Discrete Comput Geom (2009) 41: 61–76
DOI 10.1007/s00454-008-9061-2

Geometric Tomography of Convex Cones

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Received: 20 June 2007 / Revised: 10 January 2008 /
Published online: 5 February 2008

Abstract The parallel X-ray of a convex set $K \subset \mathbb{R}^n$ in a direction u is the function that associates to each line l, parallel to u, the length of $K \cap l$. The problem of finding a set of directions such that the corresponding X-rays distinguish any two convex bodies has been widely studied in geometric tomography. In this paper we are interested in the restriction of this problem to convex cones, and we are motivated by some applications of this case to the covariogram problem. We prove that the determination of a cone by parallel X-rays is equivalent to the determination of its sections from a different type of tomographic data (namely, point X-rays of a suitable order). We prove some new results for the corresponding problem which imply, for instance, that convex polyhedral cones in $\mathbb{R}^3$ are determined by parallel X-rays in certain sets of two or three directions. The obtained results are optimal.

Keywords Geometric tomography · X-ray tomography · Convex cones · Covariogram · Chord function

1 Introduction

Geometric tomography deals with the retrieval of information about a geometric object using data from some of its projections or sections. It is a geometric relative of computerized tomography, which reconstructs an image from X-rays of a human patient. The book [10] gives a fascinating, updated and very complete account of geometric tomography.

A well-studied problem in this area concerns the determination of a convex set from the knowledge of some of its X-rays. Let $K \subset \mathbb{R}^n$ be a convex body, let $u \in S^{n-1}$

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and let $u^\perp$ denote the $(n-1)$-dimensional subspace orthogonal to u. The *(parallel) X-ray of K in direction u* is defined, for each $x \in u^\perp$, as

$$X_u K(x) = \lambda_1\big(K \cap (x + l_u)\big),$$

where $x + l_u$ denotes the line through x parallel to u, and λ_1 denotes one-dimensional Lebesgue measure. Knowing $X_u K$ is equivalent to knowing the length of each chord of K parallel to u. The parallel X-ray corresponds to the X-ray transform of the characteristic function 1_K of K used in computerized tomography; see [16].

While the previous definition corresponds to X-rays taken from infinity, it is natural and of interest to also consider X-rays emanating from finite points. This corresponds to the "fan-beam" X-rays of great importance in medicine; in fact, CAT scanners use this type of X-ray. Let $p \in \mathbb{R}^n$ and $i \in \mathbb{R}$. We define the *(point) X-ray of order i of K at p* by

$$X_{i,p} K(u) = \int_{-\infty}^{\infty} 1_K(p + tu)\, |t|^{i-1}\, dt,$$

for $u \in S^{n-1}$ for which the integral exists. The X-ray of order 1 at p gives the lengths of all the intersections of the body with lines through p. Some results of this paper concern the following problem.

Problem 1.1 Find a set of points such that the X-rays of order i at these points distinguish between any two different convex bodies.

The initial motivation of this paper comes from the *covariogram problem*. The *covariogram g_K* of a convex body $K \subset \mathbb{R}^n$ is the function, defined for $x \in \mathbb{R}^n$, by

$$g_K(x) = \lambda_n\big(K \cap (K + x)\big),$$

where λ_n stands for the n-dimensional Lebesgue measure. The covariogram problem asks whether g_K determines K, among all convex bodies, up to translations and reflections in a point. This problem was posed in 1986 by G. Matheron, who conjectured a positive answer for $n = 2$. The conjecture has been recently confirmed by Averkov and Bianchi [1]. However, the covariogram problem, in the general setting, has a negative answer. Bianchi [2] proved this by finding polyhedral counterexamples in $\mathbb{R}^n$ for every $n \geq 4$. Regarding the case $n = 3$, Bianchi [3] proved that a convex three-dimensional polytope is determined by its covariogram. It is the proof of this result which first motivated the study subject of this paper.

To explain this point, let A and B be closed convex polyhedral cones in $\mathbb{R}^3$, with apex the origin o and $A \cap B = \{o\}$. The *cross covariogram* of A and B is the function, defined for $x \in \mathbb{R}^3$, by

$$g_{A,B}(x) = \lambda_3\big(A \cap (B + x)\big).$$

The following question was first posed by Mani-Levitska [15] and arises naturally in the study of the covariogram problem for three-dimensional convex polytopes: Does the cross covariogram of A and B determine the pair (A, B), among all pairs of convex cones, up to certain ambiguities which are inherent in the problem? The paper [3] gives a partial positive answer to the previous problem, whose proof relies on the fact that a suitable second-order mixed derivative of $g_{A,B}$ equals the parallel X-ray of the cones in some direction. This leads to the following problem.

Problem 1.2 Find a set of directions such that the parallel X-rays in these directions distinguish between any two different convex cones.

Gardner and McMullen [12] (see [10, Corollary 2.2.1]) proved that there are sets of four directions, contained in the same two-dimensional subspace, such that every convex body in $\mathbb{R}^n$ is determined, among all convex bodies, by its parallel X-rays in these directions, and the number four is optimal. Moreover, convex bodies in $\mathbb{R}^3$ are not determined by parallel X-rays in any set of four noncoplanar directions (see [10, Theorem 2.2.3]). Volčič [17] (see [10, Theorems 5.3.7 and 5.3.8]) proved that X-rays of order 1 at three noncollinear points distinguish between all different planar convex bodies not containing the points, while X-rays of order 1 at any four points, with no three collinear, distinguish between all different planar convex bodies. We refer to [10] for complete bibliographical information on these problems.

The following result states that Problem 1.2 is equivalent to Problem 1.1, with $i=-1$, for a section of the cones. In this paper, unless explicitly stated otherwise, a cone has apex o.

Theorem 1.3 *Let A and A' be closed convex cones contained in $\{(x_1,\dots,x_n)\in\mathbb{R}^n : x_n\ge 0\}$, with $n\ge 2$. Let $u\in S^{n-1}\cap\{x_n\neq 0\}$, with $\pm u\notin A$, let $K=A\cap\{x_n=1\}$, $K'=A'\cap\{x_n=1\}$ and $p=l_u\cap\{x_n=1\}$.*

If $X_uA=X_uA'$, then $X_{-1,p}K=X_{-1,p}K'$.

Conversely, if $X_{-1,p}K=X_{-1,p}K'$ and K and K' are in the same open halfspace bounded by an $(n-1)$-dimensional subspace containing p, then $X_uA=X_uA'$.

