab and ac form the boundary of the cone. We say that a point set Q is *in convex position*, or *convex*, when it forms the vertex set of a point set.

To investigate the lower bounds for $g(k)$, we need the concept of deficient point sets (see [2]). A point set P is called *a deficient point set of type* $P(m, s, k)$ and denoted as $P = P(m, s, k)$ if $|V(P)| = m$, $|I(P)| = s$, and for any $Q \subset P$, $i^*(Q) \neq k$. For brevity, we may use $P(m, s, k)$ to stand for the deficient set itself. The existence of a deficient point set $P = P(m, s, k)$ implies that $g(k) \geq s + 1$.

For a point set P, an edge of $CH(P)$ is also called an edge of P. If for an edge xy of $P(3, s, 3)$, there exists a subset $Q \subset P$ with $i^*(Q) = k$ such that xy is an edge of Q, we say the edge xy is of *type-k*, where $k \neq 3$ and $k \leq s - 1$.

We call a subset Q of P a *3-int subset* when $i^*(Q) = 3$, that is, when Q contains exactly 3 interior points of P.

According to [1], a configuration of a deficient point set P with certain required properties is called a *monster*.

3 Basic Lemmas

For the proof of $g(3) = 9$, we need the following properties of deficient point sets of type $P(3, s, 3)$ with $4 \leq s \leq 8$, which are proved in [18].

Lemma 1 [18] *A deficient point set* $P = P(3, 4, 3)$ *has the following properties*:

1. *Every edge of* P *is of type*-1 *or of type*-2;
2. P *has at least two edges each of which is of both type*-1 *and type*-2.

$P(3, 5, 3)$ *has the same properties.*

Lemma 2 [18] *Every edge of a deficient point set* $P(3, 6, 3)$ *is of both type*-1 *and type*-2. $P(3, 7, 3)$ *and* $P(3, 8, 3)$ *have the same property.*

In the proof of Lemma 1, we obtain two different configurations of $P(3, 5, 3)$ called a 5-*I-monster* and a 5-*II-monster* and shown in Fig. 1. In a 5-*I-monster,* one

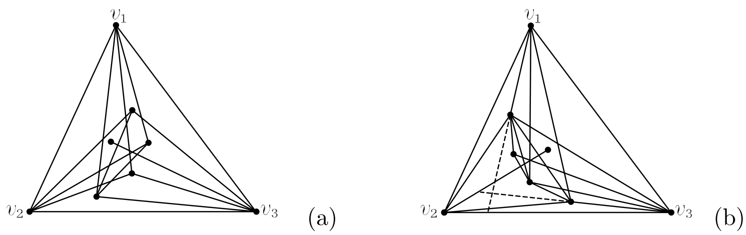

Fig. 1 (**a**) 5-*I-monster*; (**b**) 5-*II-monster*

edge v_1v_3 is of type-2 but not of type-1 (see Fig. 1a). In a 5-*II-monster,* one edge v_1v_3 is of type-1 but not of type-2 (see Fig. 1b).

In the proof of Lemma 2, we obtain three different configurations of $P(3,8,3)$ called an 8-*I-monster*, an 8-*II-monster,* and an 8-*III-monster,* respectively.

We also need the following lemma.

Reduction Lemma [2] *Let P be a planar point set with $m \geq 3$ vertices and $s \geq 0$ interior points. There exists a vertex v_i of P such that $P' = P \setminus \{v_i\}$ has m' vertices, s' interior points, and either (a) $m' = m - 1$ and $s' = s$ or (b) $m' = m + t$ and $s' = s - t - 1$, where $t = 0, 1, 2, \ldots, \lfloor s/m \rfloor$.*

4 The Main Result and Its Proof

Theorem $g(3) = 9$.

The existence of $P(3,8,3)$ (see Fig. 2) shows that $g(3) \geq 9$. Therefore, to prove that $g(3) = 9$, it suffices to prove that $g(3) \leq 9$.

According to the definition of $g(3)$, what we need to prove is that *for any point set P with $|I(P)| \geq 9$, there always exists a 3-int subset of P.*

We split our conclusion into several lemmas. For brevity, in the proofs of the following lemmas, we always assume that every triangle $\triangle$ with $i^*(\triangle) = 4, 5, 6, 7$, or 8 is a deficient point set (or monster), and do not consider (x, y)-splitters with $x = 3$ or $y = 3$, since otherwise we obtain a 3-int subset, and we are done.

Lemma 3 *If $|I(P)| = 9$ and $|V(P)| = 3$, then P has a 3-int subset.*

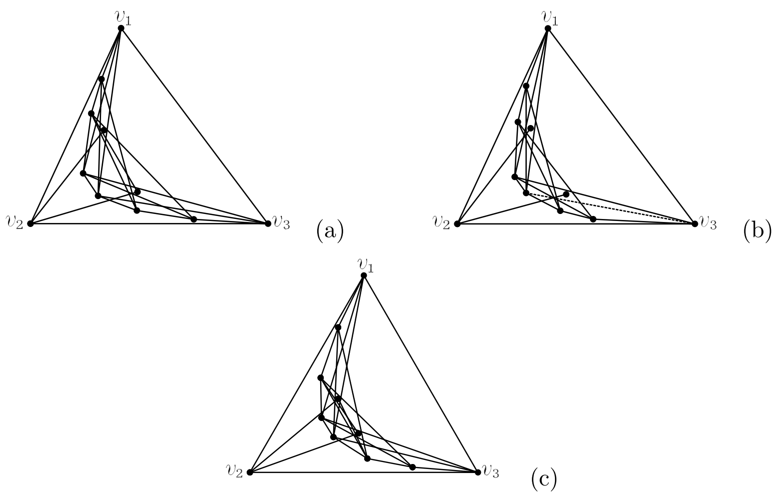

Fig. 2 (**a**) 8-*I-monster*; (**b**) 8-*II-monster*; (**c**) 8-*III-monster*

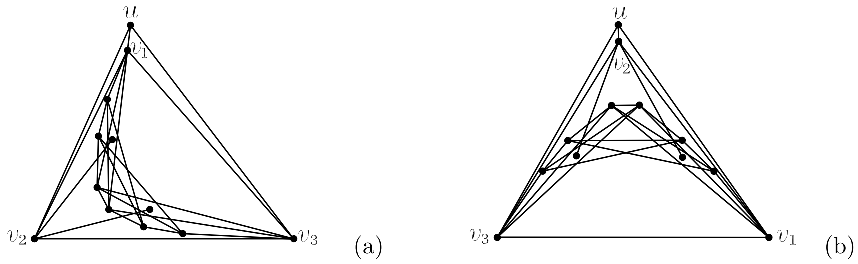

Fig. 3 $\triangle v_1v_2v_3$ is an 8-II-monster in Fig. 2: (**a**) v_1 is an $(8,0)$-splitter; (**b**) v_2 is an $(8,0)$-splitter

Proof It suffices to consider (x,y)-splitters of P of types $(8,0)$, $(7,1)$, $(6,2)$, and $(4,4)$.

