Discrete Comput Geom (2008) 40: 504–527
DOI 10.1007/s00454-007-9037-7

The Intrinsic Diameter of the Surface of a Parallelepiped

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Received: 22 April 2007 / Revised: 7 June 2007, 7 July 2007,
30 August 2007 / Published online: 29 September 2007

Abstract In the paper we obtain an explicit formula for the intrinsic diameter of the surface of a rectangular parallelepiped in 3-dimensional Euclidean space. As a consequence, we prove that an parallelepiped with relation $1 : 1 : \sqrt{2}$ for its edge lengths has maximal surface area among all rectangular parallelepipeds with given intrinsic diameter.

Keywords Convex surface · Rectangular parallelepiped · Intrinsic distance · Intrinsic diameter · Surface area

1 Introduction

A convex surface is the boundary of a bounded convex body in the 3-dimensional Euclidean space $\mathbb{E}^3$. The problem of finding the intrinsic diameter (i.e., the longest of all shortest paths on the surface between pairs of points) of a given convex surface is known to be very hard and has been solved only for surfaces of some very special kinds. For instance, it is known that the intrinsic diameter of a convex centrally symmetric surface of revolution is equal to the length of its generators [11]. It should be noted that the intrinsic diameters for any class of convex surfaces can be used to estimate extremal values of some natural functionals defined on convex bodies.

Interesting results on computing the intrinsic diameters of general polytopal surfaces were obtained in [10] and [1], where one can find also extensive references. The reader is referred to [4] and references therein for methods for approximate computing the geodesic diameters. In particular, the authors of [10] presented an algorithm for computing the intrinsic diameter of a general polytope in $\mathbb{E}^3$. On the other hand, all known methods for computing the intrinsic diameter are neither easy nor fast, therefore, one should apply some special ideas in order to express the intrinsic diameter of a given polytopal surface explicitly.

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In this paper we obtain an explicit formula for the intrinsic diameter of the surface of an arbitrary rectangular parallelepiped in $\mathbb{E}^3$.

Let us consider a rectangular parallelepiped $P = ABCDA'B'C'D'$ in $\mathbb{E}^3$ with edge lengths $|AB| = a$, $|AD| = b$, $|AA'| = c$, $0 < a \le b \le c$ (this object is commonly known as "rectangular box"). The object of our study is the intrinsic distance on the surface ∂P, the boundary of the parallelepiped P. Recall that *the intrinsic distance* $d(M, N)$ between points $M \in \partial P$ and $N \in \partial P$ is the minimal length of polygonal lines, connecting the points M and N, in ∂P. Simply speaking, the intrinsic distance is the length of the shortest path that a spider needs to overcome between two points on the boundary (walls, a floor and a ceiling) of a room. Many properties of the intrinsic distance on the parallelepiped's surface are not obvious. For instance, it is not obvious what points of the surface are the farthest from a vertex of parallelepiped. For a cube, the farthest point is the opposite vertex, whereas the opposite vertex is not the farthest point for a parallelepiped with $a : b : c = 1 : 1 : 2$. Moreover, in the paper [9] it is proved that for a parallelepiped with edge lengths $0 < a \le b \le c$, the opposite vertex is the farthest point (on the parallelepiped's surface) from a given vertex if and only if $2c^2 - 2bc - ac - ab \le 0$. Note that the same result was obtained later in [12]. One can find some remarkable results on the set of farthest points on general convex surfaces in [5, 11, 13, 14].

The intrinsic diameter of a parallelepiped (more precisely, of the surface of a parallelepiped) is the maximal intrinsic distance between pairs of points on the surface of a parallelepiped. We shall denote the intrinsic diameter of the parallelepiped P by $D(P)$. Note that the term "the geodesic diameter" is often used instead of "the intrinsic diameter".

Let us consider the following sets:

$$\mathcal{M} = \big\{(a, b, c) \in \mathbb{R}^3 \mid 0 < a \le b \le c\big\}, \tag{1}$$

$$\mathcal{ME} = \Big\{(a, b, c) \in \mathcal{M} \mid \sqrt{\max\big\{0, a^2 + 2ab - 2bc\big\}} + \sqrt{\max\big\{0, b^2 + 2ab - 2ac\big\}} \ge 2c - a - b\Big\}. \tag{2}$$

The main result of this paper is the following:

Theorem 1 *Let $D(P)$ be the intrinsic diameter of the surface of a rectangular parallelepiped P with edge lengths $0 < a \le b \le c$. Then the following are true*:

(1) *If $(a, b, c) \in \mathcal{ME}$, then $D(P) = \sqrt{(a + b)^2 + c^2}$.*
(2) *If $(a, b, c) \in \mathcal{M} \setminus \mathcal{ME}$ and $a^2b^2 \le c^2(b - a)(a + b + 2c)$, then*

$$D(P) = \sqrt{b^2 + 3c^2 + 2b(a + c) - 2c\sqrt{(b + c)^2 - 2a(c - b) - a^2}}.$$

(3) *If $(a, b, c) \in \mathcal{M} \setminus \mathcal{ME}$ and $a^2b^2 \ge c^2(b - a)(a + b + 2c)$, then $D(P) = l$, where l is a unique real solution of the equation*

$$\sqrt{l^2 - (a + c)^2} + \sqrt{l^2 - (b + c)^2} + \sqrt{2l^2 - (a + b + c)^2} = c \tag{3}$$

with the property $l \ge \max\{b + c, \sqrt{(a + b)^2 + c^2}\}$.

By using calculations with Gröbner bases, it is easy to deduce that the number l from Theorem 1 satisfies the following polynomial equation:

$$\begin{aligned}
&l^8 - \left(2a^2+4bc+2b^2+8c^2+4ac\right)l^6 + \left(a^4+18c^4+4b^3c+16c^2a^2+2a^2b^2\right.\\
&\quad \left.+b^4+16b^2c^2+24bc^3+4a^3c+8a^2bc+8ab^2c+24abc^2+24ac^3\right)l^4\\
&\quad -2c\left(17c^3a^2+12ca^3b+2b^4c+8c^5+16cb^2a^2+17c^3b^2+8c^2a^3+2a^4b\right.\\
&\quad +18c^4b+8c^2b^3+2ab^4+32bc^2a^2+2a^2b^3+18c^4a+40bc^3a+2a^4c\\
&\quad \left.+2a^3b^2+32b^2c^2a+12b^3ca\right)l^2+c^2\left(2b^2+c^2+2bc\right)\left(2ac+c^2+2a^2\right)\\
&\quad \times\left(2a^2+4ab+6ac+2b^2+6bc+5c^2\right)=0.
\end{aligned}$$

Obviously, $D(P)=d(M,N)$ for some points $M,N\in\partial P$. It can be proved that the points M and N with the property $d(M,N)=D(P)$ are symmetric each to other with respect to the center of P (Proposition 1). A detailed information on such points (for the various cases described in Theorem 1), can be found below (Theorems 4, 5). From Theorems 4 and 5 we get that $\mathcal{N}$, the number of pairs $M,N\in\partial P$ with the property $D(P)=d(M,N)$, is finite for any parallelepiped P. Note that the set of such pairs are invariant under the reflection with respect to the plane defined by the midpoints of all edges of length α, where α is either a or b. The maximal value of $\mathcal{N}$ is equal to 8 and is attained by parallelepipeds with the following properties: $a^2b^2>c^2(b-a)(a+b+2c)$, $\widetilde{l}=\sqrt{(a+b)^2+c^2}$ and $a<c$. One of possible examples is $(a,b,c)=(1,1,\sqrt{2})$. The minimal value of $\mathcal{N}$ is equal to 2 and is attained by parallelepipeds such that $a^2b^2<c^2(b-a)(a+b+2c)$ and $(a,b,c)\notin\mathcal{ME}$ (see (2)).

Consider any two (antipodal) points $M,N\in\partial P$ with the property $D(P)=d(M,N)$. It is interesting to find $\mathcal{N}(M,N)$, the number of distinct shortest paths connected these points.

In the case (1) of Theorem 1 the points M and N are opposite vertexes of P, and $\mathcal{N}(M,N)\in\{2,4,6\}$. For instance, $\mathcal{N}(M,N)=6$ for a cube, but $\mathcal{N}(M,N)=2$ for a parallelepiped with $a\neq b\neq c$ (see Fig. 1A).

In the case (2) of Theorem 1 the point M (distinct from a vertex) is determined in Theorem 5 (see also Lemma 16): $(D(P))^2=g_1(\tau_2)=g_3(\tau_2)$ (see (10)). The points M and N are on the plane π defined by the midpoints of all edges of length a. It is

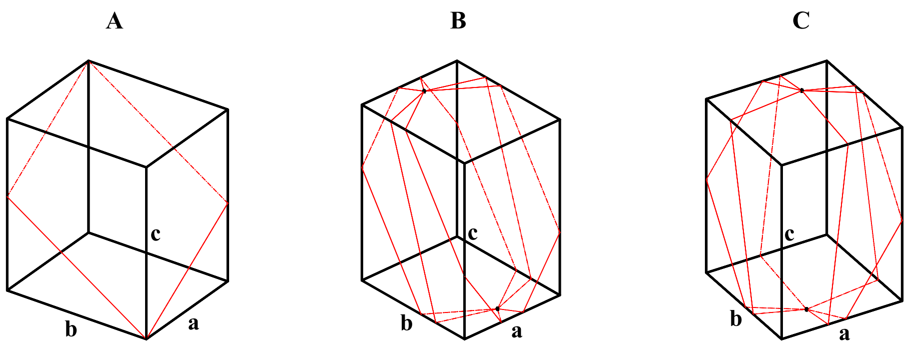


Fig. 1

easy to see that there are exactly two shortest paths which correspond to g_1 (these two paths are symmetric each to other with respect to the center of P and with respect to the plane π) and there are four shortest paths which correspond to g_3 (the set of these paths is invariant under the central symmetry of P and under the reflection with respect to the plane π). Therefore, in this case $\mathcal{N}(M, N) = 6$ (see Fig. 1B).