Observe that if $\pm u\in A$, the X-ray X_uA is infinite and gives no information about A. In view of this result we are interested in Problem 1.1 for X-rays of order -1 at points outside the two convex bodies. A polyhedral set is the intersection of finitely many closed halfspaces. We call it nondegenerate if its interior is nonempty.

Theorem 1.4 *Let $K\subset\mathbb{R}^n$ be a nondegenerate polyhedral set, let $K'\subset\mathbb{R}^n$ be a closed convex set, and let p_1, p_2 be distinct points of $\mathbb{R}^n\setminus K$ such that the line l through p_1 and p_2 meets K. Suppose that one of the following conditions holds*:

(i) *the sets K and K' meet the same component of $l\setminus\{p_1,p_2\}$*;
(ii) *the line l supports K.*

If K and K' have equal X-rays of order -1 at p_j, $j=1,2$, then $K=K'$.

Theorem 1.5 *A nondegenerate convex polygon K is determined, in the class of planar convex bodies, by its X-rays of order -1 at any set of three noncollinear points not contained in K.*

We wish to stress that very little is known about Problem 1.1 when $i\le 0$, and some evidence suggests that the answer may be somewhat different from that corresponding to the case $i>0$ (see Remark 4.1 for a discussion of this point). The previous theorems imply the following theorems for cones. In each of them, the determination holds in the class of all closed convex sets.

Theorem 1.6 *A nondegenerate convex polyhedral cone $A \subset \mathbb{R}^n$, $n \geq 2$, is determined by its parallel X-rays in two directions u_1 and u_2, if $\pm u_j \notin A$, $j = 1, 2$, $u_1 \neq \pm u_2$, and the two-dimensional subspace which contains u_1 and u_2 intersects $A \setminus \{o\}$.*

Theorem 1.7 *A nondegenerate convex polyhedral cone $A \subset \mathbb{R}^3$ is determined by its parallel X-rays in any set of three directions u_j, $j = 1, 2, 3$, which are not contained in the same two-dimensional subspace and satisfy $\pm u_j \notin A$, $j = 1, 2, 3$.*

The last four theorems are optimal in the sense explained in Remark 4.4. Moreover the next result implies that Theorems 1.4 and 1.6, without the assumption that the convex set K or cone A are polyhedral, are false (see Remark 4.2 for more comments on this point).

Theorem 1.8 *There exist planar convex bodies K and K' with equal X-rays of order -1 at distinct points p_1 and p_2, and such that $p_1, p_2 \notin K \cup K'$ and $K \cap [p_1, p_2] = K' \cap [p_1, p_2]$ is a nondegenerate segment. The line through p_1 and p_2 supports both K and K'.*

Theorems 1.5 and 1.7 can be extended, respectively, to convex bodies and general convex cones under some extra assumptions (see Theorems 4.3 and 5.1). Section 6 contains a result regarding the determination of convex bodies in $\mathbb{R}^n$ from parallel X-rays in sets of noncoplanar directions.

We conclude by mentioning two recent results. Some stability estimates regarding Problem 1.1, with $i = -1$, have been obtained in [4], while an algorithm for reconstructing a planar convex body from possibly noisy measurements of either its parallel X-rays or its point X-rays has been presented in [11].

2 Preliminaries

As usual, S^{n-1} denotes the unit sphere in $\mathbb{R}^n$, centred at the origin o. If $u \in S^{n-1}$, $u^\perp$ denotes the $(n-1)$-dimensional subspace orthogonal to u, while l_u denotes the line through the origin parallel to u. For $x, y \in \mathbb{R}^n$, we write $[x, y]$ for the line segment with endpoints x and y.

If $B \subset \mathbb{R}^n$ we denote by $\operatorname{int} B$, $\operatorname{cl} B$, ∂B and $\operatorname{conv} B$ the *interior*, *closure*, *boundary* and *convex hull* of B, respectively. The *positive hull* of B is $\operatorname{pos} B = \{\mu x : x \in B, \mu \geq 0\}$. The symmetric difference of two sets B and B' is $B \Delta B' = (B \setminus B') \cup (B' \setminus B)$. The symbol λ_i, where $i \in \{1, \ldots, n\}$, denotes *i-dimensional Lebesgue measure*.

A *convex body* K is a compact convex set with nonempty interior. The symbol $\operatorname{relint} K$ indicates the *relative interior* of K. A convex polyhedral cone is a cone (with apex o) which is the intersection of finitely many closed halfspaces. If q is a vertex of a polygon P, the *support cone of P at q* is $\operatorname{cone}(P, q) = \{\mu(y - q) : y \in P, \mu \geq 0\}$.

The next results will be used repeatedly in the proofs. If K and K' are planar closed convex sets, $p \in \mathbb{R}^2 \setminus K$, and $X_{-1,p}K = X_{-1,p}K'$, then K and K' have equal supporting lines issuing from p, since the corresponding parallel lines from the origin bound the support of $X_{-1,p}K$. It is also well known that if ϕ is an affine transformation in $\mathbb{R}^n$, $u \in S^{n-1}$ and K, K' are convex sets in $\mathbb{R}^n$, then $X_u K = X_u K'$ if and only if $X_{\phi u}\phi K = X_{\phi u}\phi K'$. The following property of the X-rays of order -1 is crucial.

Lemma 2.1 [10, Theorem 6.2.8] *Let $l \subset \mathbb{R}^2$ be a line, $p \in \mathbb{R}^2$ and let K and K' be planar convex sets not meeting l. Let ϕ be a nonsingular projective transformation taking l to the line at infinity. If $p \in l$, then $X_{-1,p}K = X_{-1,p}K'$ if and only if ϕK and $\phi K'$ have equal parallel X-rays in the direction corresponding to ϕp. If $p \notin l$, then $X_{-1,p}K = X_{-1,p}K'$ if and only if $X_{-1,\phi p}\phi K = X_{-1,\phi p}\phi K'$.*

3 Proof of Theorem 1.3

Assume first that $n = 2$. If $\mathrm{int}A = \emptyset$, then the result is trivial, since $X_u A$, $X_u A'$, $X_{-1,p}K$ and $X_{-1,p}K'$ are identically 0.