Case 1 P has an $(8,0)$-splitter.

Let $\triangle v_1v_2v_3$ be an 8-monster as shown in Fig. 2. We may assume that $\triangle v_1v_2v_3$ is an 8-II-monster. The arguments for 8-I-monster and 8-III-monster are similar. Let $V(\triangle v_1v_2v_3 \setminus \{v_2\}) = \{v_1, p_1, p_2, p_3, p_4, p_5, p_6, v_3\}$ and $I^*(\triangle v_1v_2v_3 \setminus \{v_2\}) = \{q_1, q_2\}$, where $q_1 \in C(v_2; p_2, p_3)$ and $q_2 \in C(v_2; p_4, p_5)$.

(a) v_1 is an $(8,0)$-splitter, $V(P) = \{u, v_2, v_3\}$, and $i^*(\triangle uv_2v_3) = 9$. See Fig. 3a, in which $\triangle v_1v_2v_3$ is an 8-II-monster as shown in Fig. 2. Let $r_u \in I^*(\triangle uv_2v_3)$ be such that $C(u; v_2, r_u) \approx \emptyset$.

If $r_u = v_1$, then $I^*(\triangle uv_2q_1) = \{v_1, p_1, p_2\}$. If $r_u = p_1$, then $I^*(\triangle uv_2v_3 \setminus \{v_2\}) = \{v_1, q_1, q_2\}$. If $r_u = p_2$, then $I^*(\{u, p_2, q_2, v_3\}) = \{v_1, q_1, p_1\}$. Next suppose that $r_u = p_3$. If $v_1 \in H(q_2u; v_3)$, then $I^*(\{u, p_3, q_2, v_1\}) = \{p_1, p_2, q_1\}$. Otherwise $I^*(\triangle up_2q_2) = \{v_1, p_1, q_1\}$. Now assume that $r_u = p_4$. If $p_1 \in H(q_2u; v_3)$, then $I^*(\{u, p_4, q_2, p_1\}) = \{p_2, p_3, q_1\}$. Otherwise, if $v_1 \in H(q_2u; v_3)$, then $I^*(\triangle up_3q_2) = \{p_1, p_2, q_1\}$; if $v_1 \in H(q_2u; v_2)$, then $I^*(\triangle up_2q_2) = \{v_1, p_1, q_1\}$. Lastly consider the case where $r_u = p_5$ or p_6. If $q_1 \in H(p_2u; v_2)$, then $I^*(\triangle uq_1v_3) = \{v_1, p_1, p_2\}$. Otherwise, if $q_1 \in C(u; p_2, p_1)$, then $I^*(\triangle up_5q_1) = \{p_2, p_3, p_4\}$; if $q_1 \in C(u; p_1, v_1)$, then $I^*(\triangle up_4q_1) = \{p_1, p_2, p_3\}$; if $q_1 \in H(uv_1; v_3)$, then $I^*(\triangle up_3q_1) = \{v_1, p_1, p_2\}$.

(b) v_3 is an $(8,0)$-splitter, $V(P) = \{u, v_1, v_2\}$, $i^*(\triangle uv_1v_2) = 9$. $\triangle v_1v_2v_3$ is an 8-II-monster as shown in Fig. 2. The proof is similar to that of (a).

(c) v_2 is a $(8,0)$-splitter, $V(P) = \{u, v_3, v_1\}$, and $i^*(\triangle uv_3v_1) = 9$. See Fig. 3b in which $\triangle v_1v_2v_3$ is an 8-II-monster as shown in Fig. 2.

If $v_2 \in H(p_3u; v_1)$, then $I^*(\triangle up_3v_1) = \{v_2, p_1, p_2\}$. So we can assume that $v_2 \in H(p_3u; v_3)$. If $v_2 \in H(p_6u; v_1)$, then $I^*(\triangle up_6p_3) = \{v_2, p_4, p_5\}$. Otherwise $I^*(\triangle uv_3p_4) = \{v_2, p_5, p_6\}$.

Case 2 P has a $(7,1)$-splitter but no $(8,0)$-splitter.

Let $V(P) = \{v_1, v_2, v_3\}$. Assume that u_1 is a $(7,1)$-splitter, where $i^*(\triangle u_1v_2v_3) = 7$, $i^*(\triangle v_1u_1\ v_3) = 1$ with $I^*(\triangle v_1u_1v_3) = \{u_2\}$, and $u_2 \in H(v_2u_1; v_3)$. If u_2 is a $(4,4)$-splitter of $\triangle v_1v_2v_3$, then $i^*(\triangle v_2u_2u_1) = 3$. If u_2 is a $(6,2)$-splitter of $\triangle v_1v_2v_3$, then by Lemma 2, there exists a 3-int subset. For example, suppose that $i^*(\triangle v_2v_3u_2) = 6$, then by Lemma 2, the edge v_2u_2 is of type-2 in $\triangle v_2v_3u_2$, that is, there exists a subset Q with v_2u_2 as an edge such that $i^*(Q) = 2$, as shown in the

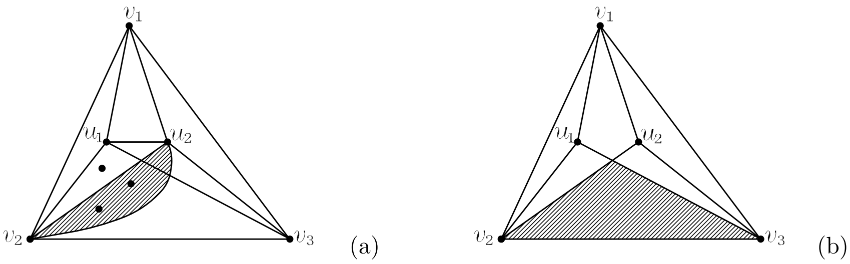

Fig. 4 (**a**) The convex hull of the union of the shaded region and u_1 contains 3 interior points; (**b**) Both $\triangle u_1v_2v_3$ and $\triangle u_2v_2v_3$ are 7-monsters

shaded region of Fig. 4a, and $i^*(Q \cup \{u_1\}) = 3$. If u_2 is a (7, 1)-splitter of $\triangle v_1v_2v_3$, suppose that $i^*(\triangle v_1v_2u_2) = 7$, then by Lemma 2, there exists a 3-int subset. So assume that $i^*(\triangle u_2v_2v_3) = 7$. Thus both $\triangle u_1v_2v_3$ and $\triangle u_2v_2v_3$ are 7-monsters. See Fig. 4b.

Let $w_1 \in I^*(\triangle v_2v_3u_2)$ be such that $C(v_2; v_3, w_1) \approx \emptyset$, where w_1 is a $(6,0)$-, $(5,1)$- or $(4,2)$-splitter of $\triangle u_2v_2v_3$, and w_1 is a $(7,1)$-, $(6,2)$-, or $(4,4)$-splitter of $\triangle v_1v_2v_3$.