In the case (3) of Theorem 1 the point M (distinct from a vertex) is determined in Theorem 4 (see also Lemmas 8 and 11): $D(P) = d_1(\widetilde{x}, \widetilde{y}) = d_2(\widetilde{x}, \widetilde{y}) = d_3(\widetilde{x}, \widetilde{y})$. It is easy to see that for any $1 \le i \le 3$ there are two shortest paths which correspond to d_i (these two paths are symmetric each to other with respect to the center of P). Therefore, in this case we have $\mathcal{N}(M, N) = 6$, too (see Fig. 1C).

It should be noted that these calculations are consistent with the fact pointed out in [10]: If a pair of points $x, y \in S$ realizes the geodesic diameter of S, then either x or y is a vertex of S, or there are at least five distinct shortest paths between x and y. Here S is the boundary of any convex polytope. It can be inferred that the convex polytopes with two points at maximal intrinsic distance connected by exactly 5 shortest paths form a dense set in the class. From this point of view all rectangular parallelepipeds are exceptional since they are centrally symmetric.

It would be helpful to present the set of points $(a, b, c) \in \mathcal{M}$ satisfied by the cases (1), (2), (3) of Theorem 1, taking $c = 1$. Let

$$E := \{(a, b) \mid (a, b, 1) \in \mathcal{ME}\},$$

$$F = \{(a, b) \mid (a, b, 1) \in \mathcal{M} \setminus \mathcal{ME},\ a^2b^2 \le (b - a)(a + b + 2)\},$$

$$G = \{(a, b) \mid (a, b, 1) \in \mathcal{M} \setminus \mathcal{ME},\ a^2b^2 \ge (b - a)(a + b + 2)\}.$$

Each of these sets is a part of a triangle determined by the inequality $0 \le a \le b \le 1$ on a coordinate plane with coordinates (a, b) (see Fig. 2).

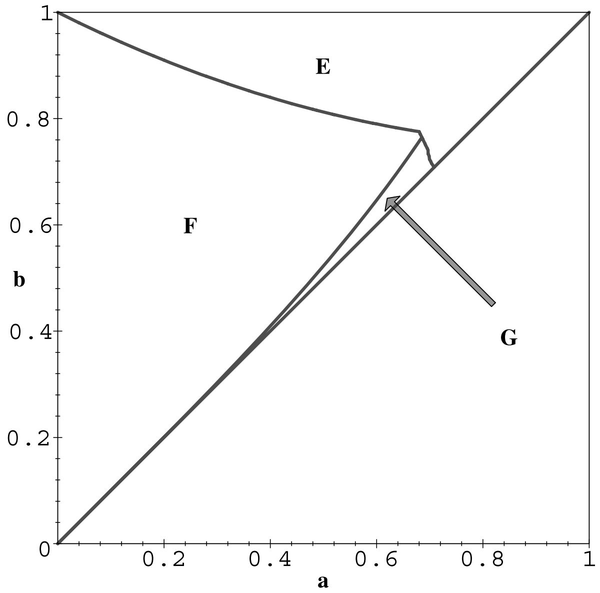
Note that Theorem 1 gives a method to look for extremal values of various functionals (see [3]), defined on the set of rectangular parallelepipeds, with restrictions on the intrinsic diameter. For instance, a natural problem of this kind is to find a parallelepiped of maximal surface area among all rectangular parallelepipeds with given intrinsic diameter. If we suppose $a = b = 1$, then from Theorem 1 we get that $D(P) = \sqrt{c^2 + 4}$ for $c \in [1, \sqrt{2}]$ and $D(P) = l = \sqrt{3c^2 + 2c - 2c\sqrt{c^2 - 1}}$ for $c \ge \sqrt{2}$. It is easy to prove that the maximal value of $A(P)/(D(P))^2$, where $A(P) = 4 + 2c$ is the surface area of ∂P, is attained by the point $c = \sqrt{2}$ in this partial case. This observation leads to the assertion of the following theorem.

Theorem 2 *Among all rectangular parallelepipeds with given intrinsic diameter the maximal surface area is attained by a parallelepiped with the relation $a : b : c = 1 : 1 : \sqrt{2}$ for edge lengths. In other words, for any rectangular parallelepiped P with edge lengths $0 < a \le b \le c$ the following inequality holds*:

$$ab + ac + bc \le \frac{1 + 2\sqrt{2}}{6}\bigl(D(P)\bigr)^2$$

with equality only when $a : b : c = 1 : 1 : \sqrt{2}$.

Fig. 2



In particular, Theorem 2 implies that $A(P)$, the surface area of any parallelepiped P with unit intrinsic diameter, satisfies the inequality $A(P) \leq (1+2\sqrt{2})/3 \approx 1.276142375$. Note that the area of a doubly covered square (a degenerate parallelepiped) with unit diagonal (the intrinsic diameter), is equal to 1. In this context it is useful to recall a well known conjecture of A.D. Alexandrov [2], that the maximal surface area of a convex surface with intrinsic diameter 1 is equal to $\pi/2 \approx 1.570796327$ and is attained by a doubly covered plane disc. Note also that in [7] it is proved that the area $A(T)$ and the geodesic diameter $D(T)$ of an arbitrary tetrahedron T in $\mathbb{E}^3$ satisfy the inequality $\frac{A(T)}{D(T)^2} \leq \frac{3\sqrt{3}}{4} \approx 1.299038106$ with equality only when T is a regular tetrahedron. Some generalizations of this result were obtained in [6].

We hope that the methods used in this article would be helpful for studying geodesic diameters of more general convex polytopes. The results of this paper could be used also for testing applied computer programs, calculating geometrical characteristics of convex polytopes. Note that the iterated calculation of the intrinsic diameter of polytopal surfaces is a very complicated procedure, and the effectiveness of any method of such calculation can be verified using Theorem 1 (see [1, 4, 10] and references therein for algorithms for computing the geodesic diameter of a general polytopal surface).

The paper is organized as follows. In Sect. 2 we give some general results on the intrinsic distance and the intrinsic diameter of ∂P. In Sect. 3 we describe the set $\mathcal{ME}$. Sections 4 and 5 are devoted to searching of the intrinsic diameter of ∂P in the cases $a^2b^2 \geq c^2(b-a)(a+b+2c)$ and $a^2b^2 \leq c^2(b-a)(a+b+2c)$, respectively. In Sect. 6 we give the proofs of Theorems 1 and 2.

2 Preliminaries

We remark that $D(P)$ can be calculated as the maximum of all intrinsic distances between pairwise symmetric points on the parallelepiped's surface. This follows from the following:

Proposition 1 ([11]) *Let F be a convex centrally symmetric surface in $\mathbb{E}^3$, and $D(F)$ its intrinsic diameter. If $M, N \in F$ are such that $D(F)$ is equal to the intrinsic distance between M and N on F, then the points M and N move from one to the other, under the central symmetry of the surface F.*

Remark 1 In the paper [8] the following generalization of the above result was obtained. Let us consider an inner metric space (M, ρ) homeomorphic to the 2-dimensional sphere S^2. Let $I : M \to M$ be an involute isometry with no fixed point, and $D(M)$ the intrinsic diameter of (M, ρ). Then there exists an $x \in M$ such that $D(M) = \rho(x, I(x))$. We do not know, whether this result is fulfilled for inner metric spaces (M, ρ) homeomorphic to the sphere S^n for $n \geq 3$.

An example of inner metric space with involute isometry is a boundary of a centrally symmetric body in $\mathbb{E}^3$. In this case the involute isometry is the restriction of the central symmetry of the body under consideration.

It is convenient to introduce Cartesian coordinates in the space with the origin at the point A and with coordinate axes AB, AD, AA'. We shall call two faces of the parallelepiped P parallel to the plane of the two first coordinates as *bases*, and all the other faces as *profile faces* of P.

The intrinsic distance between two points on the parallelepiped's surface can be calculated as the minimal length of polygonal lines that connect these points on ∂P. For any points $M, N \in \partial P$ there exists at least one polygonal line with length equal to the intrinsic distance between M and N. A polygonal line with this property is called *shortest*. It is easy to get the following:

Lemma 1 *Let γ be a shortest polygonal line connecting two given points M and N on the surface ∂P. Then the intersection of γ with any parallelepiped's face is connected.*

Lemma 1 easily implies the following:

Lemma 2 *The intrinsic distance between the points A and C' (i.e. the intrinsic distance between two opposite vertices of P) satisfies the equality $d(A, C') = \sqrt{(a+b)^2 + c^2}$.*

Proof It is clear that a shortest path between the points A and C' is a polygonal line with two segments, each of which is entirely in one of parallelepiped's faces. The shortest polygonal line of this kind (recall, that $0 < a \leq b \leq c$) has length $\sqrt{(a+b)^2 + c^2}$. $\square$