Assume $\mathrm{int}A \neq \emptyset$ and $X_u A = X_u A'$. Up to a linear transformation which maps $\{x : x_2 \geq 0\}$ into itself, we may suppose that $u = p = (0, 1)$ and that $A \subset \{x : x_1 > 0\}$. (Note that since $u \notin A$, either $A \subset \{x : x_1 > 0\}$ or $A \subset \{x : x_1 < 0\}$.) The identity $X_u A = X_u A'$ and the assumption A, $A' \subset \{x : x_2 \geq 0\}$ imply that $p \notin A'$ and $A' \subset \{x : x_1 > 0\}$ too. Let $0 \leq m_1 < m_2$, $0 \leq m_1' \leq m_2'$ be such that

$$A = \{(x_1, x_2) : x_2 \geq 0,\ m_1 x_1 \leq x_2 \leq m_2 x_1\},$$
$$A' = \{(x_1, x_2) : x_2 \geq 0,\ m_1' x_1 \leq x_2 \leq m_2' x_1\}.$$

Then, if $w = \pm(1, 0)$,

$$K \cap (p + l_w) = \left\{(x_1, 1) : \frac{1}{m_2} \leq x_1 \leq \frac{1}{m_1}\right\},$$

and

$$K \cap (p + l_w) = \left\{(x_1, 1) : \frac{1}{m_2'} \leq x_1 \leq \frac{1}{m_1'}\right\},$$

where, for $i = 1, 2$, $1/m_i$ is substituted by $+\infty$ when $m_i = 0$. The identity $X_u A = X_u A'$ implies

$$m_2 - m_1 = m_2' - m_1'.$$

The value of the X-ray of order -1 of K at p in the direction w is $m_2 - m_1$, that of K' is $m_2' - m_1'$ and, by the above identity, they coincide.

To prove the converse implication assume, as before, that $u = p = (0, 1)$. By assumption, either A, $A' \subset \{x : x_1 > 0\}$ or A, $A' \subset \{x : x_1 < 0\}$. In each case the identity $X_u A = X_u A'$ can be proved as before, expressing all X-rays in terms of m_1 and m_2.

When $n > 2$ the proof follows by the result for $n = 2$ applied to every two-dimensional subspace containing u. □

The following lemma shows that in order to determine any polyhedral set, or convex cone, among convex sets, it is enough to deal with the class of polyhedral sets, or convex cones, respectively.

Lemma 3.1 *Let K, $K' \subset \mathbb{R}^n$ be closed convex sets, $u_1, u_2 \in S^{n-1}$, $u_1 \neq \pm u_2$, and $p \in \mathbb{R}^n \setminus K$.*

If K is a nondegenerate polyhedral set and either $X_{u_1}K$ and $X_{u_1}K'$ are finite and coincide or else $X_{-1,p}K = X_{-1,p}K'$, then K' is a polyhedral set.

If K is a convex cone with nonempty interior and, for $j = 1, 2$, $X_{u_j}K$ and $X_{u_j}K'$ are finite and coincide, then K' is a convex cone.

Proof Let us start with the case of parallel X-rays. If K' is described as

$$K' = \{q + tu_1 : q \in H,\ t \in \mathbb{R},\ f(q) \le t \le g(q)\},$$

with $H \subset u_1^{\perp}$ a convex set, f a convex function, g a concave one, both defined on H, then $X_{u_1}K' = g - f$. If K is a polyhedral set, then $X_{u_1}K$ and $g - f$ are piecewise linear. This may happen only if both f and g are piecewise linear, that is, only if K' is a polyhedral set. Similarly, if K is a cone, then $X_{u_1}K$ and $g - f$ are homogeneous of degree 1, and H is a cone with apex o. This may happen only if both $f - c$ and $g - c$ are homogeneous of degree 1, for some $c \in \mathbb{R}$, that is, only if K' is a cone with apex on l_{u_1}. The information regarding u_2 implies that o is the apex of K', since $l_{u_1} \cap l_{u_2} = \{o\}$.

In the case of X-rays of order -1, let L be a hyperplane that contains p and does not meet K. The set L also does not meet K', since $X_{-1,p}K = X_{-1,p}K'$. By exchanging K' with its reflection in p, if necessary, we may suppose that K and K' are contained in the same halfspace bounded by L. Let us embed $\mathbb{R}^n$ in $\mathbb{R}^{n+1}$ in such a way that $\mathbb{R}^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = 1\}$. The cones $\operatorname{pos}(K)$ and $\operatorname{pos}(K')$ in $\mathbb{R}^{n+1}$ have the same parallel X-rays in the direction of $\operatorname{pos}(p)$, by Theorem 1.3. Since $\operatorname{pos}(K)$ is a polyhedral set, the result for parallel X-rays implies that $\operatorname{pos}(K')$ (and, consequently, K') is also a polyhedral set. □

4 X-Rays of Order -1 of Polyhedral Sets

Proof of Theorem 1.4 Observe that neither p_1 nor p_2 belong to K', because the X-rays of order -1 of K' at p_1 and p_2 are finite, since $p_1, p_2 \notin K$. Moreover, Lemma 3.1 implies that K' is a polyhedral set, because K is a polyhedral set.

Assume $n = 2$ and (1.4). Choose a Cartesian coordinate system such that $p_1 = (0, 0)$, $p_2 = (1, 0)$ and, for brevity, let $\{y > 0\} = \{(x, y) : y > 0\}$. We first prove that $K \cap \{y > 0\}$ and $K' \cap \{y > 0\}$ have the same X-rays of order -1 at p_1 and p_2. Let l_i, for $i = 1, 2$, be a line through p_i which does not intersect K. Clearly l_i does not intersect K', because otherwise K and K' have different X-rays of order -1 at p_i in the direction of l_i or in a direction close to that of l_i. Both K and K' are contained in the same component C of $\mathbb{R}^2 \setminus (l_1 \cup l_2)$. Let $u \in S^1$ and $j \in \{1, 2\}$. The set $(l_u + p_j) \cap C$ either does not intersect $\{y > 0\}$ or else is contained in $\{y > 0\}$. In the first case $X_{-1,p_j}(K\cap\{y>0\})(u)=X_{-1,p_j}(K'\cap\{y>0\})(u)=0$, while in the second case $X_{-1,p_j}(K\cap\{y>0\})(u)=X_{-1,p_j}K(u)=X_{-1,p_j}K'(u)=X_{-1,p_j}(K'\cap\{y>0\})(u)$. The claim is proved.

