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Asymmetry of Convex Polytopes and Vertex Index of Symmetric Convex Bodies

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Abstract In (Gluskin, Litvak in Geom. Dedicate 90:45–48, 2002) it was shown that a polytope with few vertices is far from being symmetric in the Banach–Mazur distance. More precisely, it was shown that Banach–Mazur distance between such a polytope and any symmetric convex body is large. In this note we introduce a new, averaging-type parameter to measure the asymmetry of polytopes. It turns out that, surprisingly, this new parameter is still very large, in fact it satisfies the same lower bound as the Banach–Mazur distance. In a sense it shows the following phenomenon: *if a convex polytope with small number of vertices is as close to a symmetric body as it can be, then most of its vertices are as bad as the worst one*. We apply our results to provide a lower estimate on the vertex index of a symmetric convex body, which was recently introduced in (Bezdek, Litvak in Adv. Math. 215:626–641, 2007). Furthermore, we give the affirmative answer to a conjecture by Bezdek (Period. Math. Hung. 53:59–69, 2006) on the quantitative illumination problem.

Keywords Asymmetry of polytopes · Illumination parameter · Measures of symmetry · Polytopes with few vertices · Vertex index

1 Introduction

Let $\mathbf{K}$ be a convex body in $\mathbb{R}^d$. How asymmetric is $\mathbf{K}$? More precisely, how far is $\mathbf{K}$ from being centrally symmetric? Of course, such a question requires a functional,

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which measures asymmetry, or a distance between convex bodies. One of the natural ways to introduce such a measure is the following functional

$$d_{\mathbf{K}} := \inf_{a\in\mathbf{K}} \sup_{x\in\mathbf{K}-a} \|-x\|_{\mathbf{K}-a},$$

where $\|\cdot\|_{\mathbf{K}}$ denotes *the Minkowski functional* of $\mathbf{K}$ or *the gauge* of $\mathbf{K}$ (see the definitions below). This functional is closely related (in fact, is equivalent to) the Banach–Mazur distance between $\mathbf{K}$ and the set of all centrally symmetric bodies in $\mathbb{R}^d$. We refer to [7] for related discussion on this and other ways to measure the asymmetry of a given convex body in $\mathbb{R}^d$.

In the present note we deal with convex d-polytopes, i.e. with convex polytopes whose interiors are not empty. Let $\mathbf{K}$ be such a polytope and $\{x_i\}_{i=1}^m$ be its vertices. It is easy to see that $d_{\mathbf{K}} = \max_i \|-(x_i - a)\|_{\mathbf{K}-a}$ for some $a \in \mathbf{K}$. Thus, the functional $d_{\mathbf{K}}$ takes into account only one, the worst, vertex of $\mathbf{K}$. We suggest another, averaging-type functional to measure the asymmetry of convex polytopes, namely we define

$$\delta_{\mathbf{K}} := \inf_{a\in\mathbf{K}} \frac{1}{m} \sum_{i=1}^{m} \big\|-(x_i - a)\big\|_{\mathbf{K}-a}.$$

Thus, our functional measures how bad vertices are in average ("x is bad" here means that $\|-x\|_{\mathbf{K}}$ is big, which in turn means that $-x$ is far away from the body). Clearly, $\delta_{\mathbf{K}} \le d_{\mathbf{K}}$. Our main result, Theorem 3.2, states:

Let $1 \le k \le d$ and $m = d + k$. Let $\mathbf{K}$ be a convex d-polytope in $\mathbb{R}^d$ with m vertices $x_1, x_2, \dots, x_m$. Then

$$\delta_{\mathbf{K}} \ge \frac{m}{2k}.$$

It shows that a convex d-polytope cannot be centrally symmetric unless it has at least $2d$ vertices and provides a quantitative lower bound. The bound should be compared with the main result from [5], which states that $d_{\mathbf{K}} \ge d/k$ and which is sharp (i.e., for every d, k there exists a convex d-polytope in $\mathbb{R}^d$ with $d + k$ vertices such that $d_{\mathbf{K}} \le \lceil d/k \rceil$, where $\lceil a \rceil$ denotes the smallest integer larger than or equal to a). Thus, in general, the lower bound for the *worst vertex* is almost the same as the lower bound for the *average vertex*. In other words, Theorem 3.2 discovers the following (high-dimensional) phenomenon: *if a convex polytope with small number of vertices is as close to a symmetric body as it can be, then most of its vertices are as bad as the worst one*.