Assume that $w_1 \in H(v_1u_1; v_2)$ (for $w_1 \in C(v_1; u_1, u_2)$ or $w_1 \in H(v_1u_2; v_3)$, the proof is similar).

Suppose that w_1 is a $(6,0)$-splitter of $\triangle u_2v_2v_3$. If $i^*(\triangle u_2w_1v_3) = 6$, then w_1 is an $(8,0)$-splitter of $\triangle v_1v_2v_3$, contradicts to our assumption. If $i^*(\triangle u_2v_2w_1) = 6$, then by Lemma 2, we can find a 3-int subset.

Suppose that w_1 is a $(5,1)$-splitter of $\triangle u_2v_2v_3$. If w_1 is a $(7,1)$- or $(6,2)$-splitter of $\triangle v_1v_2v_3$, then $i^*(\triangle v_1w_1v_3) = 7$ or 6, and we can find a 3-int subset as before. So we only need to consider the case where w_1 is a $(4,4)$-splitter of $\triangle v_1v_2v_3$. Then $i^*(\triangle u_2v_2w_1) = 5$ and $i^*(\triangle u_2w_1v_3) = 1$, say $I^*(\triangle u_2w_1v_3) = \{w_2\}$. If $w_2 \in H(v_1u_2; v_2)$, then applying Lemma 2 to $\triangle u_2v_2v_3$, we can find a 3-int subset of P. So assume that $w_2 \in H(v_1u_2; v_3)$. Then $i^*(\triangle v_1w_1w_2) = 3$.

Suppose that w_1 is a $(4,2)$-splitter of $\triangle u_2v_2v_3$. As before, if w_1 is a $(7,1)$- or $(6,2)$-splitter of $\triangle v_1v_2v_3$, then $i^*(\triangle v_1w_1v_3) = 7$ or 6, and we can find a 3-int subset. So we only need to consider the case where w_1 is a $(4,4)$-splitter of $\triangle v_1v_2v_3$. Then $i^*(\triangle u_2v_2w_1) = 4$ and $i^*(\triangle u_2w_1v_3) = 2$, say $I^*(\triangle u_2w_1v_3) = \{w_2, w_3\}$. We may assume that at least one of w_2, w_3, say $w_2 \in H(v_1u_2; v_3)$, since otherwise by applying Lemma 2 to $\triangle u_2v_2v_3$ we can find a 3-int subset. If $w_3 \in I^*(\triangle v_1w_1w_2)$, then $i^*(\triangle v_1w_1w_2) = 3$. If $w_3 \notin I^*(\triangle v_1w_1w_2)$, then $i^*(\triangle v_1w_1w_2) = 2$ and $i^*(\triangle v_1w_1u_2) = 1$. By property (1) of Lemma 1, the edge v_1w_1 is of type-1 or type-2 in $\triangle v_1v_2w_1$, that is, there exists a subset Q with v_1w_1 as an edge such that $i^*(Q) = 1$ or 2. Thus $i^*(Q \cup \{w_2\}) = 3$ or $i^*(Q \cup \{u_2\}) = 3$.

Case 3 P has a $(6,2)$-splitter, but no $(8,0)$-splitter and no $(7,1)$-splitter.

Let $V(P) = \{v_1, v_2, v_3\}$. Let u_1 be a $(6,2)$-splitter, where $i^*(\triangle u_1v_2v_3) = 6$ and $i^*(\triangle v_1u_1\ v_3) = 2$, and 2 interior points of $\triangle v_1u_1v_3$ lie in $H(v_2u_1; v_3)$. Let $u_2 \in I^*(\triangle v_1u_1v_3)$ such that $I^*(\triangle v_1u_2v_3) = \emptyset$, where u_2 is a $(1,0)$-splitter of $\triangle v_1u_1v_3$. We can assume that $i^*(\triangle v_1u_1u_2) = 1$ with $I^*(\triangle v_1u_1u_2) = \{u\}$, and u lies in $H(v_3u_2; v_2)$, since otherwise by applying Lemma 2 to $\triangle u_1v_2v_3$ we obtain a 3-int

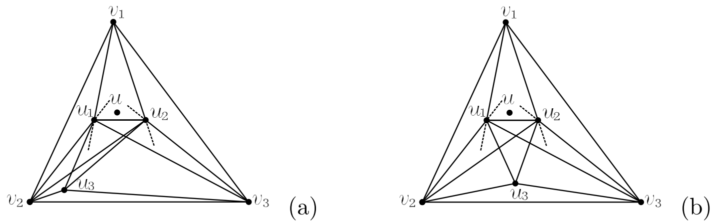

Fig. 5 $i^*(\triangle u_1v_2v_3) = 6$, and $I^*(\triangle u_3v_2v_3) = \emptyset$

subset. See Fig. 5. Let $u_3 \in I^*(\triangle v_2v_3u_2)$ be such that $C(v_2; v_3, u_3) \approx \emptyset$, where u_3 is a $(5,0)$-splitter or $(4,1)$-splitter of $\triangle u_1v_2v_3$.

(a) $u_3 \in H(v_1u_1; v_2)$ (see Fig. 5a).

Assume that u_3 is a $(4,1)$-splitter of $\triangle u_1v_2v_3$. If $i^*(\triangle u_1u_3v_3) = 4$, then u_3 is an $(8,0)$- or $(7,1)$-splitter of $\triangle v_1v_2v_3$, contradicting the assumption. So assume that $i^*(\triangle u_1v_2u_3) = 4$ and $i^*(\triangle u_1u_3v_3) = 1$ with $I^*(\triangle u_1u_3v_3) = \{w\}$. First suppose that $w \in H(u_3u_2; u_1)$. If u_2 is a $(4,4)$-splitter of $\triangle v_1v_2v_3$, then $i^*(\triangle v_2u_3u_2) = 3$. If u_2 is a $(6,2)$-splitter of $\triangle v_1v_2v_3$, then by Lemma 2, P has a 3-int subset. Secondly suppose that $w \in H(u_3u_2; v_3)$. If u_3 is a $(6,2)$-splitter of $\triangle v_1v_2v_3$, then $i^*(\triangle v_1u_3v_3) = 6$, and hence $i^*(\triangle v_1u_3u) = 3$. If u_3 is a $(4,4)$-splitter of $\triangle v_1v_2v_3$, then applying property (1) of Lemma 1 to $\triangle v_1v_2u_3$, the edge v_1u_3 is of type-1 (or type-2) in $\triangle v_1v_2u_3$, that is, there exists a subset Q with v_1u_3 as an edge such that $i^*(Q) = 1$ (or 2) and $i^*(Q \cup \{u_2\}) = 3$ (or $i^*(Q \cup \{u\}) = 3$).