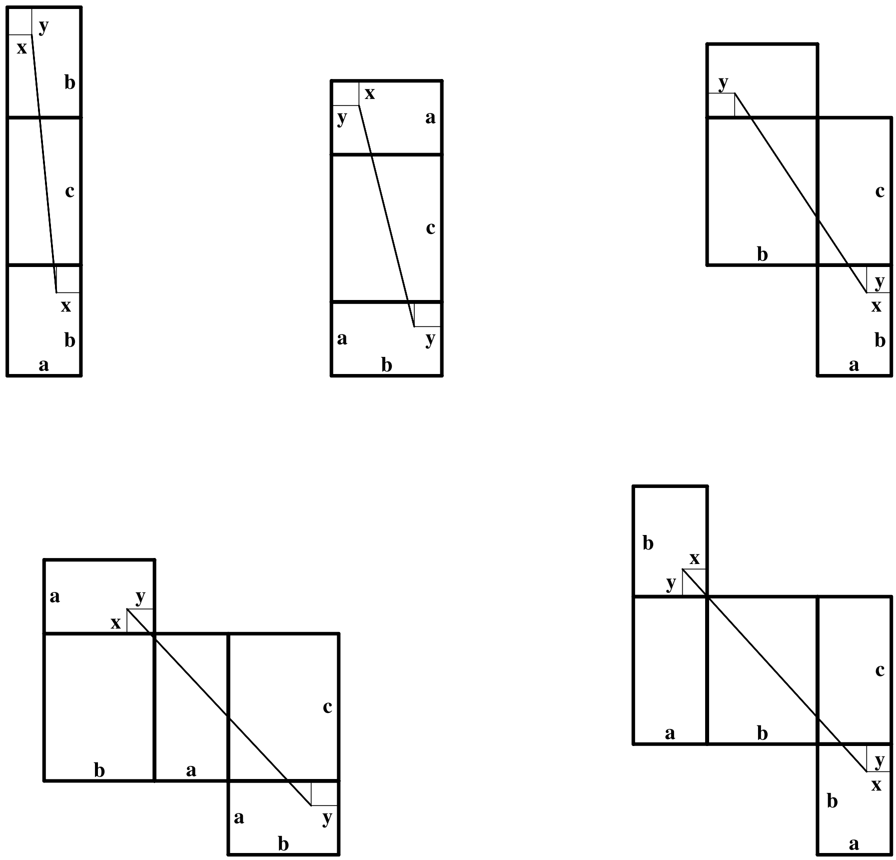


Fig. 3

Lemma 3 *Let $M, N \in \partial P$ be such that $D(P) = d(M, N)$. Then the points M and N are symmetric to each to other* (*with respect to the center of the parallelepiped P*), *and one of these points is in the face $ABCD$.*

Proof The fact that the points M and N are symmetric to each to other, follows from Proposition 1. Suppose that the points M and N are not in the faces $ABCD$ and $A'B'C'D'$. Since they are symmetric to each to other, it is possible to connect them by a polygonal line on the parallelepiped's surface with length less than $\sqrt{(a+b)^2+c^2}$. Since $d(A, C') = \sqrt{(a+b)^2+c^2}$, this means that $d(M, N) < d(A, C') \leq D(P)$. $\square$

Let us consider two points $M, N \in \partial P$ with the property $D(P) = d(M, N)$. By using Lemma 3 and obvious symmetries of the parallelepiped P, we may assume that M has the coordinates $(x, y, 0)$ and N has the coordinates $(a-x, b-y, c)$, where $0 \leq x \leq a/2$, $0 \leq y \leq b/2$.

By comparing lengths of locally shortest polygonal lines which correspond to various unfoldings of the parallelepiped's surface, one can find an explicit expression for $d(M,N)$. Let us consider the functions d_i $(1 \le i \le 5)$ defined by the following formulas:

$$\begin{aligned} d_1(x,y) &= \sqrt{(a+c)^2+(b-2y)^2}, \\ d_2(x,y) &= \sqrt{(b+c)^2+(a-2x)^2}, \\ d_3(x,y) &= \sqrt{(c+x+y)^2+(a+b-x-y)^2}, \\ d_4(x,y) &= \sqrt{(a+b)^2+(c+2y)^2}, \\ d_5(x,y) &= \sqrt{(a+b)^2+(c+2x)^2}, \end{aligned} \tag{4}$$

where $0 \le x \le a/2$, $0 \le y \le b/2$. Consider the function $\overline{D} : [0,a/2] \times [0,b/2] \to \mathbb{R}$ defined by

$$\overline{D}(x,y) = \min\{d_i(x,y) \mid 1 \le i \le 5\}. \tag{5}$$

Proposition 2 *Let $M=(x,y,0)$, $N=(a-x,b-y,c)$, $0 \le x \le a/2$, $0 \le y \le b/2$. Then $d(M,N)=\overline{D}(x,y)$.*

Proof Let γ be a shortest polygonal line on the surface of P between the points $M=(x,y,0)$ and $N=(a-x,b-y,c)$. Consider $\widetilde{\gamma}$, which is a part of γ situated on profile faces. It is clear that $\widetilde{\gamma}$ is a (connected) polygonal line consisting of at most three segments (each of which is entirely in some profile face). Consequently, γ consists of at most five segments, each of which is in some face of P. Considering various polygonal lines with this property and calculating their lengths, it is easy to see that γ is one of the polygonal lines shown on Fig. 3. The lengths of these polygonal lines are presented by formulas (4). Now the statement of the proposition follows from the definition of $\overline{D}(x,y)$ (see (5)). □

Remark 2 It could happen that (for some x and y) one of the polygonal lines in Fig. 3 does not correspond to a "real" polygonal line passing on the surface of P and crossing faces as in Fig. 3. But it is easy to see that in this case the value of $d_i(x,y)$ which corresponds to this "unreal" unfolding is greater than the length of the shortest polygonal line. Therefore, consideration of all $d_i(x,y)$ for $1 \le i \le 5$ in the formula (5) is justified (for more details see, e.g., Sect. 2.2 in [10]).

From Propositions 1 and 2 we obtain:

Proposition 3 *The intrinsic diameter of the parallelepiped P can be calculated by the formula*

$$D(P) = \max\{\overline{D}(x,y) \mid 0 \le x \le a/2,\ 0 \le y \le b/2\},$$

where the function $\overline{D}(x,y)$ is defined by the equality (5).

3 Description of the Set $\mathcal{ME}$

According to Lemma 2, the intrinsic distance between two opposite vertexes of the parallelepiped P is given by $\sqrt{(a+b)^2+c^2}$, hence for an arbitrary parallelepiped P the inequality $D(P) \geq \sqrt{(a+b)^2+c^2}$ is true. From Proposition 2 it is easy to get the following:

Lemma 4 *The condition $D(P) > \sqrt{(a+b)^2+c^2}$ is equivalent to the existence of $x \in (0, a/2]$ and $y \in (0, b/2]$ such that $d_i(x, y) > \sqrt{(a+b)^2+c^2}$ for $1 \leq i \leq 3$, where the functions d_i are defined by* (4).

From this it follows:

Lemma 5 *If $b = c$, then $D(P) = \sqrt{(a+b)^2+c^2} = \sqrt{(a+b)^2+b^2}$.*

Proof Indeed, $D(P) \geq \sqrt{(a+b)^2+c^2} = \sqrt{(a+b)^2+b^2}$. Let us consider arbitrary $x \in [0, a/2]$ and $y \in [0, b/2]$. It is clear that (see (4))

$$d_1(x, y) = \sqrt{(a+c)^2+(b-2y)^2} = \sqrt{(a+b)^2+(b-2y)^2} \leq \sqrt{(a+b)^2+b^2}.$$

Therefore, $\overline{D}(x, y) \leq \sqrt{(a+b)^2+b^2}$. From Proposition 3 we get that $D(P) = \sqrt{(a+b)^2+b^2}$. □

Lemma 6 *The function $h(x, y) = (d_3(x, y))^2 = (c+x+y)^2 + (a+b-x-y)^2$ is convex and $\sqrt{h(0,0)} = d_3(0,0) = \sqrt{(a+b)^2+c^2}$. The inequality $d_3(x, y) > \sqrt{(a+b)^2+c^2}$ holds for any $(x, y) \in [0, a/2] \times [0, b/2]$ when $a+b < c$, and it is equivalent to the inequality $x+y > a+b-c$ when $a+b \geq c$.*

Proof The first statement of the lemma follows from the convexity of the function $t \mapsto (c+t)^2 + (a+b-t)^2$. The second statement of the lemma follows from the fact that $(d_3(x, y))^2 - ((a+b)^2+c^2) = (x+y)(x+y+c-a-b)$. □

For an arbitrary parallelepiped P with edge lengths $0 < a \leq b \leq c$ we define two sets:

$$\begin{aligned} \Omega_x &= \left\{x \in [0, a/2] \mid (b+c)^2 + (a-2x)^2 \geq (a+b)^2 + c^2\right\}, \\ \Omega_y &= \left\{y \in [0, b/2] \mid (a+c)^2 + (b-2y)^2 \geq (a+b)^2 + c^2\right\}. \end{aligned} \tag{6}$$

A simple direct calculation implies the following:

Lemma 7 *If $a^2+2ab-2bc \leq 0$, then $\Omega_x = [0, a/2]$; if $a^2+2ab-2bc > 0$, then $\Omega_x = [0, (a-\sqrt{a^2+2ab-2bc})/2]$. If $b^2+2ab-2ac \leq 0$, then $\Omega_y = [0, b/2]$; if $b^2+2ab-2ac > 0$, then $\Omega_y = [0, (b-\sqrt{b^2+2ab-2ac})/2]$.*

The main result of this section is the following

Theorem 3 *The equality* $D(P)=\sqrt{(a+b)^2+c^2}$ *is equivalent to the condition* $(a,b,c)\in\mathcal{ME}$ *(see* (2)*).*

Proof We should prove that $D(P)>\sqrt{(a+b)^2+c^2}$ if and only if $(a,b,c)\in\mathcal{M}\setminus\mathcal{ME}$ (cf. formulas (1) and (2)). It is easy to check that there is no point with the property $b^2+2ab-2ac\le 0$ in the set $\mathcal{ME}$. Note also that the inequality $a^2+4c^2-2ac-4bc\le 0$ is equivalent to the inequality $(2c-a-b)^2\le b^2+2ab-2ac$.