We apply our results to provide sharp lower bounds for the vertex index of a centrally symmetric convex body $\mathbf{K}$, vein($\mathbf{K}$), and for the illumination parameter of a centrally symmetric convex body $\mathbf{K}$, ill($\mathbf{K}$) (see the precise definitions in Sect. 4). Both quantities are closely connected to some important quantities and problems in asymptotic theory of normed spaces and in convex geometry, including the problem of covering a convex body with smaller, homothetic copies of itself. In particular, the illumination parameter of a convex body was introduced in [1] as a cost function for the Boltyanski–Hadwiger illumination conjecture. We refer the interested reader to [2–4, 8] and references therein for more details.

K. Bezdek conjectured (see Problem 6.3 of [3]) that for every centrally symmetric d-dimensional body $\mathbf{K}$ one has

$$\mathrm{ill}(\mathbf{K}) \geq 2d.$$

Applying our results on the asymmetry of polytopes, in Theorem 4.1 we prove that for every d-dimensional centrally symmetric convex body $\mathbf{K}$ one has

$$\mathrm{vein}(\mathbf{K}) \geq 2d.$$

As $\mathrm{ill}(\mathbf{K}) \geq \mathrm{vein}(\mathbf{K})$, it proves Bezdek's conjecture.

2 Preliminaries and Notation

By $|\cdot|$ and $\langle\cdot,\cdot\rangle$ we denote the canonical Euclidean norm and the canonical inner product on $\mathbb{R}^d$. The canonical basis of $\mathbb{R}^d$ we denote by $e_1, \dots, e_d$. Given points $x_1, \dots, x_k$ in $\mathbb{R}^d$ we denote their convex hull by $\mathrm{conv}\{x_i\}_{i\leq k}$.

Let $\mathbf{K} \subset \mathbb{R}^d$ be a compact convex body with nonempty interior such that $0 \in \mathbf{K}$. We denote by $\mathbf{K}^\circ$ the polar of $\mathbf{K}$, i.e.,

$$\mathbf{K}^\circ = \big\{x \mid \langle x, y\rangle \leq 1 \text{ for every } y \in \mathbf{K}\big\}.$$

In particular, if 0 is on the boundary of $\mathbf{K}$, then $\mathbf{K}^\circ$ is unbounded star-shaped convex set. We will also use that if E is a linear subspace of $\mathbb{R}^d$, then the polar of $\mathbf{K} \cap E$ (taken in E) is

$$(\mathbf{K} \cap E)^\circ = P_E \mathbf{K}^\circ,$$

where P_E is the orthogonal projection onto E.

The Minkowski functional (or the gauge) of a convex body $\mathbf{K}$ containing 0 in its interior is defined as

$$\|x\|_{\mathbf{K}} = \inf\big\{\lambda > 0 \mid x \in \lambda\mathbf{K}\big\}.$$

Note that if $\mathbf{K}$ is centrally symmetric with respect to the origin, then $\|\cdot\|_{\mathbf{K}}$ is a norm on $\mathbb{R}^d$ and $\mathbf{K}$ is its unit ball. Below it will be convenient to consider the Minkowski functional even for unbounded closed star-shaped sets as well as for sets containing 0 as a boundary point under usual agreement $\inf\emptyset = \infty$. In particular, it can happen that $\|x\|_{\mathbf{K}} = 0$ for $x \neq 0$ or that $\|x\|_{\mathbf{K}} = \infty$.

Given real number a by $[a]$ we denote the largest integer not exceeding a and by $\lceil a\rceil$ we denote the smallest integer which is not smaller than a.

We also will use the following proposition, which is a combination of Weil's Theorem ([10], see also Theorem 2.3.1 of [6]) and Stinespring's result ([9], see also (3.7.8) in [6]). For the reader's convenience we outline the proof.

Proposition 2.1 *Let $T = \{t_{ij}\}$ be an $m \times m$ matrix and $\lambda_1, \lambda_2, \dots, \lambda_m$ be eigenvalues of T. Then*

$$\sum_{j=1}^{m} |\lambda_j| \leq \sum_{i,j=1}^{m} |t_{ij}|.$$

Proof Recall that singular numbers of T are defined as

$$s_j(T) = \lambda_j\big((T^*T)^{1/2}\big),$$

where $\lambda_j(A)$ denotes jth largest eigenvalue of A (counted according to multiplicities). Weil's Theorem says that

$$\sum_{j=1}^{m} \big|\lambda_j(T)\big| \le \sum_{j=1}^{m} s_j(T).$$

It is well known that $\gamma(T) := \sum_{j=1}^{m} s_j(T)$ is a norm (see, e.g., Theorem 3.7.1 in [6]). Therefore,

$$\gamma(T) \le \sum_{i,j=1}^{m} |t_{ij}|\gamma(E_{ij}),$$

where E_{ij} denotes the matrix with zero entries except one element in ith row, jth column, which is 1. It is easy to see that $\gamma(E_{ij}) = 1$ for every i and j, which completes the proof. □

