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Circumscribed Polygons of Small Area

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Abstract Given any plane strictly convex region K and any positive integer $n \geq 3$, there exists an inscribed $2n$-gon Q_{2n} and a circumscribed n-gon P_n such that

$$\frac{Area(P_n)}{Area(Q_{2n})} \leq \sec\frac{\pi}{n}.$$

The inequality is the best possible, as can be easily seen by letting K be an ellipse. As a corollary, it follows that for any convex region K and any $n \geq 3$, there exists a circumscribed n-gon P_n such that

$$\frac{Area(P_n)}{Area(K)} \leq \sec\frac{\pi}{n}.$$

This improves the existing bounds for $5 \leq n \leq 11$.

Keywords Convex regions · Circumscribed polygons

1 Introduction

Given a plane convex region K and a positive integer $n \geq 3$, let $C_n(K)$ denote the circumscribed n-gon of minimum area and let $I_n(K)$ denote the inscribed n-gon of maximum area. Notice that both $C_n(K)$ and $I_n(K)$ may not necessarily be proper n-gons. For instance if K is a triangle, then $C_n(K) = I_n(K) = K$ for every $n \geq 3$.

In order to avoid dealing with these special situations, in the following we will consider only bodies that are *strictly convex*, that is, their boundaries do not contain line segments. This restriction ensures that both $C_n(K)$ and $I_n(K)$ are proper convex

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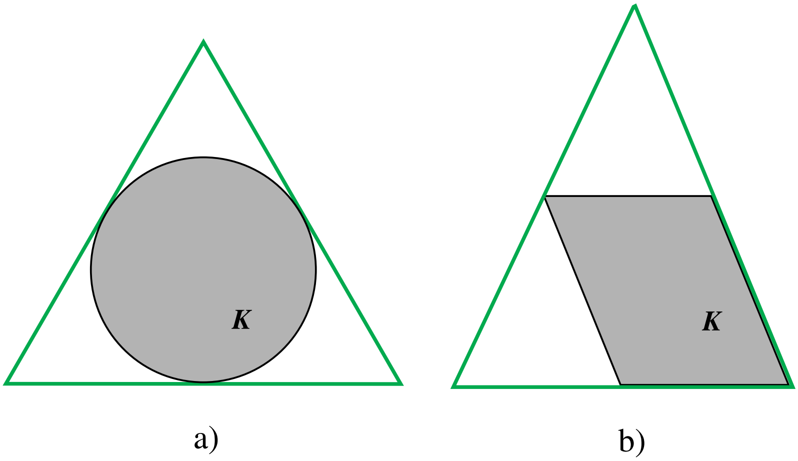

Fig. 1 (**a**) $K=$ circle of unit area, $Area(C_3(K))=\frac{3\sqrt{3}}{\pi}\approx 1.65\ldots$. (**b**) $K=$ parallelogram of unit area, $Area(C_3(K))=2$

n-gons. The results we obtain for the class of strictly convex bodies can then be extended to arbitrary convex bodies by a standard approximation argument.

The problems of finding $i_n := \inf Area(I_n(K))$ and $c_n := \sup Area(C_n(K))$ for a given integer $n \geq 3$ when K ranges over all convex domains of unit area have been studied for their intrinsic interest as well as for their connection with packing and covering problems (see, e.g., [1, Sects. 22 and 27], [15], [13], [7, Chap. II, Sect. 4], and [2]).

It is well known that these quantities are actually attained, because the space of affine equivalence classes of convex regions is compact (see [12]). This motivates the problem of finding the convex regions K that produce the extreme values i_n and c_n.

In 1939, Sas [16] proved that

$$i_n = \frac{n}{2\pi}\sin\frac{2\pi}{n}, \tag{1}$$

the infimum being reached if and only if K is an ellipse.

One would expect a similar statement to hold for circumscribed polygons. If K is an ellipse of unit area, it is easy to prove that the circumscribed n-gon of minimum area is affinely regular and its area is $\frac{n}{\pi}\tan\frac{\pi}{n}$. It would therefore be tempting to conjecture that $c_n = \frac{n}{\pi}\tan\frac{\pi}{n}$. However, one can readily check that this equality does not hold. It is sufficient to note that for $n=3$ the area of $C_3(K)$ equals 2 if K is a parallelogram of unit area while it is only $\sqrt{27}/\pi = 1.65\ldots$ if K is a circle (or an ellipse) of unit area; see Fig. 1.

It has been proven by many authors (see [4, 5, 8]) that $c_3 = 2$ and the extremal domains are the parallelograms. For $n \geq 4$ the extremal domains are not known and all we can realistically hope for is to obtain good bounds for c_n.