Later on we shall use the following representation: $\mathcal{ME}=\mathcal{M}_1\cup\mathcal{M}_2$, where

$$\mathcal{M}_1=\big\{(a,b,c)\in M,\,|\,b^2+2ab>2ac,\,a^2+2ab\le 2bc,\,a^2+4c^2-2ac-4bc\le 0\big\},$$

$$\mathcal{M}_2=\big\{(a,b,c)\in M,\,|\,a^2+2ab>2bc,\,\sqrt{a^2+2ab-2bc}+\sqrt{b^2+2ab-2ac}\\ \ge 2c-a-b\big\}.$$

It is clear that $\mathcal{M}_1\cap\mathcal{M}_2=\emptyset$.

Using the statement of Lemma 5, we assume that $b>c$ (all points $(a,b,c)\in\mathcal{M}$ with relation $b=c$ are in the set $\mathcal{ME}$, as it is easy to see). According to Lemma 4, we should clarify when there exist $x\in(0,a/2]$ and $y\in(0,b/2]$ such that $d_i(x,y)>\sqrt{(a+b)^2+c^2}$ for $1\le i\le 3$.

Let us consider successively the following three cases: (1) $b^2+2ab-2ac\le 0$, (2) $b^2+2ab-2ac>0$; $a^2+2ab-2bc\le 0$, (3) $a^2+2ab-2bc>0$.

Case 1. In this case $\Omega_x=[0,a/2]$ and $\Omega_y=[0,b/2]$. Moreover, for any $x<a/2$ and $y<b/2$, we have $d_1(x,y)>\sqrt{(a+b)^2+c^2}$ and $d_2(x,y)>\sqrt{(a+b)^2+c^2}$. Note now that $d_3(a/2,b/2)>\sqrt{(a+b)^2+c^2}$, since $a/2+b/2>a+b-c$ (see Lemma 6). Therefore, (using the continuity) for some $x<a/2$ and $y<b/2$ we get $d_i(x,y)>\sqrt{(a+b)^2+c^2}$ for $1\le i\le 3$. Consequently, $D(P)>\sqrt{(a+b)^2+c^2}$ in this case.

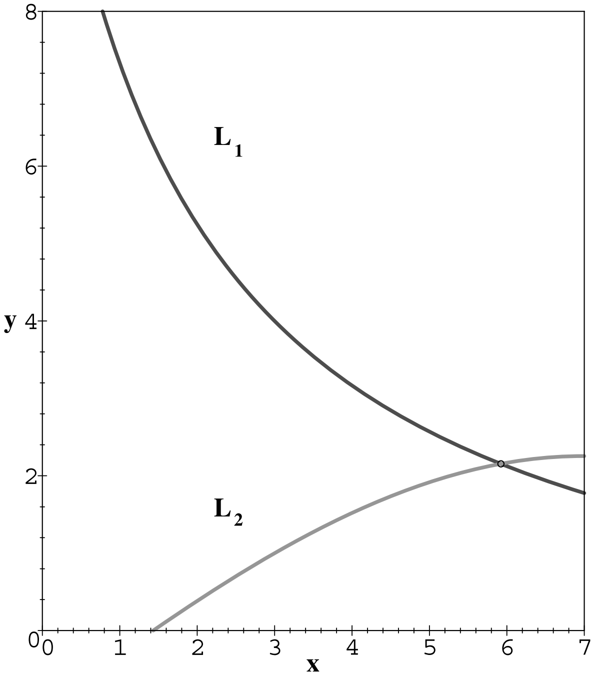
Case 2. In this case $\Omega_x=[0,a/2]$ and $\Omega_y=[0,(b-\sqrt{b^2+2ab-2ac})/2]$, moreover, $0<(b-\sqrt{b^2+2ab-2ac})/2<b/2$ (since $c>b$). Let us consider the point $(\widetilde{x},\widetilde{y})=(a/2,(b-\sqrt{b^2+2ab-2ac})/2)$. It is clear that if $x<\widetilde{x}$ and $y<\widetilde{y}$, then $d_1(x,y)>\sqrt{(a+b)^2+c^2}$ and $d_2(x,y)>\sqrt{(a+b)^2+c^2}$. Hence, the existence of a point $(x,y)\in(0,a/2]\times(0,b/2]$, which has the property $d_i(x,y)>\sqrt{(a+b)^2+c^2}$ for $1\le i\le 3$, is equivalent to the condition $d_3(\widetilde{x},\widetilde{y})>\sqrt{(a+b)^2+c^2}$ (see Lemma 6). The latter inequality is equivalent to the inequality $a^2+4c^2-2ac-4bc>0$ (and also to the inequality $(2c-a-b)^2>b^2+2ab-2ac$). Therefore, $D(P)>\sqrt{(a+b)^2+c^2}$ if and only if $a^2+4c^2-2ac-4bc>0$ in this case.

Case 3. It is clear that in this case $\Omega_x=[0,(a-\sqrt{a^2+2ab-2bc})/2]$ and $\Omega_y=[0,(b-\sqrt{b^2+2ab-2ac})/2]$; moreover, $0<(a-\sqrt{a^2+2ab-2bc})/2<a/2$ and $0<(b-\sqrt{b^2+2ab-2ac})/2<b/2$ (since $c>b\ge a$). Let us consider the point

$$(\widetilde{x},\widetilde{y})=\big((a-\sqrt{a^2+2ab-2bc})/2,\big(b-\sqrt{b^2+2ab-2ac}\big)/2\big).$$

If $x<\widetilde{x}$ and $y<\widetilde{y}$, then $d_1(x,y)>\sqrt{(a+b)^2+c^2}$ and $d_2(x,y)>\sqrt{(a+b)^2+c^2}$. Hence, the existence of a point $(x,y)\in(0,a/2]\times(0,b/2]$ with the condition

Fig. 4



$d_i(x, y) > \sqrt{(a+b)^2 + c^2}$ for $1 \le i \le 3$, is equivalent to the condition $d_3(\widetilde{x}, \widetilde{y}) > \sqrt{(a+b)^2 + c^2}$ (see Lemma 6). The latter inequality is equivalent to the following one:

$$\sqrt{a^2 + 2ab - 2bc} + \sqrt{b^2 + 2ab - 2ac} < 2c - a - b.$$

Therefore, $D(P) > \sqrt{(a+b)^2 + c^2}$ if and only if $\sqrt{a^2 + 2ab - 2bc} + \sqrt{b^2 + 2ab - 2ac} < 2c - a - b$ in this case.

Consequently, we have proved that $D(P) > \sqrt{(a+b)^2 + c^2}$ if and only if $(a, b, c) \in \mathcal{M} \setminus \mathcal{ME}$ (see (1) and (2)). The theorem is completely proved. □

4 The Case $a^2b^2 \ge c^2(b-a)(a+b+2c)$

The key idea lies in the following:

Lemma 8 *If* $a^2b^2 \ge (<) c^2(b-a)(a+b+2c)$, *then there exists a unique point* (*respectively*, *there is no point*) $(\widetilde{x}, \widetilde{y}) \in [0, a/2] \times [0, b/2]$ *such that* $d_1(\widetilde{x}, \widetilde{y}) = d_2(\widetilde{x}, \widetilde{y}) = d_3(\widetilde{x}, \widetilde{y})$.

Proof Consider functions $q_1, q_2 : [0, a/2] \times [0, b/2] \to \mathbb{R}$ defined as follows:

$$
\begin{aligned}
q_1(x,y) &= \frac{1}{8}\big(2\big(d_3(x,y)\big)^2 - \big(d_1(x,y)\big)^2 - \big(d_2(x,y)\big)^2\big) \\
&= \left(x + \frac{c-a}{2}\right)\left(y + \frac{c-b}{2}\right) - \frac{c^2-ab}{4}, \\
q_2(x,y) &= \frac{1}{4}\big(\big(d_1(x,y)\big)^2 - \big(d_2(x,y)\big)^2\big) \\
&= \left(y - \frac{b}{2}\right)^2 - \left(x - \frac{a}{2}\right)^2 - \frac{(b-a)(a+b+2c)}{4}.
\end{aligned}
\tag{7}
$$

Obviously, the condition $d_1(\widetilde{x},\widetilde{y}) = d_2(\widetilde{x},\widetilde{y}) = d_3(\widetilde{x},\widetilde{y})$ is equivalent to the equality $q_1(\widetilde{x},\widetilde{y}) = q_2(\widetilde{x},\widetilde{y}) = 0$. It is clear that the curves $q_1(x,y) = 0$ and $q_2(x,y) = 0$ are hyperbolas (the first (respectively, the second) curve is a pair of straight lines for $c = b = a$ (respectively, for $b = a$)). The first (the second) hyperbola has the center $((a-c)/2, (b-c)/2)$ (respectively, $(a/2, b/2)$). Let us describe intersections of these curves with the set $\Omega = [0, a/2] \times [0, b/2]$. Set

$$
L_1 = \{(x,y) \in \Omega \mid q_1(x,y) = 0\}, \qquad L_2 = \{(x,y) \in \Omega \mid q_2(x,y) = 0\}. \tag{8}
$$

In Fig. 4 the curves L_1 and L_2 are depicted for $(a,b,c) = (14,16,18)$. Note that for $c = b = a$ the curve L_1 is a union of two segments: with endpoints at the points $(0, b/2)$ and $(0,0)$, and also with endpoints at the points $(0,0)$ and $(a/2, 0)$. If $c > a$, then L_1 is the graph of the decreasing function $f(x) = (c^2 - ab)/(2(2x + c - a)) - (c-b)/2$ on the interval $[\frac{a(c-b)}{2c}, \frac{a}{2}]$. Note also that $f(\frac{a(c-b)}{2c}) = \frac{b}{2}$ and $f(\frac{a}{2}) = \frac{b}{2}(1 - \frac{a}{c}) = \frac{b(c-a)}{2c}$.