Assume that u_3 is a $(5,0)$-splitter of $\triangle u_1v_2v_3$. Then $i^*(\triangle u_1v_2u_3) = 5$, since otherwise u_3 is a $(8,0)$-splitter of $\triangle v_1v_2v_3$. If u_3 is a $(4,4)$-splitter of $\triangle v_1v_2v_3$, then $i^*(\triangle v_1u_3u_2) = 3$. If u_3 is a $(6,2)$-splitter of $\triangle v_1v_2v_3$, then $i^*(\triangle v_1u_3v_3) = 6$, and hence $i^*(\triangle u_1v_1u_3) = 3$.

(b) $u_3 \in H(v_1u_1; v_3)$ (see Fig. 5b).

Assume that u_3 is a $(4,1)$-splitter of $\triangle u_1v_2v_3$. Then $i^*(\triangle u_1v_2u_3) = 1$, say $I^*(\triangle u_1v_2u_3) = \{w\}$, since otherwise $i^*(\{v_1, u_1, u_3, v_3\}) = 3$. If $w \in H(v_1u_1; v_3)$, then applying Lemma 2 to $\triangle u_1v_2v_3$, we obtain a 3-int subset. So we can assume that $w \in H(v_1u_1; v_2)$. If w is a $(4,4)$-splitter of $\triangle v_1v_2v_3$, then $i^*(\triangle wu_3v_3) = 3$. If w is a $(6,2)$-splitter of $\triangle v_1v_2v_3$, then $i^*(\triangle v_1wv_3) = 6$, and hence $i^*(\triangle u_1wv_3) = 3$.

Assume that u_3 is a $(5,0)$-splitter of $\triangle u_1v_2v_3$. Then $i^*(\triangle u_1v_2u_3) = 5$, since otherwise by applying Lemma 2 to $\triangle u_1v_2v_3$ we obtain a 3-int subset. If u_2 is a $(4,4)$-splitter of $\triangle v_1v_2v_3$, then $i^*(\triangle v_2u_3u_2) = 3$. If u_2 is a $(6,2)$-splitter, and if $u_3 \in H(v_1u_2; v_2)$, then due to Lemma 2, there exists a 3-int subset; and if $u_3 \in H(v_1u_2; v_3)$, then u_3 is an $(8,0)$-splitter of $\triangle v_1v_2v_3$, a contradiction to our assumption.

Case 4 Every (x, y)-splitter is a $(4,4)$-splitter.

Let $V(P) = \{v_1, v_2, v_3\}$. Let u_1 be a $(4,4)$-splitter, where $i^*(\triangle v_1u_1v_3) = 4$ and $i^*(\triangle u_1v_2\ v_3) = 4$. No point lies in $H(v_2u_1; v_1)$, and no point lies in $H(v_1u_1; v_2)$. Let $u_2 \in I^*(\triangle v_1u_1v_3)$ such that $I^*(\triangle v_1u_2v_3) = \emptyset$, where u_2 is a $(2,1)$-splitter of $\triangle v_1u_1v_3$. No point lies in $H(v_3u_2; v_1)$, and no point lies in $H(v_1u_2; v_3)$. Let

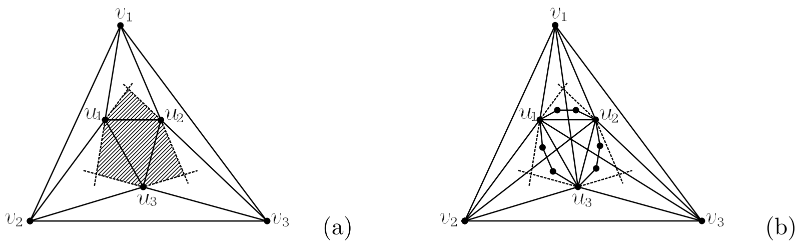

Fig. 6 (**a**) The points of $I(P)$ lie in the *shaded region*; (**b**) $I(P)$ is a convex 9-gon

$u_3 \in I^*(\triangle u_1v_2v_3)$ be such that $I^*(\triangle u_3v_2v_3) = \emptyset$, where u_3 is a $(2,1)$-splitter of $\triangle u_1v_2v_3$. No point lies in $H(v_3u_3; v_2)$, and no point lies in $H(v_2u_3; v_3)$. See Fig. 6a. So we may assume that either $i^*(\triangle u_1u_3v_3) = i^*(\triangle u_1v_3u_2) = 1$ or $i^*(\triangle u_1u_3v_3) = i^*(\triangle u_1v_3u_2) = 2$, since otherwise $i^*(\{u_1, u_3, v_3, u_2\}) = 3$.

(a) $i^*(\triangle u_1u_3v_3) = i^*(\triangle u_1v_3u_2) = 2$.
Then $i^*(\triangle v_1u_1u_2) = i^*(\triangle v_2u_3u_1) = 1$. If $i^*(\triangle u_1u_3u_2) = 4$, then $i^*(\{v_1, u_1, w, u_2\}) = 3$ or $i^*(\{v_2, u_3, w, u_1\}) = 3$, where $w \in I^*(\triangle u_1u_3u_2)$ such that $I^*(\triangle wu_3u_2) = \emptyset$, and w is a $(2,1)$-splitter of $\triangle u_1u_3u_2$. If $i^*(\triangle u_1u_3u_2) = 3$, then we are done. If $i^*(\triangle u_1u_3u_2) = 2$, then $i^*(\{v_1, u_1, u_3, u_2\}) = 3$. If $i^*(\triangle u_1u_3u_2) = 1$, then $i^*(\triangle u_3v_3u_2) = 3$. If $i^*(\triangle u_1u_3u_2) = 0$, then $i^*(\{v_1, v_2, u_3, u_2\}) = 3$.

(b) $i^*(\triangle u_1u_3v_3) = i^*(\triangle u_1v_3u_2) = 1$.
Then $i^*(\triangle v_1u_1u_2) = i^*(\triangle v_2u_3u_1) = 2$. If $i^*(\triangle u_1u_3u_2) = 2$, then $i^*(\{v_1, u_1, w, u_2\}) = 3$ or $i^*(\{v_2, u_3, w, u_1\}) = 3$, where $w \in I^*(\triangle u_1u_3u_2)$ such that $I^*(\triangle wu_3u_2) = \emptyset$, and w is a $(1,0)$-splitter of $\triangle u_1u_3u_2$. If $i^*(\triangle u_1u_3u_2) = 1$, then $i^*(\{v_1, u_1, u_3, u_2\}) = 3$. If $i^*(\triangle u_1u_3u_2) = 0$, then $I(P)$ is a convex 9-gon, see Fig. 6b, since otherwise there exists a 3-int subset. Then it is easy to see that P has a 3-int subset. □

Lemma 4 *If* $|I(P)| = 9$ *and* $|V(P)| \geq 4$, *then* P *has a* 3*-int subset.*

Proof Let $V(P) = \{v_1, v_2, \dots, v_m\}$, where $m = |V(P)|$. Triangulate P by joining vertex v_1 to each of the other vertices in $V(P)$, and we obtain $m - 2$ triangles. If there exists a triangle $\triangle$ such that $i^*(\triangle) = 3$ or 9, we are done. If there exists a triangle $\triangle$ such that $i^*(\triangle) = 6, 7$, or 8, due to Lemma 2, there exists a 3-int subset. If for any triangle $\triangle$, $i^*(\triangle) \leq 4$, then due to Lemma 1 it is easy to verify that we may find a 3-int subset of P by concatenating a set of adjacent triangles. If one triangle contains 5 interior points and each of the other triangles contains at most 2 interior points, then by Lemma 1, it is easy to verify that we may find a 3-int subset. So we can assume that $i^*(\triangle v_1v_iv_{i+1}) = 5$ and $i^*(\triangle v_1v_jv_{j+1}) = 4$, and the other remaining triangles contain no interior point, where $i + 1 \leq j$.