3 Asymmetry of Polytopes

Lemma 3.1 *Let $\Lambda = \{\lambda_{ij}\}$ be an $m \times m$ matrix of rank k with nonnegative entries such that $\lambda_{ii} \ge 1$ for every $i \le m$. Then*

$$\sum_{i,j=1}^{m} \lambda_{ij} \ge 3m - 2k.$$

Moreover, if $m \ge 2k$, then

$$\sum_{i,j=1}^{m} \lambda_{ij} \ge m + \frac{m(m-1)}{2k-1}.$$

Remark Note that the estimates of Lemma 3.1 are asymptotically sharp. Indeed, consider a block-diagonal matrix with k blocks $[m/k] \times [m/k]$ or $\lceil m/k \rceil \times \lceil m/k \rceil$ of rank one, such that each block has entries of 1 only. Then we have

$$\sum_{i,j=1}^{m} \lambda_{ij} \le \frac{m^2}{k} + \frac{k}{4}.$$

Proof of Lemma 3.1 First let us note that without loss of generality we can assume that $\lambda_{ii} = 1$ for every i (otherwise we pass to the matrix $\{\lambda_{ij}/\lambda_{ii}\}_{ij}$).

Consider $T = \Lambda - I$, where I is the identity and denote its entries by t_{ij}. Clearly, $t_{ij} \ge 0$ and $t_{ii} = 0$ for every i, j. By λ_j denote the eigenvalues of T.

Since Λ is of rank k, at least $m-k$ of eigenvalues of T are equal to -1 (indeed, $T=-I$ on $\operatorname{Ker}\Lambda$). Since

$$0=\sum_{i=1}^{m} t_{ii}=\operatorname{Trace} T=\sum_{i=1}^{m}\lambda_i,$$

we obtain

$$\sum_{i=1}^{m}|\lambda_i|\geq 2m-2k.$$

Proposition 2.1 implies

$$\sum_{i,j=1}^{m} t_{ij}\geq 2m-2k,$$

which shows

$$\sum_{i,j=1}^{m}\lambda_{ij}\geq 3m-2k.$$

For $m<2k$ it proves the result.

Now we assume that $m\geq 2k$. Let $\sigma\subset\{1,2,\dots,m\}$ be of cardinality l for some $2k\leq l\leq m$. Let

$$\bar{\Lambda}=\{\lambda_{ij}\}_{i,j\in\sigma}.$$

Clearly the rank of $\bar{\Lambda}$ does not exceed k, so, by the first part, we have

$$\sum_{i,j\in\sigma}\lambda_{ij}\geq 3l-2k.$$

Using averaging argument, we obtain

$$\begin{aligned}\sum_{i,j=1}^{m}\lambda_{ij}&=m+\sum_{\substack{i,j=1\\ i\neq j}}^{m}\lambda_{ij}=m+\binom{m-2}{l-2}^{-1}\sum_{\substack{\sigma\subset\{1,2,\dots,m\}\\ |\sigma|=l}}\sum_{\substack{i,j\in\sigma\\ i\neq j}}\lambda_{ij}\\ &\geq m+\binom{m-2}{l-2}^{-1}\binom{m}{l}(2l-2k)=m+2\frac{m(m-1)}{l(l-1)}(l-k).\end{aligned}$$

The choice $l=2k$ completes the proof. □

Theorem 3.2 *Let $1\leq k\leq d$ and $m=d+k$. Let* $\mathbf{K}$ *be a convex d-polytope in $\mathbb{R}^d$ with m vertices $x_1,x_2,\dots,x_m$. Then*

$$\sum_{i=1}^{m}\|-x_i\|_{\mathbf{K}}\geq\frac{m^2}{2k}\geq\max\left\{2d,\frac{md}{2k}\right\}.$$

Remark 1 Since a shift of a convex d-polytope with m vertices is still a convex d-polytope with m vertices, Theorem 3.2 implies immediately

$$\delta_{\mathbf{K}} \geq \frac{m}{2k}.$$

Remark 2 As was noticed in [5], the estimate is asymptotically sharp. Geometrically one can consider the following example. Let

$$\mathbb{R}^d = \bigoplus_{i=1}^{k} E_i,$$

where E_i's are orthogonal coordinate subspaces of $\mathbb{R}^d$ of dimension $[d/k]$ or $\lceil d/k \rceil$. In each E_i consider the regular simplex, denoting its vertices by x_{ij}, $j \leq \dim E_i + 1$ and set $\mathbf{K} = \operatorname{conv}\{x_{ij}\}_{ij}$. Clearly,

$$\|-x_{ij}\| = \dim E_i.$$

Therefore

$$\frac{1}{m}\sum_{i,j} \|-x_{ij}\| \leq \max_{i,j} \|-x_{ij}\| = \max_i \dim E_i = \lceil d/k \rceil,$$

which means $\delta_{\mathbf{K}} \leq d_{\mathbf{K}} \leq \lceil d/k \rceil$.