To illustrate how difficult it is to find the *exact* value of c_n for $n \geq 4$, let us mention that the problem of finding c_6 is related to the still unsolved problem of determining the centrally symmetric convex region of lowest packing density. An old conjecture of Reinhardt [15] states that the extremal domain is the so-called "smoothed octagon", a convex region resulting from a regular octagon by cutting off each vertex v_i with a hyperbolic arc from the hyperbola that is tangent to $v_i v_{i-1}$ and $v_i v_{i+1}$ and has the supporting lines $v_{i-1}v_{i-2}$ and $v_{i+1}v_{i+2}$ as asymptotes. Since by a result of Fejes Tóth (see [7]) the packing density of any centrally symmetric convex region K equals the ratio $Area(K)/Area(C_6(K))$ solving Reinhardt's conjecture is equivalent to determining $c_6' := \sup Area(C_6(K))$ when K ranges over all *centrally symmetric* convex domains of unit area.

Modifying the averaging technique Sas employed to prove (1), Fejes Tóth [6] showed that

$$c_n \le \frac{n-2}{\pi} \tan \frac{\pi}{n-2} \quad \text{for every } n \ge 5. \tag{2}$$

This is currently the best asymptotic result. Fejes Tóth also proved the following theorem.

Theorem 1.1 ([7, Chap. II, Sect. 4]) *Let K be a convex domain whose boundary contains from a circle of perimeter L two diametrically opposite arcs with total length $4L/n$. Then one can circumscribe about K an n-gon U_n such that $Area(U_n) \le Area(K) \frac{n}{\pi} \tan \frac{\pi}{n}$.*

This suggests that for large values of n the extremal domains are approximately ellipses. The problem of estimating c_n is therefore more interesting for small values of n. Chakerian [4] proved that $c_4 \le \sqrt{2}$ (the equality sign being later removed by Kuperberg in [9]) and that

$$c_n \le \frac{2\pi}{n} \csc \frac{2\pi}{n} \quad \text{for every } n \ge 5. \tag{3}$$

Notice that Chakerian's estimate (3) is better that Fejes Tóth's bound (2) only for $n \le 7$.

The main result of this paper is given by the following theorem.

Theorem 1.2 *For every plane strictly convex region K there exists an inscribed $2n$-gon Q_{2n} and a circumscribed n-gon P_n such that*

$$\frac{Area(P_n)}{Area(Q_{2n})} \le \sec \frac{\pi}{n}.$$

The inequality is the best possible, as can be seen by letting K be an (unit area) ellipse. It can be shown that, in this case, both the circumscribed n-gon of minimum area and the inscribed $2n$-gon of maximum area are affinely regular and their areas are $\frac{n}{\pi} \tan \frac{\pi}{n}$ and $\frac{n}{\pi} \sin \frac{\pi}{n}$, respectively.

It follows that the upper bound in Theorem 1.2 cannot be improved. Besides extending the known results $c_3 = 2$ and $c_4 \le \sqrt{2}$, Theorem 1.2 provides a new upper bound for c_n.

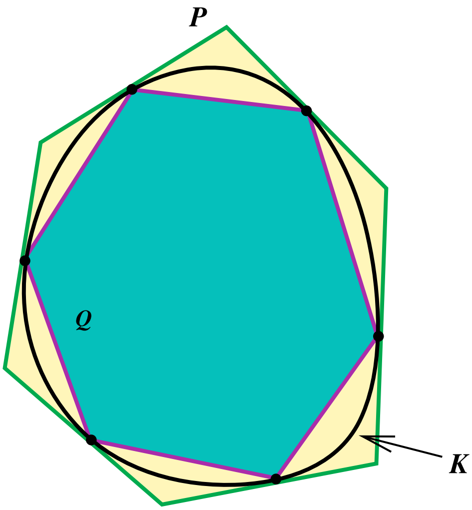

Fig. 2 The inscribed n-gon Q cuts off triangles of equal area from the circumscribed n-gon P. For each such pair inequality (4) holds

Corollary 1.3

$$c_n \le \sec \frac{\pi}{n} \quad \text{for every } n \ge 3.$$

This improves Chakerian's estimate (3) for every $n \ge 5$ and it is better than Fejes Tóth's bound (2) for all $5 \le n \le 11$.

2 The Tools

For the proof of Theorem 1.2 we are going to need two results. The first one is due to Lázár ([10], see also [7]).

Theorem 2.1 *Given a plane strictly convex region K and a positive integer $n \ge 3$, there exists a circumscribed convex n-gon P such that*

$$\frac{Area(P)}{Area(Q)} \le \sec^2 \frac{\pi}{n}, \tag{4}$$

where Q is the n-gon spanned by the points where P touches the boundary of K.

Lázár's original result is actually stronger. He proves that if the circumscribed n-gon P has the property that the triangles cut off from P by the sides of the n-gon Q have the same area, then the above inequality holds (see Fig. 2).

The other result to be used later is a corollary of the Brunn–Minkowski inequality. This inequality is valid in a much more general setting than used here. We refer the interested reader to the original works of Brunn [3] and Minkowski [14]; for a modern treatment one should consult [17].