Case 1 The two triangles $\triangle v_1v_iv_{i+1}$ and $\triangle v_1v_jv_{j+1}$ are adjacent.

Without loss of generality, we may assume that $i^*(\triangle v_1v_2v_3) = 5$ and $i^*(\triangle v_1v_3v_4) = 4$. By Lemma 1, it suffices to consider the case where $\triangle v_1v_2v_3$ is a 5-I-monster, that is, the edge v_1v_3 is of type-2 but not of type-1 in $\triangle v_1v_2v_3$, or

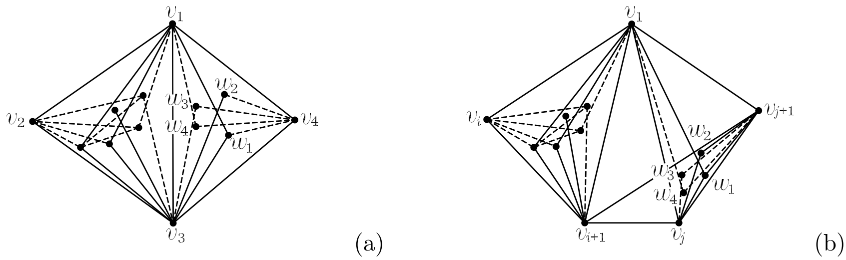

Fig. 7 $\triangle v_1v_2v_3$ is a 5-I-monster

a 5-II-monster, that is, the edge v_1v_3 is of type-1 but not of type-2 in $\triangle v_1v_2v_3$. If $i^*(\triangle v_1v_2v_4) \geq 6$ or ≤ 3, then by Lemma 3 or Lemma 2, P has a 3-int subset. So assume that $i^*(\triangle v_1v_2v_4) = 4$ or 5.

(a) $\triangle v_1v_2v_3$ is a 5-I-monster. See Fig. 1a.

Rotate the edge v_1v_3 counter clockwise with v_3 as center in $\triangle v_1v_2v_3$, and label the points the edge v_1v_3 meets consecutively by a_1, a_2, a_3, a_4, a_5.

Let $w_1 \in I^*(\triangle v_1v_3v_4)$ such that $C(v_3; v_4, w_1) \approx \emptyset$, where w_1 is a (2, 1)-splitter of $\triangle v_1v_3v_4$. If $i^*(\triangle v_1v_3w_1) = 1$, then $i^*(\{v_1, a_3, v_3, w_1\}) = 3$. So assume that $i^*(\triangle v_1w_1v_4) = 1$ with $I^*(\triangle v_1w_1v_4) = \{w_2\}$ and $i^*(\triangle v_1v_3w_1) = 2$ with $I^*(\triangle v_1v_3w_1) = \{w_3, w_4\}$. Then $\{v_1, v_3, w_4, w_3\}$ is convex, since otherwise $I^*(\{v_1, a_3, v_3, w_3\}) = \{a_1, a_2, w_4\}$ or $I^*(\{v_1, a_3, v_3, w_4\}) = \{a_1, a_2, w_3\}$. Suppose that $w_3, w_4 \in H(v_3w_2; v_1)$, since otherwise $I^*(\{v_1, a_3, v_3, w_2\}) = \{a_1, a_2, w_3\}$ or $I^*(\triangle v_3v_4w_2) = \{w_1, w_3, w_4\}$. Then $w_3, w_4 \in C(v_4; w_1, w_2)$, since otherwise $I^*(\triangle v_3v_4w_3) = \{w_1, w_2, w_4\}$ or $I^*(\triangle v_1w_4v_4) = \{w_1, w_2, w_3\}$. See Fig. 7a.

Suppose that $a_3 \in H(v_2v_4; v_1)$ and $w_4 \in H(v_2v_4; v_3)$. If $a_2 \in H(a_3w_4; v_3)$, then $I^*(\{v_2, v_3, w_4, a_3\}) = \{a_2, a_4, a_5\}$. Otherwise $I^*(\{v_1, a_3, w_4, w_2\}) = \{a_1, a_2, w_3\}$. Suppose that $a_3 \in H(v_2v_4; v_3)$ and $w_4 \in H(v_2v_4; v_1)$. If $w_3, w_4 \in H(a_2w_2; v_1)$, then $I^*(\{v_1, a_3, a_2, w_2\}) = \{a_1, w_3, w_4\}$; if $w_3, w_4 \in H(a_2w_2; v_3)$, then $I^*(\{v_3, v_4, w_2, a_2\}) = \{w_1, w_3, w_4\}$; otherwise $I^*(\{v_1, v_2, a_2, w_2\}) = \{a_1, a_3, w_3\}$. Suppose that $a_3, w_4 \in H(v_2v_4; v_1)$. Then $I^*(\triangle v_1v_2v_4) = \{a_1, a_3, w_2, w_3, w_4\}$. If $w_3, w_4 \in H(a_1w_1; v_1)$, then $I^*(\{v_1, a_1, w_1, v_4\}) = \{w_2, w_3, w_4\}$; if $w_3, w_4 \in H(a_1w_1; v_3)$, then $I^*(\{a_1, a_3, v_3, w_1\}) = \{a_2, w_3, w_4\}$; otherwise $I^*(\{a_1, a_5, v_3, w_1\}) = \{a_2, a_4, w_4\}$. Suppose that $a_3, w_4 \in H(v_2v_4; v_3)$. Then $i^*(\triangle v_2v_3v_4) \geq 6$, and we are done.

(b) $\triangle v_1v_2v_3$ is a 5-II-monster (see Fig. 1b).

Rotate the edge v_1v_3 counter clockwise with v_3 as center in $\triangle v_1v_2v_3$, and label the points the edge v_1v_3 meets consecutively by a_1, a_2, a_3, a_4, a_5.