It is easy to see that for $b = a$ the curve L_2 is an intersection of the straight line $y = x + (b-a)/2$ with the set Ω. If $b > a$, then L_2 is either the empty set or it is a graph of the increasing function

$$
\widetilde{f}(x) = \frac{b}{2} - \sqrt{\left(x - \frac{a}{2}\right)^2 + \frac{(b-a)(a+b+2c)}{4}}.
$$

Note, that $\widetilde{f}(\frac{a}{2}) = \frac{b - \sqrt{(b-a)(a+b+2c)}}{2}$.

It is clear, that there exists a point $(\widetilde{x},\widetilde{y}) \in \Omega$ with the property $q_1(\widetilde{x},\widetilde{y}) = q_2(\widetilde{x},\widetilde{y}) = 0$ if and only if $\widetilde{f}(\frac{a}{2}) \geq f(\frac{a}{2})$; moreover, it is obvious that such a point is unique. Now it suffices to note that the latter inequality is equivalent to the inequality $a^2b^2 \geq c^2(b-a)(a+b+2c)$. □

Later on, in the case $a^2b^2 \geq c^2(b-a)(a+b+2c)$, we shall denote by $(\widetilde{x},\widetilde{y})$ the unique point in $\Omega = [0, a/2] \times [0, b/2]$ from the statement of Lemma 8. Moreover, we shall use the notation

$$
l := d_1(\widetilde{x},\widetilde{y}) = d_2(\widetilde{x},\widetilde{y}) = d_3(\widetilde{x},\widetilde{y}). \tag{9}
$$

Remark 3 It follows from (7) that $\widetilde{x} = a/2$ is equivalent to $a^2b^2 = c^2(b-a)(a+b+2c)$. Moreover, in this case we get $(\widetilde{x},\widetilde{y}) = (a/2, \tau_2) = (a/2, \frac{b(c-a)}{2c})$ (see Theorems 4 and 5). Lemma 10 implies that $(\widetilde{x},\widetilde{y}) = (0,0)$ if and only if $a = b = c$.

Lemma 9 *Suppose that the inequality* $a^2b^2 \geq c^2(b-a)(a+b+2c)$ *holds. Then*:

(1) $a^2(b^2+c^2) \geq b^2c^2$,
(2) $a(a+b+c) > bc$,
(3) $\frac{a(a+b+c)-bc}{a+c} \geq \frac{\sqrt{a^2b^2-c^2(b-a)(a+b+2c)}}{c}$.

Proof The first inequality is proved by the following computation:

$$a^2b^2 \geq c^2(b-a)(a+b+2c) \geq c^2(b-a)(a+b) = c^2\bigl(b^2-a^2\bigr).$$

Using Inequality (1), we get

$$a+b+c > b+c > \sqrt{b^2+c^2} \geq bc/a,$$

which proves (2). In force of Inequality (2), Inequality (3) is equivalent to the following:

$$\bigl(a(a+b+c)-bc\bigr)^2c^2 \geq (a+c)^2\bigl(a^2b^2-c^2(b-a)(a+b+2c)\bigr).$$

The latter inequality is proved by the following chain of pairwise equivalent inequalities:

$$\begin{aligned} &a^2(a+b)^2c^2+(a-b)c^3\bigl(c(a-b)+2a(a+b)\bigr)\\ &\quad\geq (a+c)^2a^2b^2-c^2(b-a)(a+c)^2(a+b+2c), \end{aligned}$$

$$\begin{aligned} &a^2\bigl((a+b)^2c^2-(a+c)^2b^2\bigr)+c^2(b-a)\bigl(\bigl(a^2+2ac+c^2\bigr)(a+b+2c)\\ &\quad-c\bigl(2a^2+2ab+ac-bc\bigr)\bigr) \geq 0, \end{aligned}$$

$$a^2\bigl(a^2\bigl(c^2-b^2\bigr)+2abc(c-b)\bigr)+c^2(b-a)\bigl(a^3+a^2b+2bc^2+2a^2c+4ac^2+2c^3\bigr) \geq 0,$$

where the latter one is obvious. □

Lemma 10 *Let* $a^2b^2 \geq c^2(b-a)(a+b+2c)$ *and let* $(\widetilde{x},\widetilde{y})$ *be the point in* $\Omega = [0,a/2]\times[0,b/2]$ *as in Lemma 8. Then*

$$\widetilde{x} \geq \frac{b(c-a)}{2(c+a)}, \qquad \widetilde{y} \geq \frac{b(c-a)}{2c} \geq \frac{a(c-b)}{2(c+b)}.$$

Proof In the case $c=b=a$ all are clear. Later on we shall assume that $c>a$. Recall (see the proof of Lemma 8), that $(\widetilde{x},\widetilde{y})$ is the point of intersection of the curves L_1 and L_2. Since L_1 is a graph of the decreasing function $f(x)=(c^2-ab)/(2(2x+c-a))-(c-b)/2$ on the interval $[\frac{a(c-b)}{2c},\frac{a}{2}]$, then $\widetilde{y}=f(\widetilde{x}) \geq f(\frac{a}{2})=\frac{b(c-a)}{2c}$. Note also that the inequality $\frac{b(c-a)}{2c} \geq \frac{a(c-b)}{2(c+b)}$ is equivalent to the (obvious) inequality $(b-a)c^2+b^2(c-a) \geq 0$.

Now we should prove that $\widetilde{x} \geq \frac{b(c-a)}{2(c+a)}$. Since the point $(\widetilde{x}, \widetilde{y})$ is on the curve L_2, then

$$\left(\frac{a}{2} - \widetilde{x}\right)^2 = \left(\frac{b}{2} - \widetilde{y}\right)^2 - \frac{(b-a)(a+b+2c)}{4}.$$

As we have shown above, $\widetilde{y} \geq \frac{b(c-a)}{2c}$; therefore, $(\frac{b}{2} - \widetilde{y})^2 \leq \frac{a^2b^2}{4c^2}$. Furthermore,

$$\left(\frac{a}{2} - \widetilde{x}\right)^2 \leq \frac{a^2b^2}{4c^2} - \frac{(b-a)(a+b+2c)}{4} = \frac{a^2b^2 - c^2(b-a)(a+b+2c)}{4c^2}.$$

Consequently,

$$\widetilde{x} \geq \frac{a}{2} - \frac{\sqrt{a^2b^2 - c^2(b-a)(a+b+2c)}}{2c}.$$

Then it suffices to show that $\frac{a}{2} - \frac{\sqrt{a^2b^2-c^2(b-a)(a+b+2c)}}{2c} \geq \frac{b(c-a)}{2(c+a)}$. But, as it is easy to see that the latter inequality is equivalent to Inequality (3) of Lemma 9, and the lemma has been proved. □

Lemma 11 *Let $a^2b^2 \geq c^2(b-a)(a+b+2c)$ and let $(\widetilde{x}, \widetilde{y})$ be the point in $\Omega = [0, a/2] \times [0, b/2]$ as in Lemma 8. Then (see (5))*

$$\overline{D}(\widetilde{x}, \widetilde{y}) = l = d_1(\widetilde{x}, \widetilde{y}) = d_2(\widetilde{x}, \widetilde{y}) = d_3(\widetilde{x}, \widetilde{y}).$$

Proof By definition of the function $\overline{D}$ (see (5)) it suffices to show that $d_4(\widetilde{x}, \widetilde{y}) \geq l$ and $d_5(\widetilde{x}, \widetilde{y}) \geq l$ (see (4)). It is easy to verify that

$$\big(d_2(x,y)\big)^2 - \big(d_5(x,y)\big)^2 = -4(a+c)\left(x - \frac{b(c-a)}{2(c+a)}\right).$$

By Lemma 10 it follows that $\widetilde{x} \geq \frac{b(c-a)}{2(c+a)}$, hence $l = d_2(\widetilde{x}, \widetilde{y}) \leq d_5(\widetilde{x}, \widetilde{y})$.

Analogously,

$$\big(d_1(x,y)\big)^2 - \big(d_4(x,y)\big)^2 = -4(b+c)\left(y - \frac{a(c-b)}{2(c+b)}\right).$$

By Lemma 10 it follows that $\widetilde{y} \geq \frac{a(c-b)}{2(c+b)}$, hence $l = d_1(\widetilde{x}, \widetilde{y}) \leq d_4(\widetilde{x}, \widetilde{y})$. □

Now we can state the main result of this section.

Theorem 4 *If $a^2b^2 \geq c^2(b-a)(a+b+2c)$, then the intrinsic diameter of the parallelepiped P can be calculated by the formula*

$$D(P) = \max\left\{\sqrt{(a+b)^2 + c^2},\, l\right\},$$

where l is defined by (9). Moreover, if $\sqrt{(a+b)^2+c^2} > l$ (respectively, $< l$), then $D(P) = \overline{D}(x,y)$ if and only if $(x,y) = (0,0)$ (respectively, $(x,y) = (\widetilde{x}, \widetilde{y})$). If $\sqrt{(a+b)^2+c^2} = l$, then $D(P) = \overline{D}(x,y)$ if and only if $(x,y) \in \{(0,0), (\widetilde{x}, \widetilde{y})\}$.