Proof of Theorem 3.2 Consider the linear operator $T : \mathbb{R}^m \longrightarrow \mathbb{R}^d$ defined by $Te_i = x_i$. Denote the kernel of T by L. Clearly, L is a k-dimensional subspace of $\mathbb{R}^m$. The orthogonal projection onto $L^{\perp}$ we denote by P.

Since $\mathbf{K}^{\circ} = \{f \in \mathbb{R}^d \mid \langle f, x_j \rangle \leq 1 \text{ for every } j \leq m\}$, we have

$$A := \sum_{i=1}^{m} \|-x_i\|_{\mathbf{K}} = \sum_{i=1}^{m} \sup\bigl\{\langle f, -x_i \rangle \mid f \in \mathbb{R}^d, \langle f, x_j \rangle \leq 1 \text{ for every } j \leq m\bigr\}.$$

Using $\langle f, x_i \rangle = \langle f, Te_i \rangle = \langle T^* f, e_i \rangle$, we obtain

$$A = \sum_{i=1}^{m} \sup\bigl\{\langle h, -e_i \rangle \mid h \in \mathbb{R}^m, h \in L^{\perp}, \langle h, e_j \rangle \leq 1 \text{ for every } j \leq m\bigr\}.$$

Now denote

$$S := \bigl\{h \in \mathbb{R}^m \mid \langle h, e_j \rangle \leq 1 \text{ for every } j \leq m\bigr\},$$

and for every $i \leq m$ denote

$$Q_i := \bigl\{h \in \mathbb{R}^m \mid \langle h, e_i \rangle \geq -1\bigr\}.$$

Then

$$S^{\circ} = \left\{h \in \mathbb{R}^m \mid 0 \leq \langle h, e_j \rangle \text{ for every } j \leq m, \text{ and } \sum_{j=1}^{m} \langle h, e_j \rangle \leq 1\right\},$$

and

$$Q_i^\circ = \{h \in \mathbb{R}^m \mid -1 \le \langle h, e_i\rangle \le 0,\ \langle h, e_j\rangle = 0 \ \text{ for } \ j \ne i\}.$$

Therefore

$$\|z\|_{S^\circ} := \begin{cases} \sum_{j=1}^m \langle z, e_j\rangle & \text{if } \langle z, e_j\rangle \ge 0 \text{ for every } j \le m, \\ \infty & \text{otherwise,} \end{cases}$$

which implies

$$\begin{aligned} \|z\|_{PS^\circ} &= \inf_{y\in L} \|z+y\|_{S^\circ} \\ &= \inf\left\{\sum_{j=1}^m \langle z+y, e_j\rangle \;\middle|\; y \in L, \langle y, e_j\rangle \ge -\langle z, e_j\rangle \text{ for every } j \le m\right\}. \end{aligned}$$

Using duality and our notation, we observe

$$\begin{aligned} A &= \sum_{i=1}^m \sup_{h\in S\cap L^\perp} \langle h, -e_i\rangle = \sum_{i=1}^m \sup_{h\in S\cap L^\perp} \|h\|_{Q_i} \\ &= \sum_{i=1}^m \sup_{h\in Q_i^\circ} \|h\|_{PS^\circ} = \sum_{i=1}^m \|-e_i\|_{PS^\circ} \\ &= \sum_{i=1}^m \inf\left\{\sum_{j=1}^m \langle y, e_j\rangle - 1 \;\middle|\; y \in L, \langle y, e_i\rangle \ge 1, \langle y, e_j\rangle \ge 0 \text{ for every } j \le m\right\}. \end{aligned}$$

Assume that for every $i \le m$ the latter infimum attains on $y_i \in L$. Denoting $y_{ij} := \langle y_i, e_j\rangle$, we observe that $y_{ij} \ge 0$ and $y_{ii} \ge 1$ for every $i \le m$, $j \le m$, and that the matrix $\{y_{ij}\}$ has rank at most k. Since $m = d + k \ge 2k$, applying Lemma 3.1, we obtain

$$A = \sum_{i=1}^m \sum_{j=1}^m y_{ij} - m \ge \frac{m(m-1)}{2k-1} \ge \frac{m^2}{2k}.$$

This completes the proof. □

4 A Lower Bound on the Vertex Index

In this section we apply our results to provide a sharp lower estimate for the vertex index of a centrally symmetric convex body and, in particular, to prove a conjecture of K. Bezdek on the lower bound for the illumination parameter of a centrally symmetric convex body.