Let $w_1 \in I^*(\triangle v_1v_3v_4)$ be such that $C(v_3; v_4, w_1) \approx \emptyset$, where w_1 is a (2, 1)-splitter of $\triangle v_1v_3v_4$. Then assume that $i^*(\triangle v_1v_3w_1) = 1$ with $I^*(\triangle v_1v_3w_1) = \{w_2\}$ and $i^*(\triangle v_1w_1v_4) = 2$ with $I^*(\triangle v_1w_1v_4) = \{w_3, w_4\}$, since otherwise $i^*(\{v_1, a_1, a_5, v_3, w_1\}) = 3$. Suppose that $|V(\triangle v_1v_3v_4 \setminus \{v_4\})| = 5$ with $V(\triangle v_1v_3v_4 \setminus \{v_4\}) = \{v_1, v_3, w_1, w_4, w_3\}$, since otherwise there exists a 3-int subset. Suppose that $w_2 \in H(v_3w_3; v_1)$, since otherwise $I^*(\triangle v_3v_4w_3) = \{w_1, w_2, w_4\}$. See Fig. 8a.

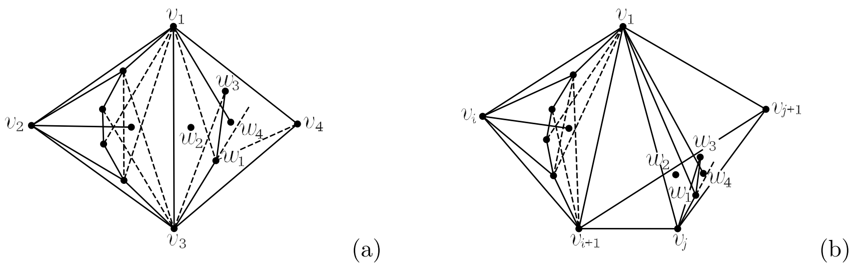

Fig. 8 $\triangle v_1v_2v_3$ is a 5-II-monster

When $a_2, w_4 \in H(v_2v_4; v_1)$, $I^*(\{a_2, v_2, v_3, w_1\}) = \{a_4, a_5, w_2\}$ or $I^*(\{v_1, a_2, w_1, v_4\}) = \{w_2, w_3, w_4\}$. When $a_2, w_4 \in H(v_2v_4; v_3)$, $I^*(\{v_1, v_2, a_2, w_3\}) = \{a_1, a_3, w_2\}$ or $I^*(\{a_2, v_3, v_4, w_3\}) = \{w_1, w_2, w_4\}$. When $a_2 \in H(v_2v_4; v_3)$ and $w_4 \in H(v_2v_4; v_1)$, $I^*(\{v_1, v_2, a_2, w_3\}) = \{a_1, a_3, w_2\}$ or $I^*(\{a_2, v_3, v_4, w_3\}) = \{w_1, w_2, w_4\}$. When $a_2 \in H(v_2v_4; v_1)$ and $w_4 \in H(v_2v_4; v_3)$, $I^*(\{v_1, a_2, w_1, v_4\}) = \{w_2, w_3, w_4\}$ or $I^*(\{a_2, v_2, v_3, w_1\}) = \{w_2, a_4, a_5\}$.

Case 2 The two triangles $\triangle v_1v_iv_{i+1}$ and $\triangle v_1v_jv_{j+1}$ are not adjacent.

Due to Lemma 1, it suffices to consider the case where $\triangle v_1v_iv_{i+1}$ is a 5-I-monster or a 5-II-monster. Connect v_{i+1} and v_{j+1}. If $i^*(\triangle v_1v_{i+1}v_{j+1}) \geq 1$, then the conclusion is correct as before. So suppose that $i^*(\triangle v_{i+1}v_jv_{j+1}) = 4$. Rotate the edge v_1v_{i+1} counter clockwise with v_{i+1} as center in $\triangle v_1v_iv_{i+1}$, and label the points the edge v_1v_{i+1} meets consecutively by a_1, a_2, a_3, a_4, a_5.

(a) $\triangle v_1v_iv_{i+1}$ is a 5-I-monster.

The proof is similar to that of Case 1(a), and we label the four interior points of $\triangle v_1v_jv_{j+1}$ as shown in Fig. 7b. Then $I^*(\{v_1, a_3, v_{i+1}, w_3, v_{j+1}\}) = \{a_1, a_2, w_2\}$.

(b) $\triangle v_1v_iv_{i+1}$ is a 5-II-monster.

The proof is similar to that of Case 1(b), and we label the four interior points of $\triangle v_1v_jv_{j+1}$ as shown in Fig. 8b. If $w_2 \in H(w_1v_{i+1}; v_1)$, then $I^*(\triangle v_{i+1}w_1v_{j+1}) = \{w_2, w_3, w_4\}$. Otherwise $I^*(\{v_1, a_1, a_5, v_{i+1}, w_1, v_{j+1}\}) = \{a_2, w_3, w_4\}$. □

Lemma 5 *If* $|I(P)| = 10$ *and* $|V(P)| = 3$, *then* P *has a* 3-*int subset.*

Proof Let $V(P) = \{v_1, v_2, v_3\}$. If P has a $(9,0)$-splitter, then by Lemma 3, we are done. So it remains to consider (x, y)-splitters of P of types (8, 1), or (7, 2), or (5, 4).

Case 1 P has an (8, 1)-splitter.

The proof is similar to that of Case 1 of Lemma 3.

Case 2 P has a (7, 2)-splitter but no (8, 1)-splitter.

Let u_1 be a $(7,2)$-splitter, where $i^*(\triangle u_1v_2v_3) = 7$, and $i^*(\triangle v_1u_1v_3) = 2$ with $\triangle v_1u_1v_3) = \{u_2, u_3\}$, $u_2, u_3 \in H(v_2u_1; v_3)$. Let $u_2 \in I^*(\triangle v_1u_1v_3)$ be such that $I^*(\triangle v_1u_2v_3) = \emptyset$, where u_2 is a $(1,0)$-splitter of $\triangle v_1u_1v_3$. Assume that $i^*(\triangle v_1u_1u_2) = 1$ with $I^*(\triangle v_1u_1u_2) = \{u_3\}$, and u_3 lies in $H(v_3u_2; u_1)$, since otherwise by applying Lemma 2 to $\triangle u_1v_2v_3$ we obtain a 3-int subset.

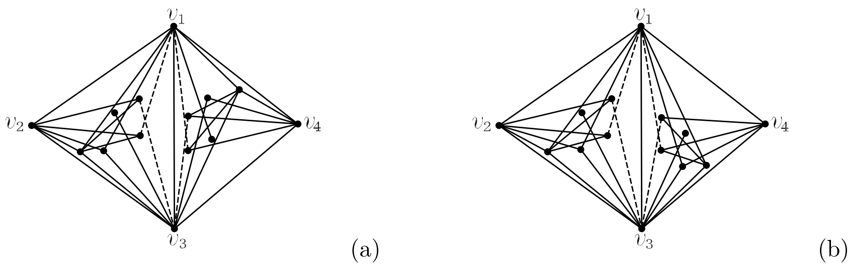

Fig. 9 Two triangles are adjacent, both $\triangle v_1v_2v_3$ and $\triangle v_1v_3v_4$ are 5-I-monsters

(a) u_2 is a $(5,4)$-splitter of $\triangle v_1v_2v_3$.
The proof is similar to that in the previous discussion and is omitted.
(b) u_2 is a $(7,2)$-splitter of $\triangle v_1v_2v_3$.