To prove that $K\cap\{y>0\}=K'\cap\{y>0\}$ we apply the projective transformation ϕ defined by

$$\phi(x,y)=\left(\frac{1-x}{y},\frac{x}{y}\right), \tag{4.1}$$

which takes p_1 and p_2 to points on the line at infinity, precisely p_1 to the direction $u_1=(1,0)$ and p_2 to the direction $u_2=(0,1)$. If we set $H=\phi(K\cap\{y>0\})$ and $H'=\phi(K'\cap\{y>0\})$, H and H' are unbounded convex polygonal regions contained in $\{x+y\ge 0\}$. Observe that if $\alpha x+\beta y+\gamma=0$ is the equation of a line l' containing an unbounded edge of H or of H', then $(\beta-\alpha)x+\gamma y+\alpha=0$ is the equation of a line containing a segment of ∂K or of $\partial K'$, with this segment intersecting the x-axis. Since p_1 and p_2 do not belong to K or to K', we have $\alpha\neq 0$ and $\beta\neq 0$. Thus, the slope of l' is finite and different from 0. Let $y=m_1x+q_1$ and $y=m_2x+q_2$ be the equations of the lines which contain the two unbounded edges of H, for suitable m_1, m_2, q_1 and q_2 which satisfy $m_1\le m_2$ and, when $m_1=m_2$, $q_1<q_2$. Let $y=m'_1x+q'_1$ and $y=m'_2x+q'_2$ be the equations of the lines which contain the two unbounded edges of H', for suitable m'_1, m'_2, q'_1 and q'_2 which satisfy $m'_1\le m'_2$ and, when $m'_1=m'_2$, $q'_1<q'_2$.

Let r be a line through p_2 different from the x-axis. Lemma 2.1, applied to the segments $r\cap K\cap\{y>0\}$ and $r\cap K'\cap\{y>0\}$, implies that H and H' intersect the line ϕr, parallel to u_2, in segments of equal length. Due to the arbitrariness of r, this is equivalent to

$$X_{u_2}H(x)=X_{u_2}H'(x)\quad \forall x\in\mathbb{R}. \tag{4.2}$$

Similar arguments prove $X_{u_1}H=X_{u_1}H'$.

The set of $c\in\mathbb{R}$ with the property that $\{x=c\}$ intersects all the unbounded edges of H and H', is an unbounded interval I. The identity (4.2), for $x\in I$, implies

$$(m_2-m_1)x+q_2-q_1=(m'_2-m'_1)x+q'_2-q'_1.$$

A similar expression holds for the X-ray in direction u_1: for all y in an unbounded interval

$$\left(\frac{1}{m_2}-\frac{1}{m_1}\right)y-\frac{q_2}{m_2}+\frac{q_1}{m_1}=\left(\frac{1}{m'_2}-\frac{1}{m'_1}\right)y-\frac{q'_2}{m'_2}+\frac{q'_1}{m'_1}.$$

Assume that $m_1\neq m_2$. The two last identities imply $m_1=m'_1$, $m_2=m'_2$, $q_1=q'_1$ and $q_2=q'_2$. As a consequence the lines $y=m_1x+q_1$ and $y=m'_1x+q'_1$ coincide, the same happens for the other two lines and "H and H' coincide at infinity". If $H\neq H'$, then $\partial H\cap\partial H'$ has two unbounded components. Let $z_1\in\mathbb{R}^2$ and $z_2\in\mathbb{R}^2$, with $z_1\neq z_2$, denote the endpoints of these components. Identity (4.2) implies that z_1 and z_2 have the same x-coordinate, because otherwise (4.2) would be false for some values of x in the interval whose endpoints are the x-coordinates of z_1 and z_2. The points z_1 and z_2 also have the same y-coordinate and therefore coincide, which gives a contradiction.

When $m_1 = m_2$, the previous equations imply $q_2 - q_1 = q'_2 - q'_1$ and $m'_1 = m'_2 = m_1$. If $q_1 = q'_1$, the proof is concluded as before. Otherwise, if say $q_1 < q'_1$, let us choose (x_0, y_0) in such a way that $y_0 \in [m_1 x_0 + \min(q_2, q'_1), m_1 x_0 + \max(q_2, q'_1)]$ and both lines $\{x = x_0\}$ and $\{y = y_0\}$ intersect all the unbounded edges of H and H'. Assume $m_1 > 0$. Then $(H \setminus H') \cap \{x \le x_0\}$ and $(H' \setminus H) \cap \{x \le x_0\}$ have the same finite area, as it is proved integrating both sides of (4.2) for $x \in [-\infty, x_0]$. Similarly, the areas of $(H \setminus H') \cap \{y \le y_0\}$ and $(H \setminus H') \cap \{y \le y_0\}$ coincide. We get a contradiction from the strict inclusions

$$(H' \setminus H) \cap \{y \le y_0\} \subset (H' \setminus H) \cap \{x \le x_0\},$$

and

$$(H \setminus H') \cap \{x \le x_0\} \subset (H \setminus H') \cap \{y \le y_0\},$$

which are a consequence of our choice of (x_0, y_0). A similar argument applies when $m_1 < 0$.

The identity $K \cap \{y < 0\} = K' \cap \{y < 0\}$ can be proved similarly.

Let us now consider case (ii). The line l supports K' too, for otherwise K and K' could not have the same supporting lines through p_1. The proof of this case is similar to that of the previous one. However, it is not necessary to prove $X_{-1,p_j}(K \cap \{y > 0\}) = X_{-1,p_j}(K' \cap \{y > 0\})$, for $j = 1, 2$, since in this case it is obvious.

When $n > 2$, let L be any two-dimensional plane that contains p_1 and p_2. The sets $K \cap L$ and $K' \cap L$ have the same X-rays of order -1 at p_1 and p_2. Therefore if one of these sets has nonempty relative interior, then the same is true for the other one. Moreover, the result for $n = 2$ implies that $K \cap L = K' \cap L$, whenever these sets have nonempty relative interiors. This implies that $\mathrm{int} K = \mathrm{int} K'$, and concludes the proof. $\square$

Remark 4.1 Falconer [5, 6] and Gardner [7, 8] proved results analogous to Theorem 1.4 for general convex bodies and X-rays of order $i > 0$. (These results are expressed, when $i \neq 1$, in terms of *i-chord functions at a point p* and not in terms of X-rays of order i at p; see [10] for the definition. However, when $i > 0$, and also when $i \le 0$ and p does not belong to the body, these two notions coincide.) Gardner also wrote, in [8], that the uniqueness results of Volčič [17] on three or four sources—mentioned in the introduction—can be generalized to any positive i.

For X-rays of order $i \le 0$, a result corresponding to Theorem 1.4 is known only under the extra assumption

$$\int_{K \Delta K'} |y|^{i-2}\, dx\, dy < \infty, \tag{4.3}$$

where K, K' and l are as in Theorem 1.4 and l is chosen as the x-axis of a Cartesian coordinate system; see [10, Theorem 6.2.2]. Since the weight function $|y|^{i-2}$ is unbounded near l, (4.3) requires that K and K' are "very close" near l. Theorems 1.8 and 1.4 prove that, as a matter of fact, this assumption cannot be removed for general convex bodies, while it can be removed for polyhedral sets.