The vertex index of a centrally symmetric (with respect to the origin) convex body $\mathbf{K}$, introduced in [4], is defined as

$$\operatorname{vein}(\mathbf{K}) = \inf\left\{\sum_i \|p_i\|_{\mathbf{K}} \;\middle|\; \mathbf{K} \subset \operatorname{conv}\{p_i\}_i\right\}.$$

In other words, given $\mathbf{K} = -\mathbf{K}$ one looks for the convex polytope that contains $\mathbf{K}$ and whose vertex set has the smallest possible closeness to 0 in the metric generated by $\mathbf{K}$.

The illumination parameter of a centrally symmetric convex body $\mathbf{K}$ was introduced in [1] as a cost function for the Boltyanski–Hadwiger illumination conjecture in the following way

$$\mathrm{ill}(\mathbf{K}) = \inf\left\{\sum_i \|p_i\|_{\mathbf{K}} \,\Big|\, \{p_i\}_i \subset \mathbb{R}^d \text{ illuminates } \mathbf{K}\right\},$$

where "illuminates" means that for every q on the boundary of $\mathbf{K}$ there exists a point p_i such that the ray starting at p_i and passing through q intersects the interior of $\mathbf{K}$ (after the point q).

Let us note that both the vertex index and the illumination parameter are affine invariants of $\mathbf{K}$; that is, if $T : \mathbb{R}^d \to \mathbb{R}^d$ is an invertible linear map, then $\mathrm{vein}(\mathbf{K}) = \mathrm{vein}(T(\mathbf{K}))$ and $\mathrm{ill}(\mathbf{K}) = \mathrm{ill}(T(\mathbf{K}))$. It is also easy to see that $\mathrm{vein}(\mathbf{K}) \le \mathrm{ill}(\mathbf{K})$. Thus both parameters are closely related and, as we mentioned in the introduction, they are closely connected to some important quantities and problems in asymptotic theory of normed spaces and in convex geometry.

In [3] K. Bezdek conjectured (see Problem 6.3) that for every centrally symmetric d-dimensional body $\mathbf{K}$ one has

$$\mathrm{ill}(\mathbf{K}) \ge 2d.$$

The next theorem answers in affirmative the Bezdek's conjecture and provides the lower bound on the vertex index of the centrally symmetric convex bodies.

Theorem 4.1 *Let* $\mathbf{K}$ *be* d*-dimensional centrally symmetric* (*with respect to the origin*) *convex body. Then*

$$\mathrm{ill}(\mathbf{K}) \ge \mathrm{vein}(\mathbf{K}) \ge 2d.$$

Proof As we mentioned earlier the left-hand side inequality is simple. We now show the right-hand side inequality.

Let $\mathbf{K} \subset \mathbf{L} = \mathrm{conv}\{p_i\}_{i \le m}$. Without loss of generality we can assume that $\|p_i\|_K \ge 1$ for every i. If $m \ge 2d$, then we trivially have

$$\sum_{i=1}^m \|p_i\|_{\mathbf{K}} \ge m \ge 2d.$$

Assume $m < 2d$. Since $\mathbf{K} = -\mathbf{K} \subset \mathbf{L}$, we have $\|-x\|_{\mathbf{L}} \le \|x\|_{\mathbf{K}}$ for every $x \in \mathbb{R}^d$. Therefore, applying Theorem 3.2, we obtain

$$\sum_{i=1}^m \|p_i\|_{\mathbf{K}} \ge \sum_{i=1}^m \|-p_i\|_{\mathbf{L}} \ge 2d,$$

which completes the proof. □

Remark Let $\mathbf{K}$ be a cross-polytope in $\mathbb{R}^d$. Since $\mathbf{K}$ has $2d$ vertices we have $\mathrm{ill}(\mathbf{K}) \leq 2d$, which implies

$$\mathrm{ill}(\mathbf{K}) = \mathrm{vein}(\mathbf{K}) = 2d$$

(see [4] for a direct proof of this equality). It shows that Theorem 4.1 is sharp.

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