If $i^*(\triangle v_1v_2u_2)=7$, then applying Lemma 2 to $\triangle v_1v_2u_2$ we obtain a 3-int subset. So we may assume that $i^*(\triangle v_2v_3u_2)=7$. Thus both $\triangle u_1v_2v_3$ and $\triangle u_2v_2v_3$ are 7-monsters. As in Case 2 of Lemma 3, it is easy to prove that P has a 3-int subset.

Case 3 Every (x,y)-splitter is a $(5,4)$-splitter.

Let u_1 be a $(5,4)$-splitter, where $i^*(\triangle u_1v_2v_3)=5$, $i^*(\triangle v_1u_1v_3)=4$, and 4 interior points of $\triangle v_1u_1v_3$ lie in $H(v_2u_1;v_3)$. By Lemma 1, it suffices to consider the case where $\triangle u_1v_2v_3$ is a 5-I-monster or a 5-II-monster. As in Case 1 of Lemma 4, we can obtain a 3-int subset. □

Lemma 6 *If* $|I(P)|=10$ *and* $|V(P)|\geq 4$, *then* P *has a* 3*-int subset.*

Proof Let $V(P)=\{v_1,v_2,\ldots,v_m\}$, where $m=|V(P)|$. Similarly to Lemma 4, we need only to consider the case where there are two triangles, each of which contains 5 interior points.

Case 1 The two triangles are adjacent.

Without loss of generality, we assume that $i^*(\triangle v_1v_2v_3)=i^*(\triangle v_1v_3v_4)=5$. Due to Lemma 1, it suffices to consider the cases where both $\triangle v_1v_2v_3$ and $\triangle v_1v_3v_4$ are 5-I-monsters or 5-II-monsters. If $i^*(\triangle v_1v_2v_4)\geq 6$ or ≤ 4, then by Lemmas 5, 3, or 2, P has a 3-int subset. So assume that $i^*(\triangle v_1v_2v_4)=5$, and hence $i^*(\triangle v_2v_3v_4)=5$ as well.

(a) Both $\triangle v_1v_2v_3$ and $\triangle v_1v_3v_4$ are 5-I-monsters.
Then the edge v_1v_3 is of type-2 but not of type-1 in $\triangle v_1v_2v_3$, and the edge v_1v_3 is of type-2 but not of type-1 in $\triangle v_1v_3v_4$. We obtain two configurations as shown in Fig. 9.
In Fig. 9a, for $\triangle v_1v_2v_3$, we rotate the edge v_1v_3 counter clockwise with v_3 as center and label the points the edge v_1v_3 meets consecutively by a_1,a_2,a_3,a_4,a_5; and for $\triangle v_1v_3v_4$, rotate the edge v_1v_4 counter clockwise with v_4 as center and label the points the edge v_1v_4 meets consecutively by a_6,a_7,a_8,a_9,a_0.
Suppose that $a_1,a_9\in H(v_2v_4;v_3)$. Then $i^*(\triangle v_2v_3v_4)\geq 7$, and we are done. Suppose that $a_1,a_9\in H(v_2v_4;v_1)$. Then $I^*(\triangle v_1v_2v_4)=\{a_1,a_6,a_7,a_8,a_9\}$. If $a_0\in H(a_2a_9;v_3)$, then $I^*(\{a_2,a_9,v_1,v_2\})=\{a_1,a_3,a_8\}$. Otherwise

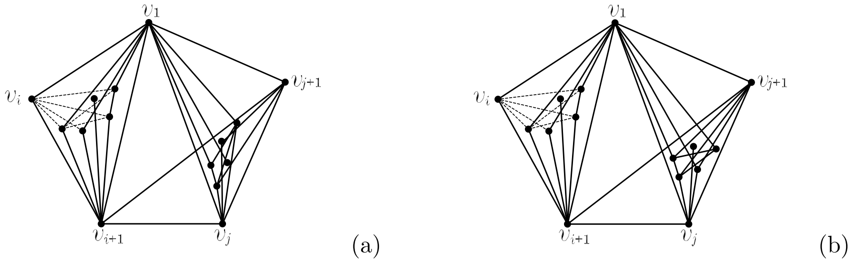

Fig. 10 Two triangles are not adjacent, both $\triangle v_1v_2v_3$ and $\triangle v_1v_3v_4$ are 5-I-monsters

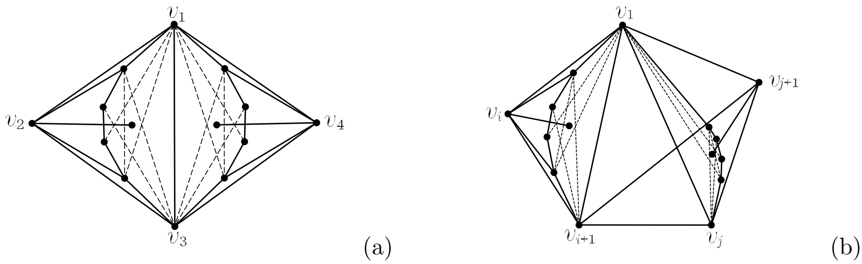

Fig. 11 Both $\triangle v_1v_2v_3$ and $\triangle v_1v_3v_4$ are 5-II-monsters

$I^*(\{v_1, a_3, a_2, a_9\}) = \{a_1, a_0, a_8\}$. Next, suppose that $a_1 \in H(v_2v_4; v_1)$ and $a_9 \in H(v_2v_4; v_3)$. If $a_8 \in H(a_1a_9; v_1)$, then $I^*(\{v_1, a_1, a_9, v_4\}) = \{a_6, a_7, a_8\}$. Otherwise $I^*(\{a_1, a_4, v_3, a_9\}) = \{a_2, a_0, a_8\}$. Lastly suppose that $a_1 \in H(v_2v_4; v_3)$ and $a_9 \in H(v_2v_4; v_1)$. Then $I^*(\triangle v_1v_2v_4) = \{a_6, a_7, a_8, a_9, a_0\}$. If $a_1 \in H(a_3a_8; v_3)$, then $I^*(\{a_3, v_3, a_9, a_8\}) = \{a_1, a_2, a_0\}$. Otherwise $I^*(\{a_3, v_3, v_4, a_8\}) = \{a_2, a_0, a_9\}$.

The proof for the case of Fig. 9b is similar to that of Fig. 9a, and we can obtain a subset $Q \subset P$ such that $i^*(Q) = 3$.

(b) Both $\triangle v_1v_2v_3$ and $\triangle v_1v_3v_4$ are 5-II-monsters.