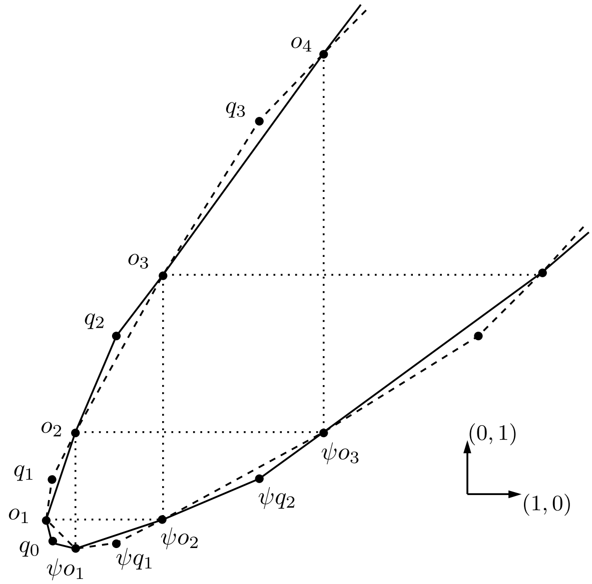

Fig. 1 Two unbounded convex sets with equal (parallel) X-rays in directions $(1,0)$ and $(0,1)$. They give rise, via a projective transformation, to the convex bodies K and K' of Theorem 1.8

Another difference between the cases $i>0$ and $i=-1$ is that while convex polygons are determined by X-rays of order $i>0$ at any two points (see [9] or [10, Theorem 6.2.7]), this is false for X-rays of order -1, as the examples described in Remark 4.4 show. Theorem 1.5 is the best possible for convex polygons when $i=-1$. We mention also Lam and Solmon [13], who studied the algorithmic reconstruction of convex polygons from their X-ray of order 1 at a single point.

Proof of Theorem 1.8 We construct distinct closed convex unbounded sets H and H' with equal parallel X-rays in directions $(1,0)$ and $(0,1)$; see Fig. 1. If ϕ denotes the projective transformation defined in (4.1), and H and H' are contained in $\{(x,y):x+y>0\}$, then $K=\phi^{-1}H$ and $K'=\phi^{-1}H'$ are convex bodies with equal X-rays of order -1 at the points $(1,0)$ and $(0,0)$. Let

$$\alpha_1=\frac{10}{3},\qquad \alpha_i=\frac{10}{3}-\sum_{j=0}^{i-2}4^{-j},\quad \text{for each } i>1.$$

The sequence (α_i) is decreasing and converges to 2. Let

$$o_1=(0,1),\qquad o_i=\left(\sum_{j=0}^{i-2}\alpha_1\cdots\alpha_j,\sum_{j=0}^{i-1}\alpha_1\cdots\alpha_j\right),\quad \text{for each } i>1,$$

where we use the agreement that $\alpha_1\cdots\alpha_j=1$, when $j=0$. Let ψ denote the reflection with respect to the line $\{(x,y):x=y\}$, and $L=\operatorname{conv}(\{o_i,\psi o_i:i\ge 1\})$. Then $[o_i,o_{i+1}]\subset\partial L$, for each $i\ge 1$ (because α_i is the slope of $[o_i,o_{i+1}]$ and (α_i) is decreasing to 2); see Fig. 1. For $h>0$ and $i\ge 1$, let

$$h_i=\frac{h}{\alpha_1\cdots\alpha_{i-1}},\qquad \mu_1=\frac{1-h}{2},\quad\text{and}\quad \mu_{i+1}=\mu_i+\frac{h}{\alpha_1^2\cdots\alpha_{i-1}^2\alpha_i}.$$

Consider the points

$$q_o=(\mu_1,\mu_1)\quad\text{and}\quad q_i=(1-\mu_i)o_i+\mu_i o_{i+1}+h_i(0,1).$$

If h is sufficiently small, then $\mu_i\in(1/3,2/3)$ for each i. We define $H=\operatorname{conv}(L,q_0,q_{2i},\psi q_{2i}:i\ge 1)$ and $H'=\operatorname{conv}(L,q_{2i+1},\psi q_{2i+1}:i\ge 0)$.

To prove that $L=H\cap H'$ or, equivalently, that $o_i\in\partial H\cap\partial H'$ for each i, it suffices to check that the slope of $[q_i,o_{i+1}]$ is larger than that of $[o_{i+1},o_{i+2}]$, and that the slope of $[o_i,o_{i+1}]$ is larger than that of $[o_{i+1},q_{i+1}]$. These inequalities amount to

$$\alpha_i-\alpha_{i+1}\ge\frac{h}{\alpha_1^2\cdots\alpha_{i-1}^2}\max\left(\frac{1}{1-\mu_i},\frac{1}{\mu_i\alpha_i^2}\right),$$

which is satisfied, since $\alpha_i-\alpha_{i+1}=4^{1-i}$, $\alpha_i>2$, $\mu_i\in(1/3,2/3)$ and we may assume $h<1/3$.

To prove that $X_{(1,0)}H=X_{(1,0)}H'$, it suffices to prove that

$$X_{(1,0)}\operatorname{conv}(o_1,q_0,\psi o_1)=X_{(1,0)}\operatorname{conv}(\psi o_1,\psi q_1,\psi o_2),\tag{4.4}$$

and, for each $i\ge 1$,

$$X_{(1,0)}\operatorname{conv}(o_i,q_i,o_{i+1})=X_{(1,0)}\operatorname{conv}(\psi o_{i+1},\psi q_{i+1},\psi o_{i+2}).\tag{4.5}$$

To prove (4.5), since the involved sets are triangles, it suffices to prove that o_j and ψo_{j+1} have the same y-coordinate, for each $j=i,i+1$, that the same is true for q_i and ψq_{i+1} and to prove that the line $q_i+l_{(1,0)}$ intersects the two triangles in segments of equal length. The y-coordinates of o_j and ψo_{j+1} coincide by definition. The y-coordinates of q_i and ψq_{i+1} are, respectively, $\sum_{j=0}^{i-1}\alpha_1\cdots\alpha_j+\mu_i\alpha_1\cdots\alpha_i+h_i$ and $\sum_{j=0}^{i-1}\alpha_1\cdots\alpha_j+\mu_{i+1}\alpha_1\cdots\alpha_i$ and, again, they coincide by definition of h_i and μ_i. Finally

$$\lambda_1\big((q_i+l_{(1,0)})\cap\operatorname{conv}(o_i,q_i,o_{i+1})\big)=\frac{h_i}{\alpha_i},$$

and

$$\lambda_1\big((q_i+l_{(1,0)})\cap\operatorname{conv}(\psi o_{i+1},\psi q_{i+1},\psi o_{i+2})\big)=h_{i+1},$$

and the right-hand sides of these equalities coincide by definition of h_i. The identity (4.4) can be proved similarly. This concludes the proof of $X_{(1,0)}H=X_{(1,0)}H'$. The symmetry of H and H' with respect to $\{(x,y):x=y\}$ implies $X_{(0,1)}H=X_{(0,1)}H'$.