Proof According to Lemmas 2 and 11 the following equalities are true (see (5)):

$$\overline{D}(0,0)=\sqrt{(a+b)^2+c^2},\qquad \overline{D}(\widetilde{x},\widetilde{y})=l.$$

According to Proposition 3, it follows that

$$D(P)=\max\bigl\{\overline{D}(x,y)\mid 0\le x\le a/2,\ 0\le y\le b/2\bigr\},$$

therefore, it suffices to show that for any point $(x,y)\in\Omega=[0,a/2]\times[0,b/2]$, different from $(0,0)$ and $(\widetilde{x},\widetilde{y})$, the inequality $\overline{D}(x,y)<\max\{\sqrt{(a+b)^2+c^2},\,l\}$ is true. In other words, we should prove that for such a point (x,y) there is $1\le i\le 5$ such that $d_i(x,y)<\max\{\sqrt{(a+b)^2+c^2},\,l\}$ (see (4)).

Let us consider an arbitrary point $(x,y)\in\Omega$, different from $(0,0)$ and $(\widetilde{x},\widetilde{y})$. There are 3 cases: (1) $x>\widetilde{x}$, (2) $y>\widetilde{y}$, (3) $0<x+y<\widetilde{x}+\widetilde{y}$. We consider these cases successively.

In the first case

$$l=d_2(\widetilde{x},\widetilde{y})=\sqrt{(b+c)^2+(a-2\widetilde{x})^2}>\sqrt{(b+c)^2+(a-2x)^2}=d_2(x,y),$$

i.e. $d_2(x,y)<l$.

In the second case

$$l=d_1(\widetilde{x},\widetilde{y})=\sqrt{(a+c)^2+(b-2\widetilde{y})^2}>\sqrt{(a+c)^2+(b-2y)^2}=d_1(x,y),$$

i.e. $d_1(x,y)<l$.

Consider now the third case. The function $h(t)=2t^2+2(c-a-b)t+(a+b)^2+c^2$ is strictly convex. Therefore,

$$\max_{0\le t\le\widetilde{x}+\widetilde{y}}h(t)=\max\bigl\{h(0),\ h(\widetilde{x}+\widetilde{y})\bigr\}=\max\bigl\{(a+b)^2+c^2,\ l^2\bigr\},$$

since $h(\widetilde{x}+\widetilde{y})=(d_3(\widetilde{x},\widetilde{y}))^2=l^2$. Consequently,

$$\bigl(d_3(x,y)\bigr)^2=h(x+y)<\max_{0\le t\le\widetilde{x}+\widetilde{y}}h(t)=\max\bigl\{(a+b)^2+c^2,\ l^2\bigr\}$$

in this case, i.e. $d_3(x,y)<\max\{\sqrt{(a+b)^2+c^2},\,l\}$. The theorem is proved. □

Lemma 12 *Let $a^2b^2\ge c^2(b-a)(a+b+2c)$ and $l\ge\sqrt{(a+b)^2+c^2}$, where l is defined by (9). Then l is the unique solution of the equation*

$$\sqrt{l^2-(a+c)^2}+\sqrt{l^2-(b+c)^2}+\sqrt{2l^2-(a+b+c)^2}=c$$

under the condition $l\ge\max\{b+c,\sqrt{(a+b)^2+c^2}\}$.

Proof The inequality $l\ge b+c\ge a+c$ is obvious ($d_2(\widetilde{x},\widetilde{y})\ge b+c$). If in addition $l\ge\sqrt{(a+b)^2+c^2}$, then

$$\begin{aligned}2l^2&\ge\bigl(b^2+2bc+c^2\bigr)+\bigl(a^2+2ab+b^2+c^2\bigr)=a^2+b^2+c^2+2ab+2ac+2bc\\&=(a+b+c)^2.\end{aligned}$$

The number l (see (4) and (9)) satisfies the equations

$$l = d_1(\widetilde{x}, \widetilde{y}), \qquad l = d_2(\widetilde{x}, \widetilde{y}),$$

from which we get

$$\widetilde{x} = \frac{a - \sqrt{l^2 - (b+c)^2}}{2}, \qquad \widetilde{y} = \frac{b - \sqrt{l^2 - (a+c)^2}}{2}.$$

Now, substituting the obtained expressions into the equality $l^2 = (d_3(\widetilde{x}, \widetilde{y}))^2$, after some simple calculations we get

$$\sqrt{l^2 - (a+c)^2} + \sqrt{l^2 - (b+c)^2} + \sqrt{2l^2 - (a+b+c)^2} = c.$$

Since the function $t \mapsto \sqrt{t^2 - (a+c)^2} + \sqrt{t^2 - (b+c)^2} + \sqrt{2t^2 - (a+b+c)^2}$ increases on the interval $[\max\{b+c, \sqrt{(a+b)^2 + c^2}\}, \infty)$, then such a number l is unique. $\square$

5 The Case $a^2b^2 \le c^2(b-a)(a+b+2c)$

In this case we should study in detail the points $(x, y) = (a/2, t)$, where $0 \le t \le a/2\,(\le b/2)$. Put

$$\begin{aligned}
g_1(t) &= \big(d_1(a/2, t)\big)^2 = (a+c)^2 + (b-2t)^2,\\
g_2(t) &= \big(d_2(a/2, t)\big)^2 = (b+c)^2,\\
g_3(t) &= \big(d_3(a/2, t)\big)^2 = 2t^2 + 2(c-b)t + (b+a/2)^2 + (c+a/2)^2,\\
g_4(t) &= \big(d_4(a/2, t)\big)^2 = (a+b)^2 + (c+2t)^2,\\
g_5(t) &= \big(d_5(a/2, t)\big)^2 = (a+b)^2 + (a+c)^2.
\end{aligned} \tag{10}$$

The following lemma is obvious.

Lemma 13 *For any $t \in [0, a/2]$ the inequality $g_5(t) \ge g_4(t)$ is true.*

By direct calculations we get

Lemma 14 *The function g_1 is decreasing, but the functions g_3 and g_4 are increasing on the interval $[0, a/2]$. Moreover, the following inequalities are true:*

$$g_1(0) > g_3(0), \qquad g_1(a/2) < g_3(a/2), \qquad g_1(0) \ge g_4(0), \qquad g_1(a/2) < g_4(a/2).$$

Lemma 15 *Let $\tau_1 = \frac{a(c-b)}{2(c+b)}$. Then $g_1(t) > g_4(t)$ (respectively, $g_1(t) < g_4(t)$) for $t \in [0, \tau_1)$ (respectively, $t \in (\tau_1, a/2]$). Moreover,*

$$g_1(\tau_1) = g_4(\tau_1) = (b^2 + c^2)\left(1 + \frac{2a}{b+c} + \frac{2a^2}{(b+c)^2}\right)$$
$$> (b^2 + c^2)\left(1 + \frac{a}{b+c}\right)^2 = g_3(\tau_1).$$

Proof The point τ_1 is the unique point $t \in [0, a/2]$ with the property (see Lemma 14) $g_1(t) = g_4(t)$. All other relations are proved by direct calculations. □

Later on we shall need the following values:

$$\tau_2 = \frac{b + c - \sqrt{(b+c)^2 - 2a(c-b) - a^2}}{2}, \tag{11}$$

$$\widetilde{l} = \sqrt{b^2 + 3c^2 + 2b(a+c) - 2c\sqrt{(b+c)^2 - 2a(c-b) - a^2}}. \tag{12}$$

Lemma 16 *For the values* (11) *and* (12) *the following are true*: $\tau_2 \in (\tau_1, a/2)$, $g_1(\tau_2) = g_3(\tau_2) = \widetilde{l}^2$, $g_1(t) > g_3(t)$ (*respectively*, $g_1(t) < g_3(t)$) *for* $t \in [0, \tau_2)$ (*respectively*, $t \in (\tau_2, a/2]$), $g_5(\tau_2) \geq g_4(\tau_2) > \widetilde{l}^2$. *If in addition* $a^2b^2 \leq c^2(b-a)(a+b+2c)$, *then* $g_2(\tau_2) = (b+c)^2 \geq \widetilde{l}^2$ *and* $\overline{D}(a/2, \tau_2) = \widetilde{l}$.

Proof By Lemma 14 there exists a number $\eta \in (0, a/2)$ such that $g_1(\eta) = g_3(\eta)$ and $g_1(t) > (<) g_3(t)$ for $t \in [0, \eta)$ (respectively, $t \in (\eta, a/2]$). Direct computations show that $\eta = \tau_2$. The equality $g_1(\tau_2) = g_3(\tau_2) = \widetilde{l}^2$ is verified directly.

According to Lemma 15 $g_1(\tau_1) > g_3(\tau_1)$, hence $\tau_2 > \tau_1$. Further, $g_4(\tau_2) > g_4(\tau_1) = g_1(\tau_1) > g_1(\tau_2) = \widetilde{l}$, since g_4 increases, but g_1 decreases. By Lemma 13 $g_5(\tau_2) \geq g_4(\tau_2) > \widetilde{l}$.

Finally, note that the inequality $g_2(\tau_2) = (b+c)^2 \geq \widetilde{l}^2$ is equivalent to the inequality $a^2b^2 \leq c^2(b-a)(a+b+2c)$.

The above reasoning shows that for $a^2b^2 \leq c^2(b-a)(a+b+2c)$ the equality $\overline{D}(a/2, \tau_2) = \widetilde{l}$ holds. The lemma is proved. □

Now we state the main result of this section.