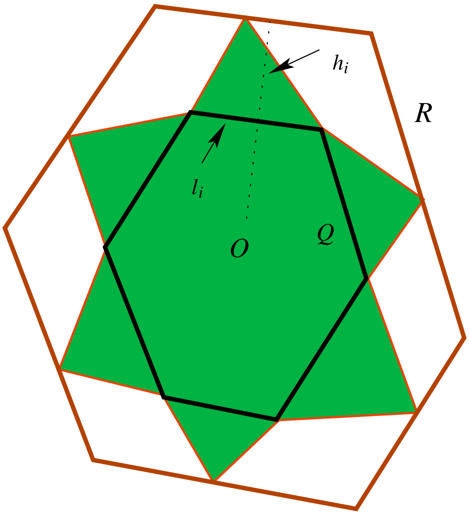

Fig. 3 $Area(Q,R)=\frac{1}{2}\sum_{i=1}^{n} l_i\, h_i=$ Area of the shaded polygon

Given two convex regions K and L, recall that $Area(K,L)$, the *mixed area* of K and L, is defined by the equality

$$Area\,(sK+(1-s)L)=s^2\,Area(K)+2s(1-s)\,Area(K,L)+(1-s)^2\,Area(L).$$

The Brunn–Minkowski inequality states that

$$Area(K,L)\ge\sqrt{Area(K)\cdot Area(L)}. \tag{5}$$

Let R and Q be two convex n-gons whose corresponding sides are parallel (see Fig. 3). Then $Area(Q,R)$ can be easily described as follows. Let O be a fixed point inside Q. If l_i is the length of a side of Q, denote by h_i the distance from O to the corresponding side of R. Then

$$Area(Q,R)=\frac{1}{2}\sum_{i=1}^{n} l_i\, h_i. \tag{6}$$

Lyusternik [11] presented an excellent introductory account of the properties of mixed areas as well as proofs of both (5) and (6).

3 Proof of the Main Theorem

Given K a plane strictly convex body consider the following polygons:

$P :=$ a circumscribed n-gon with the property from Theorem 2.1.
$Q :=$ the inscribed n-gon spanned by the points where P touches K.
$R :=$ the circumscribed n-gon parallel to Q.
$S :=$ the inscribed $2n$-gon spanned by the vertices of Q and by the points where R touches K; Fig. 4 illustrates the case $n=5$.

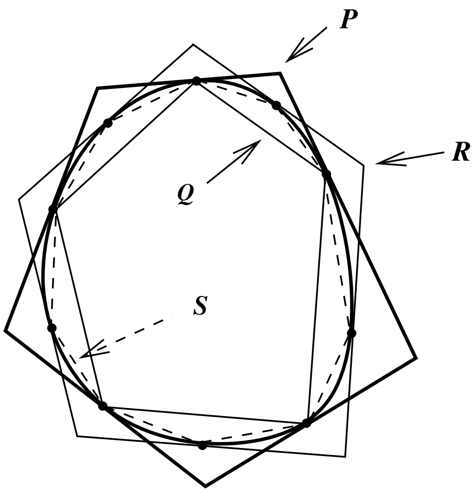

Fig. 4 The polygons considered in the proof of the main theorem

By our choice of P we have that

$$Area(P) \leq Area(Q) \cdot \sec^2 \frac{\pi}{n}. \tag{7}$$

On the other hand, by our previous argument it is easy to see that

$$Area(S) = \frac{1}{2} \sum_{i=1}^{n} l_i \, h_i = Area(Q, R)$$

which by using (5) gives

$$Area(S) \geq \sqrt{Area(Q) \cdot Area(R)}. \tag{8}$$

From (7) and (8) we obtain that

$$\frac{\min(Area(P), Area(R))}{Area(S)} \leq \frac{\sqrt{Area(P) \cdot Area(R)}}{Area(S)} \leq \frac{\sqrt{Area(Q) \cdot Area(R)} \cdot \sec \frac{\pi}{n}}{Area(S)},$$

that is,

$$\frac{\min(Area(P), Area(R))}{Area(S)} \leq \sec \frac{\pi}{n}.$$

This proves the theorem.

4 Conclusions and Open Problems

In this paper we proved a new upper bound for c_n which improves the existing bounds for small values of n. As mentioned in the introduction, the problem of determining the exact value of c_n is extremely difficult. It is apparent from the reference list that the progress has been slow and sporadic.

The only known exact value is $c_3 = 2$. One may consider the situation when $n = 4$ as it seems to be the easiest unsolved case. It is known that $c_4 < \sqrt{2} \approx 1.414\ldots$. On the other hand, Kuperberg [9] asked the following question:

Open Problem *Is it true that the (affinely) regular pentagon is the extremal region for which c_4 is attained? In other words, is it true that $c_4 = \frac{3}{\sqrt{5}} \approx 1.341\ldots$?*

While proving this may be difficult, it would be of interest if one can narrow the gap between the lower and upper bounds for c_4.

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