Since H and H' are contained in $\{(x,y):x+y>0\}$ and α_i converges to 2, then K and K' are supported by the x-axis and $K\cap\{y=0\}=K'\cap\{y=0\}=[(1/3,0),(2/3,0)]$. □

Remark 4.2 The unique determination, in the context of Theorem 1.4, depends on the behavior of K near l, and it is sensitive even to small perturbations. For instance the proof of the theorem can be repeated, almost without changes, if we assume that K *is polyhedral near* l, i.e. if there exists a neighborhood U of l such that $K \cap U$ is the intersection of U with a polyhedral set. Moreover we believe that Theorem 1.4 holds, for arbitrary convex bodies, if l supports K at a single point. On the other hand, it does not hold if we assume that l supports K and intersects it in a segment (as shown by Theorem 1.8).

Theorem 4.3 *Let p_1, p_2 and p_3 be noncollinear points in the plane. A planar convex body K is determined by its X-rays of order -1 at p_j, $j = 1, 2, 3$, if K does not intersect any line which contains two of the points p_1, p_2 and p_3.*

Proof Let $K' \neq K$ be a convex body with $X_{-1,p_j}K = X_{-1,p_j}K'$, $j = 1, 2, 3$. Neither p_1 nor p_2 nor p_3 belong to K', because the X-rays of order -1 of K' at p_1, p_2 and p_3 are finite. Moreover K' does not intersect any line which contains two of the points p_1, p_2 and p_3, for otherwise K and K' could not have the same supporting lines through the corresponding two points.

Up to affine transformations, we may assume that $p_1 = (0, 0)$, $p_2 = (1, 0)$ and $p_3 = (1/2, 1)$. Let ϕ be defined as in (4.1). The sets $H = \phi K$ and $H' = \phi K'$ are convex bodies which have equal parallel X-rays in the directions $u_1 = (1, 0)$ and $u_2 = (0, 1)$, and equal X-rays of order -1 at $q = (1/2, 1/2)$, by Lemma 2.1. Moreover H and H' do not intersect the lines $\{x = 1/2\}$ and $\{y = 1/2\}$. They are contained in the same component of $\mathbb{R}^2 \setminus (\{x = 1/2\} \cup \{y = 1/2\})$, for otherwise the supports of their parallel X-rays could not coincide. For simplicity we assume that $H, H' \subset \{x > 1/2, y > 1/2\}$, since in the other cases the proof is similar.

We introduce some notation according to [10, Sects. 1.2 and 5.3]. Assume $H \neq H'$. Since $X_{u_1}H = X_{u_1}H'$, the centroids of H and H' are aligned in the direction u_1, by [10, Lemma 1.2.3]. Since $X_{u_2}H = X_{u_2}H'$, these centroids coincide. The latter implies $\mathrm{int}H \cap \mathrm{int}H' \neq \emptyset$. Suppose that C is a component of $\mathrm{int}(H \setminus H')$ and let $j \in \{1, 2\}$. Let u_jC be the set of all $z \in \mathrm{int}(H' \setminus H)$ such that the line $z + l_{u_j}$ meets C. It is known that u_jC is a component of $\mathrm{int}(H' \setminus H)$ and it has the same area as C. Moreover $X_{u_j}C = X_{u_j}u_jC$ and the centroids of C and u_jC are aligned in the direction u_j. With similar ideas one associates to C a component qC of $\mathrm{int}(H' \setminus H)$, with the property that $X_{-1,q}C = X_{-1,q}qC$.

Let C be a component of $\mathrm{int}(H \Delta H')$ of maximal area and let

$$\mathcal{C} = \left\{u_{j_m} \dots u_{j_1}C : m \in \mathbb{N}, j_k \in \{1, 2\}\right\}$$

be the *system of components associated to* C. Reference [10, Lemma 1.2.8] proved that the centroids of the components in the system $\mathcal{C}$ form the vertices of a convex polygon P. Let $z = (x_z, y_z)$ be a vertex of P with the property that $(-1, -1)$ is an outer normal vector to P in z. Since $H, H' \subset \{x > 1/2, y > 1/2\}$, we have $x_z > 1/2$ and $y_z > 1/2$. Each vertex of P is a centroid of a component in $\mathcal{C}$, by definition. Assume, for instance, that the component D in $\mathcal{C}$ whose centroid is z is contained in $H \setminus H'$. We claim that D is nearer to q than qD. Let u_jz be the centroid of u_jD, for $j = 1, 2$. Since $P \subset \{x + y \geq x_z + y_z\}$ and u_jz is a vertex of P, the x-coordinate of

u_1z is larger than x_z and the y-coordinate of u_2z is larger than y_z. Therefore the line l through q and z meets relint$[u_1z, u_2z]$ in a point, which we denote by p. Moreover q, z and p are in this order on l. The point $p \in H'$, because $u_1z, u_2z \in H'$, and therefore each point of $l \cap (H' \setminus H)$ is farther from q than z.