Theorem 5 *If* $a^2b^2 \leq c^2(b-a)(a+b+2c)$, *then* $D(P) = \max\{\sqrt{(a+b)^2 + c^2}, \widetilde{l}\}$, *where* $\widetilde{l}$ *is defined by the equality* (12). *Moreover, if* $\sqrt{(a+b)^2 + c^2} > \widetilde{l}$ (*respectively*, $\sqrt{(a+b)^2 + c^2} < \widetilde{l}$), *then* $D(P) = \overline{D}(x, y)$ *if and only if* $(x, y) = (0, 0)$ (*respectively*, $(x, y) = (a/2, \tau_2)$, *where* τ_2 *is defined by* (11)). *If* $\sqrt{(a+b)^2 + c^2} = \widetilde{l}$, *then* $D(P) = \overline{D}(x, y)$ *if and only if* $(x, y) \in \{(0, 0), (a/2, \tau_2)\}$.

Proof According to Lemmas 2 and 16 the following equalities (see (5)) hold:

$$\overline{D}(0, 0) = \sqrt{(a+b)^2 + c^2}, \qquad \overline{D}(a/2, \tau_2) = \widetilde{l}.$$

Note that the function

$$h(t) = (c+t)^2 + (a+b-t)^2 = 2t^2 + 2(c-a-b)t + (a+b)^2 + c^2$$

is strictly convex on the interval $[0, a/2+\tau_2]$. Besides this, $h(0) = (a+b)^2 + c^2$ and $h(a/2+\tau_2) = g_3(\tau_2) = \widetilde{l}^{\,2}$. Consequently, $h(t) < \max\{(a+b)^2 + c^2, \widetilde{l}^{\,2}\}$ for $t \in (0, a/2+\tau_2)$.

Let us now consider an arbitrary point $(x, y) \in \Omega = [0, a/2] \times [0, b/2]$, different from $(0, 0)$ and $(a/2, \tau_2)$. If $y > \tau_2$, then

$$\begin{aligned}\overline{D}(x,y) \le d_1(x,y) &= \sqrt{(a+c)^2 + (b-2y)^2} < \sqrt{(a+c)^2 + (b-2\tau_2)^2}\\ &= \sqrt{g_1(\tau_2)} = \widetilde{l}.\end{aligned}$$

If $y \le \tau_2$, then $0 < x + y < a/2 + \tau_2$, and

$$\overline{D}(x,y) \le d_3(x,y) = \sqrt{h(x+y)} < \max\left\{\sqrt{(a+b)^2+c^2}, \widetilde{l}\right\},$$

and the theorem is proved. □

6 Proofs of the Main Theorems

Note that the proof of Theorem 1 follows directly from Theorems 3, 4, 5, and from Lemma 12.

In order to prove Theorem 2 we need some auxiliary results.

Lemma 17 *If* $0 \le a \le b \le 0.77c$ *and* $c \ne 0$, *then*

$$ab + ac + bc < \frac{1+2\sqrt{2}}{6}\left(b^2 + 3c^2 + 2b(a+c) - 2c\sqrt{(b+c)^2 - 2a(c-b) - a^2}\right).$$

Proof Using the homogeneity, we may assume that $c = 1$. Therefore, we should show that for $0 \le a \le b \le 0.77$ the inequality

$$3 + b^2 + 2ab + 2b - 2\sqrt{1 + b^2 + 2ab + 2b - 2a - a^2} > \frac{6}{1+2\sqrt{2}}(ab + a + b)$$

holds. Solving the equation $3 + b^2 + 2ab + 2b - 2\sqrt{1 + b^2 + 2ab + 2b - 2a - a^2} = \frac{6}{1+2\sqrt{2}}(ab + a + b)$ for a, we get that

$$a = \frac{F_1(b) - 7\sqrt{F_2(b)}}{F_3(b)} \quad \text{or} \quad a = \frac{F_1(b) + 7\sqrt{F_2(b)}}{F_3(b)},$$

where

$$F_1(b) = \left(70 - 42\sqrt{2}\right)b^3 + \left(365 - 282\sqrt{2}\right)b^2 + \left(316 - 282\sqrt{2}\right)b + 161 - 126\sqrt{2},$$

$$\begin{aligned}F_2(b) = {} & \left(359 - 312\sqrt{2}\right)b^4 + \left(732 - 456\sqrt{2}\right)b^3 + \left(1858 - 1392\sqrt{2}\right)b^2\\ & + \left(2028 - 1032\sqrt{2}\right)b + 527 - 648\sqrt{2},\end{aligned}$$

$$F_3(b) = \big(240\sqrt{2} - 344\big)b^2 + \big(312\sqrt{2} - 408\big)b + 72\sqrt{2} - 260.$$

In order to get real solutions, the inequality $F_2(b) \ge 0$ must hold. The polynomial $F_2(b)$ has exactly two real roots: $r_1 = 0.7858653434\dots$ and $r_2 = 1.698023859\dots$. Since $(359 - 312\sqrt{2}) < 0$, then $F_2(b) < 0$ for $b \le 0.77 < r_1$. Consequently, for $0 \le a \le b \le 0.77$ the expression $3 + b^2 + 2ab + 2b - 2\sqrt{1 + b^2 + 2ab + 2b - 2a - a^2} - \frac{6}{1+2\sqrt{2}}(ab + a + b)$ does not change the sign. Substituting the point $(a, b) = (0, 0)$, we conclude that it is positive for $0 \le a \le b \le 0.77$. The lemma is proved. □

Lemma 18 *If $a + b \le \sqrt{2}c$ and $a^2b^2 \ge c^2(b - a)(a + b + 2c)$, then $a^2 + 4c^2 - 2ac - 4bc > 0$ for any $0 < a \le b \le c$.*

Proof It suffices to consider the case $c = 1$. The conditions of the lemma can be rewritten as follows: $b \le h_1(a)$ and $b \le h_2(a)$, where

$$h_1(a) = \sqrt{2} - a, \qquad h_2(a) = \frac{\sqrt{1 + 2a + a^2 - 2a^3 - a^4} - 1}{1 - a^2}.$$

The function h_1 decreases. It is easy to verify that the function h_2 increases on the interval [0, 1]. The graphs of these two functions intersect each other at the point

$$(a_1, b_1) = (0.6706890957\dots, 0.7435244663\dots)$$

(more precisely, a_1 is a root of the equation $t^4 - 2\sqrt{2}t^3 + 2t^2 + (4 + 2\sqrt{2})t - 2\sqrt{2} - 2 = 0$). Therefore, for any point $(a, b, 1)$ satisfying the conditions of the lemma, the inequality $b < 0.745$ holds.

On the other hand, the condition $a^2 + 4 - 2a - 4b \le 0$ is equivalent to the inequality $h_3(a) \ge 0$, where $h_3(a) = (a - 1)^2/4 + 3/4$. Obviously, $h_3(a) \ge h_3(1) = 3/4 = 0.75$ for any $a \in \mathbb{R}$.

Therefore, for any point $(a, b, 1)$ satisfying the conditions of the lemma, the inequality $a^2 + 4 - 2a - 4b > 0$ holds. □

Lemma 19 *If $a + b \le \sqrt{2}c$, $a^2b^2 \ge c^2(b - a)(a + b + 2c)$ and $(a, b, c) \in \mathcal{ME}$ (see (2)), then $a : b : c = 1 : 1 : \sqrt{2}$. In particular, the intrinsic diameter of the parallelepiped P with edge lengths $0 < a \le b \le c$ satisfying $a + b \le \sqrt{2}c$ and $a^2b^2 \ge c^2(b - a)(a + b + 2c)$ is equal to l from the statement of Theorem 1.*

Proof Since $(a, b, c) \in \mathcal{ME}$, the inequality

$$\sqrt{\max\{0, a^2 + 2ab - 2bc\}} + \sqrt{\max\{0, b^2 + 2ab - 2ac\}} \ge 2c - a - b$$

holds. Since $\sqrt{\alpha} + \sqrt{\beta} \le \sqrt{2(\alpha + \beta)}$ for any $\alpha, \beta \ge 0$ (here the equality holds if and only if $\alpha = \beta$), then

$$(2c - a - b)^2 \le 2\big(\max\{0, a^2 + 2ab - 2bc\} + \max\{0, b^2 + 2ab - 2ac\}\big).$$

From this we deduce that $b^2 + 2ab - 2ac > 0$. Now we consider two cases: (1) $a^2 + 2ab - 2bc < 0$ and (2) $a^2 + 2ab - 2bc \ge 0$.

In the first case, by Lemma 18 we get that $a^2+4c^2-2ac-4bc>0$, but the latter contradicts to the inequality $(2c-a-b)^2 \le b^2+2ab-2ac$. Therefore, $(a,b,c) \notin \mathcal{ME}$.

In the second case we get that

$$3(a+b)^2-4(a+b)c \ge 2\big((a^2+2ab-2bc)+(b^2+2ab-2ac)\big) \ge (2c-a-b)^2.$$

Put $t=(a+b)/c$. According to the conditions of the lemma, $t \in [0,\sqrt{2}]$. It follows from the latter inequality that $3t^2-4t \ge (2-t)^2$, i.e. $t \ge \sqrt{2}$. Consequently, $(a,b,c) \in \mathcal{ME}$ if and only if $a:b:c=1:1:\sqrt{2}$.

The second statement of the lemma follows directly from the first one and from Theorem 1. □

Lemma 20 *Suppose that $a+b \le \sqrt{2}c$, $a^2b^2 \ge c^2(b-a)(a+b+2c)$, and a number l satisfies the condition* (*see* (3))

$$\sqrt{l^2-(a+c)^2}+\sqrt{l^2-(b+c)^2}+\sqrt{2l^2-(a+b+c)^2}=c.$$

Then the inequality

$$ab+ac+bc \le \frac{1+2\sqrt{2}}{6}\, l^2$$

holds with equality only when $a:b:c=1:1:\sqrt{2}$.