Then the edge v_1v_3 is of type-1 but not of type-2 in $\triangle v_1v_2v_3$, and the edge v_1v_3 is of type-1 but not of type-2 in $\triangle v_1v_3v_4$ too. See Fig. 11a. By an argument similar to that of Fig. 9a, we can find a subset $Q \subset P$ such that $i^*(Q) = 3$.

Case 2 The two triangles are not adjacent.

Without loss of generality, we assume that $i^*(\triangle v_1v_iv_{i+1}) = 5$ and $i^*(\triangle v_1v_jv_{j+1}) = 5$, where $i + 1 < j$. Due to Lemma 1, it suffices to consider the cases where both $\triangle v_1v_iv_{i+1}$ and $\triangle v_1v_jv_{j+1}$ are 5-I-monsters (or 5-II-monsters), as in Case 1. Connect v_{i+1} and v_{j+1}. If $i^*(\triangle v_1v_{i+1}v_{j+1}) \geq 1$, the conclusion is correct as before, and we are done. So assume that $i^*(\triangle v_{i+1}v_jv_{j+1}) = 5$.

(a) Both $\triangle v_1v_iv_{i+1}$ and $\triangle v_1v_jv_{j+1}$ are 5-I-monsters.

Then, as in Case 1 (a), we obtain two configurations as shown in Fig. 10. It is easy to find 3-int subsets.

(b) Both $\triangle v_1v_iv_{i+1}$ and $\triangle v_1v_jv_{j+1}$ are 5-II-monster.

Then as shown in Fig. 11b, it is clear that P has a 3-int subset. □

Lemma 7 *If* $|I(P)| = 11$ *and* $|V(P)| = 3$, *then* P *has a* 3-*int subset.*

Proof Let $V(P) = \{v_1, v_2, v_3\}$. If P has a (10, 0)- or (9, 1)-splitter, then by Lemma 5 or Lemma 3, we are done. So we need only to consider (x, y)-splitters of P of types $(8, 2)$ or $(6, 4)$, or $(5, 5)$. In each case, we obtain a 3-int subset. The proof is simple on the basis of the previous discussion and is omitted. □

Lemma 8 *If* $|I(P)| = 11$ *and* $|V(P)| \geq 4$, *then* P *has a* 3-*int subset.*

Proof Let $|V(P)| = m$ and $|I(P)| = s$. If $m = 4$ or 5, it is easy to prove that P has a 3-int subset as before. If $m \geq 6$, then the Reduction Lemma implies the existence of a proper subset $P' \subset P$ with $9 \leq i^*(P') < |I(P)|$: if (a) of the Reduction Lemma holds, we have a proper subset $P' \subset P$ with $i^*(P') = 10$; and if (b) of the Reduction Lemma holds, we have a proper subset $P' \subset P$ with $i^*(P') = 11 - \lfloor \frac{11}{m} \rfloor - 1 \geq 9$. Then by using the previous results we obtain a subset $Q \subset P'$ with $i^*(Q) = 3$, and hence P has a 3-int subset. □

Lemma 9 *If* $|I(P)| \geq 12$ *and* $|V(P)| \geq 3$, *then* P *has a* 3-*int subset.*

Proof Let $|V(P)| = m$ and $|I(P)| = s$. Consider the case $s = 12$. If $m = 3$ or 4, due to Lemmas 1 and 2, it is easy to prove the conclusion. For $m \geq 5$, we may apply the Reduction Lemma and an argument similar to that in Lemma 8. In the case $s = 13$, the proof is similar to that of the case $s = 12$. Finally, if $m \geq 3$ and $s \geq 14$, we have $s - \lfloor \frac{s}{m} \rfloor - 1 \geq 9$, and so the result follows by using the Reduction Lemma and the similar argument as above. □

Combining Lemmas 3–9, we finish the proof for $g(3) \leq 9$ and reach the conclusion that $g(3) = 9$ at last.

References

1. Avis, D., Hosono, K., Urabe, M.: On the existence of a point subset with 4 or 5 interior points. In: JCDCG 1998. LNCS, vol. 1763, pp. 57–64 (2000)
2. Avis, D., Hosono, K., Urabe, M.: On the existence of a point subset with a specified number of interior points. Discrete Math. **241**, 33–40 (2001)
3. Bisztriczky, T., Hosono, K., Károlyi, G., Urabe, M.: Constructions from empty polygons. Period. Math. Hung. **49**, 1–8 (2004)
4. Bonnice, W.E.: On convex polygons determined by a finite planar set. Am. Math. Mon. **81**, 749–752 (1974)
5. Brass, P., Moser, W., Pach, J.: Research Problems in Discrete Geometry. Springer, Berlin (2005)
6. Erdős, P., Szekeres, G.: A combinatorial problem in Geometry. Compos. Math. **2**, 463–470 (1935)
7. Erdős, P., Szekeres, G.: On some extremum problems in elementary geometry. Ann. Univ. Sci. Bp. **3–4**, 53–62 (1960–1961)
8. Fevens, T.: A note on point subset with a specified number of interior points. In: JCDCG 2002. LNCS, vol. 2866, pp. 152–158 (2003)
9. Hosono, K.: On the existence of a convex point subset containing one triangle in the plane. Discrete Math. **305**, 201–218 (2005)
10. Hosono, K., Károlyi, G., Urabe, M.: On the existence of a convex polygon with a specified number of interior points. In: Discrete Geometry, In Honor of W. Kuperberg's 60th Birthday, pp. 351–358. Dekker, New York (2003)

11. Kalbfleisch, J.D., Kalbfleisch, J.G., Santon, R.G.: A combinatorial problem on convex n-gons. In: Proc. Louisiana Conf. Comb. Graph Theory and Computing, pp. 180–188. Baton Rouge (1970)
12. Károlyi, G., Pach, J., Tóth, G.: A modular version of the Erdős–Szekeres theorem. Stud. Sci. Math. Hung. **38**, 245–259 (2001)
13. Morris, W., Soltan, V.: The Erdős–Szekeres problem on point in convex position-a survey. Bull. New Ser. Am. Math. Soc. **37**, 437–458 (2000)
14. Pach, J., Tóth, G.: A generalization of the Erdős–Szekeres theorem to disjoint convex sets. Discrete Comput. Geom. **19**, 437–445 (1998)
15. Pach, J., Tóth, G.: Erdős–Szekeres-type theorems for segments and non-crossing convex sets. Geom. Dedic. **81**, 1–12 (2000)
16. Valtr, P.: A sufficient condition for the existence of large empty convex polygons. Discrete Comput. Geom. **28**, 671–682 (2002)
17. Wei, X., Ding, R.: More on planar point subsets with a specified number of interior points. Math. Not. **83**, 684–687 (2008) (in English). Mat. Zametki **83**, 752–756 (2008) (in Russian)
18. Wei, X., Ding, R.: An Erdős–Szekeres type problem for interior points. In: Surveys on Discrete and Computational Geometry. Contemporary Mathematics, vol. 453, pp. 515–528 (2008)