We use this information to prove that the area of qD is larger than that of D. This contradicts the maximality of the area of C, because $\lambda_2(C) = \lambda_2(D) < \lambda_2(qD)$, and concludes the proof. In a polar coordinate system centred at q, let

$$D = \{(r, \theta) : 0 < r_1(\theta) \le r \le s_1(\theta),\ \alpha \le \theta \le \beta\},$$

and

$$qD = \{(r, \theta) : 0 < r_2(\theta) \le r \le s_2(\theta),\ \alpha \le \theta \le \beta\},$$

for suitable $0 < \alpha < \beta < \pi/2$, $r_j(\theta)$ and $s_j(\theta)$. Then

$$\lambda_2(D) = \int_\alpha^\beta \int_{r_1(\theta)}^{s_1(\theta)} r\,dr\,d\theta = \int_\alpha^\beta \int_{1/s_1(\theta)}^{1/r_1(\theta)} t^{-3}\,dt\,d\theta,$$

where we have used the substitution $r = 1/t$. A similar expression holds for $\lambda_2(qD)$. What has been proved above implies that $s_1(\theta) \le s_2(\theta)$, for each θ. The equality of the X-rays of order -1 at q implies that $1/r_1(\theta) - 1/s_1(\theta) = 1/r_2(\theta) - 1/s_2(\theta)$, for each θ. Since t^{-3} is decreasing and $1/s_2 \le 1/s_1$, $\lambda_2(D) < \lambda_2(qD)$. □

Proof Proof of Theorem 1.5 If no line through two of the points meets K, then the result follows from Theorem 4.3. Suppose that the line l through two of the points, p_1 and p_2, say, meets K. If $K' \ne K$ is a bounded convex polygon such that $X_{-1,p_j}K = X_{-1,p_j}K'$, $j = 1, 2$, then l also meets K', for otherwise K and K' could not have the same supporting lines through p_1 and p_2. By Theorem 1.4, the polygons K and K' must be separated by p_1 or p_2. It is now impossible for K and K' to have common supporting lines through p_j, $j = 1, 2, 3$, and this contradiction proves the theorem. □

Remark 4.4 Given any set of three points on a line l there are two different triangles T and T' with equal X-rays of order -1 at each of the points; see [10, Theorem 6.2.9 and Fig. 6.2]. Both T and T' are contained in one of the open halfplanes bounded by l. Moreover, there exist different triangles S and S' with equal X-rays of order -1 at two points p_1 and p_2, outside the triangles, and such that the line through p_1 and p_2 meets int$S$ and intS'; see [10, Theorem 6.2.10 and Fig. 6.3]. One of the triangles intersects $[p_1, p_2]$, while the other does not.

These examples imply that Theorem 1.4 is false if l does not meet K, or if K and K' meet different components of $l \setminus \{p_1, p_2\}$; they imply that Theorem 1.5 is false if the three points are collinear. These examples also prove—via the connection among parallel X-rays of cones and point X-rays of order -1 of their sections expressed in Theorem 1.3—that Theorem 1.6 is false if the two-subspace which contains u_1 and u_2 does not intersect $A \setminus \{o\}$, and that Theorem 1.7 is false, if u_1, u_2 and u_3 are contained in the same 2-subspace.

Theorem 4.5 *Let K and K' be nondegenerate convex polygons with equal X-rays of order -1 at $p_1, p_2, \dots, p_s \in \mathbb{R}^2 \setminus K$. If*

$$\mathrm{conv}(K, p_1, \dots, p_s) = \mathrm{conv}(K', p_1, \dots, p_s), \tag{4.6}$$

then $K = K'$.

Proof Assume $K \neq K'$ and let $P = \mathrm{conv}(K, p_1, \dots, p_s)$. Without loss of generality, let $p_1, \dots, p_m$, for some $1 \le m \le s$, be the points of $\{p_1, \dots, p_s\}$ which are vertices of P.

We may assume $m \le 2$, because otherwise the result follows by Theorem 1.5. We may also assume that the vertices p_1 and p_m are consecutive, since otherwise $[p_1, p_m]$ meets K. In this case, the segment $[p_1, p_m]$ meets K' too, by the equality of the X rays at p_1 and p_m and (4.6), so that by Theorem 1.4 $K = K'$.

Let $p_1, \dots, p_m, q_{m+1} \dots, q_d$ denote the vertices of P, in counterclockwise order. Each q_i is a vertex of K and K', by (4.6), and $d > m$, because $m \le 2$.

Assume $m = 1$. Let l be any line through p_1, with $l \cap (P \setminus \{p_1\}) \neq \emptyset$. Let $K \cap l = [x_1, x_2]$, with p_1, x_1 and x_2 in this order on l; let $K' \cap l = [x_1', x_2']$, with p_1, x_1' and x_2' in this order on l. One endpoint of $l \cap P$ belongs to $\cup_{j=2}^{d-1}[q_j, q_{j+1}]$, and since $\cup_{j=2}^{d-1}[q_j, q_{j+1}]$ is contained in $\partial K \cap \partial K'$, x_2 and x_2' coincide with this endpoint. The equality of the X-rays of order -1 at p_1 implies that $x_1 = x_1'$ too. The arbitrariness of l implies that $K = K'$.

Now assume $m = 2$. Let C be the component of $\partial K \cap \partial K'$ which contains q_3. We prove that $C = \partial K = \partial K'$. First assume that

$$C = \{q_3\}. \tag{4.7}$$

Observe that in this case $d = 3$. Moreover K and K' meet $[p_2, q_3]$ only in q_3, because if, for instance, $K \cap [p_2, q_3]$ strictly contains $\{q_3\}$, the same is true for K', by the equality of the X-rays of order -1 at p_2, and (4.7) is violated. Similar arguments prove that $K \cap [p_1, q_3] = K' \cap [p_1, q_3] = \{q_3\}$.

Let l be any line through p_2 that meets the two edges of K adjacent to q_3 and the two edges of K' adjacent to q_3, and which does not contain q_3. Let u be the direction of l, let $T = \mathrm{cone}(K, q_3)$ and $T' = \mathrm{cone}(K', q_3)$. The identity $X_{-1,p_2}K(u) = X_{-1,p_2}K'(u)$ clearly implies $X_{-1,p_2}(q_3 + T)(u) = X_{-1,p_2}(q_3 + T')(u)$, and this implies, by elementary computations, that the entire X-rays of order -1 of the cones $q_3 + T$ and $q_3 + T'$ at p_2 coincide. Similar arguments prove that the X-rays of order -1 of these cones at p_1 coincide too. Since both $q_3 + T$ and $q_3 + T'$ meet $[p_1, p_2]$, Theorem 1.4 applies and proves that $T = T'$. In particular the edges of K and K' adjacent to q_3 coincide, contradicting (4.7).