Proof Since for any $\alpha,\beta \ge 0$ we have that $\sqrt{a}+\sqrt{b} \le \sqrt{2(\alpha+\beta)}$, then

$$\begin{aligned}
&\sqrt{l^2-(a+c)^2}+\sqrt{l^2-(b+c)^2}\\
&\quad \le \sqrt{4l^2-2\big(a^2+b^2+c^2+2ac+2bc+c^2\big)}\\
&\quad \le \sqrt{4l^2-2\big(a^2+b^2+c^2+2ac+2bc+2ab\big)}=\sqrt{4l^2-2(a+b+c)^2},
\end{aligned}$$

because $2ab \le c^2$. The latter inequality holds since, according to the conditions of the lemma, $2\sqrt{ab} \le a+b \le \sqrt{2}c$.

Further,

$$\begin{aligned}
c&=\sqrt{l^2-(a+c)^2}+\sqrt{l^2-(b+c)^2}+\sqrt{2l^2-(a+b+c)^2}\\
&\le (1+\sqrt{2})\sqrt{2l^2-(a+b+c)^2},
\end{aligned}$$

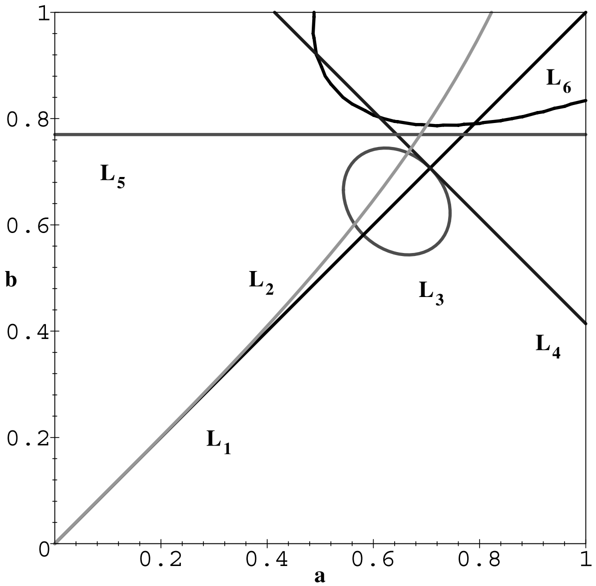
therefore,

$$c^2 \le \big(3+2\sqrt{2}\big)\big(2l^2-(a+b+c)^2\big). \tag{13}$$

Now we consider the polynomial $f(t)=(1+\sqrt{2})t^2-(7+4\sqrt{2})t+6+5\sqrt{2}$, which has the roots $t_1=\sqrt{2}$ and $t_2=1+2\sqrt{2}$. It is clear that $f(t) \ge (>)\,0$ for $t \le \sqrt{2}$ (for $t<\sqrt{2}$). If we set $t=(a+b)/c$, then from the above inequality we get

$$\big(1+\sqrt{2}\big)(a+b)^2-\big(7+4\sqrt{2}\big)(a+b)c+\big(6+5\sqrt{2}\big)c^2 \ge (>)0 \tag{14}$$

Fig. 5



for $a+b \le \sqrt{2}c$ (respectively, $a+b < \sqrt{2}c$). Note also that the inequality (14) is equivalent to the next one:

$$\frac{c^2}{6+4\sqrt{4}} + \frac{(a+b+c)^2}{2} \ge (>) \frac{6}{1+2\sqrt{2}} \left(ac + bc + \left(\frac{a+b}{2} \right)^2 \right).$$

According to the inequality (13), $l^2 \ge \frac{c^2}{6+4\sqrt{4}} + \frac{(a+b+c)^2}{2}$, therefore,

$$l^2 \ge (>) \frac{6}{1+2\sqrt{2}} \left(ac + bc + \left(\frac{a+b}{2} \right)^2 \right),$$

for $a+b \le \sqrt{2}c$ (respectively, for $a+b < \sqrt{2}c$). Since $(a+b)^2 \ge 4ab$ with equality only when $a=b$, we get the lemma. □

Proof of Theorem 2 For any parallelepiped P with the condition $a:b:c = 1:1:\sqrt{2}$ we have $D(P) = \sqrt{6}a$ (see Theorem 1), and, consequently, in this case $ab+ac+bc = (1+2\sqrt{2})a^2 = \frac{1+2\sqrt{2}}{6}(D(P))^2$.

Further we shall show that if the condition $a:b:c = 1:1:\sqrt{2}$ does not hold, then the inequality

$$ab+ac+bc < \frac{1+2\sqrt{2}}{6}(D(P))^2 \tag{15}$$

holds.

In order to prove the inequality (15) it suffices to choose $(x, y) \in [0, a/2] \times [0, b/2]$ such that

$$ab + ac + bc < \frac{1+2\sqrt{2}}{6}\big(\overline{D}(x, y)\big)^2, \tag{16}$$

since $\overline{D}(x, y) \le D(P)$ (see Proposition 3). The most useful pair for this goal is the pair $(x, y) = (0, 0)$ which correspond to a vertex of the parallelepiped. By Lemma 2, $\overline{D}(0, 0) = \sqrt{(a+b)^2 + c^2}$.

Using the similarity, we may suppose $c = 1$. Let us describe the set of points (a, b) satisfying the inequality

$$ab + a + b < \frac{1+2\sqrt{2}}{6}\big((a+b)^2 + 1\big). \tag{17}$$

Note that the set E of points (a, b) satisfying the condition

$$(a+b)^2 + 1 - \frac{6}{1+2\sqrt{2}}(ab + a + b) = 0,$$

is an ellipse symmetric relative to the straight line $b = a$ and with the center at the point (u, u), where $u = (3 + 12\sqrt{2})/31 = 0.6442117015\ldots$ (see the curve L_3 on Fig. 5). Respectively, the points (a, b) satisfying the inequality (17) constitute the exterior of the ellipse E.

Therefore, we should prove the inequality (15) for all points on the ellipse E and for all points in its interior. Let IE be a set of these points, i.e. a set of points (a, b) satisfying the inequalities $(a+b)^2 + 1 \le \frac{6}{1+2\sqrt{2}}(ab + a + b)$.

It is easy to verify that $IE \subset [u_1, u_2] \times [u_1, u_2]$, where

$$u_1 = \frac{3}{31} + \frac{12}{31}\sqrt{2} - \frac{1}{93}\sqrt{276\sqrt{2} - 303} = 0.5437313296\ldots,$$

$$u_2 = \frac{3}{31} + \frac{12}{31}\sqrt{2} + \frac{1}{93}\sqrt{276\sqrt{2} - 303} = 0.7446920734\ldots$$

(sides of the square are tangent to the ellipse E). Besides this, the straight line $a + b = \sqrt{2}$ is tangent to the ellipse E at the point $(1/\sqrt{2}, 1/\sqrt{2})$, and all the points in IE satisfying the inequality $a + b \le \sqrt{2}$.

These arguments show that for points $(a, b) \in IE$ the inequalities $b < 0.745$ and $a + b \le \sqrt{2}$ hold.

Further we consider two cases: (1) $a^2b^2 \le c^2(b-a)(a+b+2c) = (b-a)(a+b+2)$ and (2) $a^2b^2 \ge c^2(b-a)(a+b+2c) = (b-a)(a+b+2)$.

In the first case, by Lemma 17 we get the inequality (we have chosen $c = 1$)

$$ab + a + b < \frac{1+2\sqrt{2}}{6}\big(b^2 + 3 + 2b(a+1) - 2\sqrt{(b+1)^2 - 2a(1-b) - a^2}\,\big).$$

According to Theorem 5, $b^2 + 3 + 2b(a + 1) - 2\sqrt{(b+1)^2 - 2a(1-b) - a^2} \le (D(P))^2$, for the parallelepiped P with edge lengths $0 < a \le b \le c = 1$. Therefore,

$$ab + a + b < \frac{1+2\sqrt{2}}{6}\big(D(P)\big)^2.$$

Consequently, we have proved the inequality (15) in this case.

Let us consider the second case. According to Lemma 19 $D(P) = l$ in this case, where l is taken from the statement of Theorem 1. Now by Lemma 20 we get the inequality (recall, that $c = 1$)

$$ab + a + b \le \frac{1+2\sqrt{2}}{6} l^2 = \frac{1+2\sqrt{2}}{6}\big(D(P)\big)^2,$$

which becomes an equality if and only if $a = b = 1/\sqrt{2}$. The theorem is completely proved.

In order to clarify the proof of Theorem 2, one can use Fig. 5. In this picture the straight line $b = a$ is denoted by L_1, the curve $a^2b^2 - (b-a)(a+b+2) = 0$ by L_2, the curve $(a+b)^2 + 1 - \frac{6}{1+2\sqrt{2}}(ab + a + b) = 0$ (i.e. the ellipse E) by L_3, the straight line $a + b = \sqrt{2}$ by L_4, the straight line $b = 0.77$ by L_5 and, finally, the curve $3 + b^2 + 2ab + 2b - 2\sqrt{1 + b^2 + 2ab + 2b - 2a - a^2} = \frac{6}{1+2\sqrt{2}}(ab + a + b)$ by L_6. $\square$

Acknowledgements This project was supported in part by the Russian Foundation for Basic Research (grant 06-01-81002) and the State Maintenance Program for the Leading Scientific Schools and Young Russian Scientists of the Russian Federation (grants NSH-8526.2006.1 and MD-5179.2006.1). We thank A. Arvanitoyeorgos, V.N. Berestovskii, V.K. Ionin, K.O. Kizbikenov, J. O'Rourke, E.D. Rodionov, V.V. Schipunov, and V.V. Slavskii for helpful discussions concerning this project.

We are grateful to both referees, whose comments and suggestions permitted us to improve the presentation of this article.

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