Now drop (4.7) and assume that C is strictly contained in ∂K. Let x_1, x_2, with $x_1 \neq x_2$, be the endpoints of C. The equality of the X-rays of order -1 at p_1 implies that p_1, x_1 and x_2 are collinear. Similarly, p_2, x_1 and x_2 are collinear. Therefore both K and K' intersect $[p_1, p_2]$. Theorem 1.4 applies and proves that $K = K'$. □

5 Parallel X-Rays of Convex Cones

Proof of Theorem 1.6 Let $A' \neq A$ be a convex cone with $X_{u_j}A = X_{u_j}A'$, $j = 1, 2$. Let L_i be an $(n-1)$-dimensional subspace which contains u_i and intersects A only in o. (The existence of L_i follows from the assumption $\pm u_i \notin A$ and standard separation theorems.) Let L_i^+ be the halfspace bounded by L_i which contains A. Then $u_2 \notin L_1$ and $u_1 \notin L_2$, because $A \setminus \{o\}$ intersects the two-dimensional subspace containing u_1 and u_2, by assumption. Therefore we may choose a Cartesian coordinate system with origin at o and such that $L_1^+ \cap L_2^+ \subset \{(x_1, \dots, x_n) : x_n \geq 0\}$ and $u_1, u_2 \notin \{x_n = 0\}$.

Let $p_j = l_{u_j} \cap \{x_n = 1\}$, $j = 1, 2$, and let l be the line through p_1 and p_2. The support of $X_{u_i}A$ is contained in L_i^+, and therefore $A' \subset L_i^+$, for $i = 1, 2$. Since A and A' are contained in $L_1^+ \cap L_2^+$, they intersect the same component of $l \setminus \{p_1, p_2\}$. Theorems 1.3 and 1.4 imply that $A \cap \{x_n = 1\} = A' \cap \{x_n = 1\}$, that is, $A = A'$. □

Proof of Theorem 1.7 Let $A' \neq A$ be a convex cone with $X_{u_j}A = X_{u_j}A'$, $j = 1, 2, 3$. Let L_{ij} be the two-dimensional subspace which contains u_i and u_j, for $i, j = 1, 2, 3$, $i \neq j$. If one of the L_{ij} intersects $A \setminus \{o\}$, the result follows from Theorem 1.6. Otherwise, let L_{ij}^+ be the halfspace bounded by L_{ij} which contains A.

The support of $X_{u_i}A$ is contained in L_{ij}^+, and therefore $A' \subset L_{ij}^+$, for each i and j. Choose a Cartesian coordinate system with origin at o and such that the intersection

$$\bigcap_{i,j=1,2,3,\ i\neq j} L_{ij}^+$$

is contained in $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 \geq 0\}$ and meets $\{x_3 = 0\}$ only in o. This choice implies that $A \cap \{x_3 = 1\}$ and $A' \cap \{x_3 = 1\}$ are bounded. Theorems 1.3 and 1.5 imply that $A \cap \{x_3 = 1\} = A' \cap \{x_3 = 1\}$, that is, $A = A'$. □

Theorem 5.1 *Let u_1, u_2 and $u_3 \in S^2$ be directions not contained in the same two-dimensional subspace. A closed convex cone $A \subset \mathbb{R}^3$ with nonempty interior is determined, in the class of all closed convex sets, by its parallel X-rays in direction u_j, $j = 1, 2, 3$, if $\pm u_j \notin A$, $j = 1, 2, 3$, and $A \setminus \{o\}$ does not intersect any two-dimensional subspace containing two of the directions u_1, u_2 and u_3.*

Theorem 5.1 can be proved as Theorem 1.7, substituting Theorem 1.5 with Theorem 4.3 in the proof.

6 Parallel X-Rays of Convex Bodies

It is of interest to study the determination of convex sets from parallel X-rays, when the directions are not contained in the same two-dimensional subspace, as explained in [10, Sect. 2.2] (see also Problems 2.1 and 2.2 in [10]). Very little is known about this problem. The next result shows that if a convex body $K \subset \mathbb{R}^n$ has a vertex q then the X-rays of K in certain sets of n directions suffice to determine K within the class of convex bodies containing q.

Theorem 6.1 *Let $K \subset \mathbb{R}^n$ be a convex body and $q \in \partial K$. Assume that there exist $u_j \in S^{n-1}$, $j = 1, \dots, n$, such that $K \subset q + \mathrm{conv}(\mathrm{pos}(u_1), \dots, \mathrm{pos}(u_n))$. The set K is determined by its X-rays in direction u_j, $j = 1, \dots, n$, in the class of convex bodies which contain q.*

Proof Up to an affine transformation, we may assume that $q = o$ and u_j is parallel to the x_j-axis and points in the positive direction, $j = 1, \dots, n$. Therefore $K \subset \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_j \geq 0, j = 1, \dots, n\}$. Let $K' \neq K$ be a convex body, which contains o and with $X_{u_j} K = X_{u_j} K'$, $j = 1, \dots, n$. Then $K' \subset \{x_j \geq 0, j = 1, \dots, n\}$, because $X_{u_j} K$ and $X_{u_j} K'$ have the same support.

Let $\mathcal{C}$ be the system of components associated—with respect to the directions u_j, $j = 1, \dots, n$—to a fixed component of $\mathrm{int} K \Delta K'$. (We use the terminology introduced in [10] and described in the proof of Theorem 4.3.) Let C be a component in $\mathcal{C}$, whose centroid z minimizes $x_1 + \cdots + x_n$, among all the centroids of components in $\mathcal{C}$. This minimum point exists, for $\mathcal{C}$ is finite.

Let $y \in C$ and $j \in \{1, \dots, n\}$. We may assume, up to exchanging the roles of K and K', that $C \subset \mathrm{int}(K \setminus K')$, and we prove that the ray $y + \mathrm{pos}(u_j)$ contains a point, say y_j, of K'. If l is any line parallel to u_j meeting C, then l also meets $u_j C \subset \mathrm{int}(K' \setminus K)$. If $y' \in l \cap \mathrm{int}(K' \setminus K)$ and $y'' \in l \cap \mathrm{int}(K \setminus K')$, then either the x_j-coordinate of y' is larger than that of y'' or it is smaller, and which of the two alternatives occurs does not depend on the choice of l, y' and y''. The x_j-coordinate of the centroid of $u_j C$ is larger than the x_j-coordinate of z, for otherwise z would not be a minimum point. Therefore the x_j-coordinate of y' is larger that of y''. When $l = y + l_{u_j}$ this property implies the claim.

The point y belongs to the simplex $\mathrm{conv}(o, y_1, \dots, y_n)$, and this simplex is contained in K', because so are all its vertices. Therefore $y \in K'$. This contradicts the choice of $y \in C \subset \mathrm{int}(K \setminus K')$. □

A similar result has been proved, with different methods, by G. Michelacci [14] for $n = 2$ and for X-rays taken from finite points.

Acknowledgements We are indebted to M. Longinetti and C. Peri for a careful reading of the first draft of this paper and for many suggestions that helped us to improve it.

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