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Pebble Sets in Convex Polygons

Kevin Iga · Randall Maddox

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Abstract Lukács and András posed the problem of showing the existence of a set of $n-2$ points in the interior of a convex n-gon so that the interior of every triangle determined by three vertices of the polygon contains a unique point of S. Such sets have been called *pebble sets* by De Loera, Peterson, and Su. We seek to characterize all such sets for any given convex polygon in the plane.

We first consider a certain class of pebble sets, called peripheral because they consist of points that lie close to the boundary of the polygon. We characterize all peripheral pebble sets, and show that for regular polygons, these are the only ones. Though we demonstrate examples of polygons where there are other pebble sets, we nevertheless provide a characterization of the kinds of points that can be involved in non-peripheral pebble sets. We furthermore describe algorithms to find such points.

1 Introduction

Lukács and András posed the following in [1]: Prove that there exists a set S of $n-2$ points in the interior of a convex n-gon such that for any three vertices of the n-gon, the interior of the triangle determined by the three vertices contains exactly one element of S. Many solutions to this problem were given, one of which was published in [2]. In [3], De Loera, Peterson, and Su employ analogous sets in d-dimensional polytopes to prove a generalization of Sperner's Lemma. Following the terminology in [3], we will call a solution to the question posed in [1] a *pebble set*.

In this paper, we consider the problem of characterizing all pebble sets in a given convex n-gon. We begin by characterizing a certain class of pebble sets, which we call *peripheral*, since the points are near the boundary of the polygon. For some kinds of

K. Iga (✉) · R. Maddox
Natural Science Division, Pepperdine University, 24255 Pacific Coast Hwy, Malibu,
CA 90263-4321, USA
e-mail: kiga@pepperdine.edu

polygons, such as regular polygons, these are in fact the only pebble sets, but for other kinds of polygons, there are others. If we take a certain natural notion of equivalence of pebble sets, then this provides a lower bound to the number of pebble sets as a function of n.

In some cases, there may be pebble sets that are not peripheral, and we study necessary and sufficient conditions for such to exist. We also give a construction for analyzing polygons in which such pebble sets exist, breaking any such set down into pebble sets in smaller polygons. This allows us to characterize pebble sets for many polygons.

We begin in Sect. 2 with a more precise statement of the problem and some preliminary remarks about pebble sets. This section also contains terminology and notation that will be used throughout.

Section 3 introduces a construction that provides peripheral pebble sets, and proves that all peripheral pebble sets are of this type. Section 4 deals with non-peripheral pebble sets, giving a necessary and sufficient condition for such to exist. Here we also prove that there are no such pebble sets when the polygon is regular. In Sect. 5, we provide an efficient algorithm for determining if this condition is satisfied. Finally, in Sect. 6, we consider questions for further research.

2 Preliminaries

In this paper all polygons are assumed to be convex. In general, we will label the vertices and edges of an n-gon in the counterclockwise direction with the sequence $\langle v_0, e_0, \ldots, v_{n-1}, e_{n-1}\rangle$. We consider the subscripts for the vertices and edges as integers modulo n, so that for instance $v_n = v_0$ and $e_n = e_0$.

We will say that three vertices are *consecutive* if they are of the form v_k, v_{k+1}, v_{k+2}, and that two vertices are *adjacent* if they are of the form v_k and v_{k+1}. A vertex is *incident* with an edge if it is an endpoint of the edge. Two edges are *incident* if they are of the form e_k and e_{k+1}. By a *chord* of a polygon, we mean the segment joining two non-adjacent vertices of the polygon.

Definition 1 A *pebble set* S in a convex n-gon P in the plane is a set of $n-2$ points in the interior of P so that every triangle determined by vertices of P contains exactly one point of S in its interior.

First we note the following.

Proposition 1 *If S is a pebble set in an n-gon P, then no point in S lies on a chord of P.*

Proof Let p be a point on a chord f in P, and consider a triangulation of P that includes f. This triangulation contains $n-2$ triangles with pairwise disjoint interiors. Thus no pebble set can contain p. □

By a *chamber* of a polygon, we mean a maximal connected subset of the polygon that does not intersect any edges or chords of the polygon. We will say that two subsets of the polygon are *equivalent* if the sets of chambers containing points in the subsets are identical. Clearly if S is a pebble set that contains a point p in a chamber C, and if q is any point in C, then $[S - \{p\}] \cup \{q\}$ is also a pebble set. Therefore the property of being a pebble set depends only on the equivalence class.

With a slight abuse of notation, we will refer to an entire equivalence class of pebble sets as a pebble set, and note that we could construct pebble sets by merely specifying an appropriate set of chambers.

For convenience, when we refer to a triangle as a subset of the plane, we actually mean the interior of the triangle, since points in a pebble set by definition lie in the interiors of the triangles.

If u, v and w are distinct vertices in a polygon, we will refer to the triangle they form as Δuvw. We will refer to the triangle formed by an edge or chord e and vertex v as Δev, and the triangle formed by two incident edges or chords e and f as Δef.

If v_{k-1}, v_k, v_{k+1} are consecutive vertices, then we call $\Delta v_{k-1}v_kv_{k+1}$ a *border triangle*, and we will refer to this triangle as the *border triangle at* v_k. We call the union of the border triangles the *periphery* of the polygon. The complement of the periphery in the polygon will be called the *core* of the polygon.

Since there are n border triangles, then the fact that a pebble set contains $n-2$ points necessitates that at least two points in a pebble set will lie in the intersection of two overlapping border triangles. We will say that such a point lies *close to an edge* of the polygon. If a pebble set contains precisely two points that are close to an edge of the polygon, then the remaining points in the pebble set must be distributed among the remaining border triangles in a one-to-one correspondence, revealing that all points in the pebble set lie in the periphery. More generally, the pigeonhole property leads us to the following observation.

Proposition 2 *If a pebble set in an n-gon contains $k \geq 2$ points that are close to an edge of the polygon, then there are precisely $k-2$ points in the pebble set that lie in the core.*

The following will be helpful in showing that a set of points is a pebble set.

Lemma 3 *Suppose S is a set of $n-2$ points in an n-gon P, none of which lies on a chord. Let $\mathcal{T}$ be the set of triangles in P of the form Δev, where e is an edge and v a vertex of P. If every triangle in $\mathcal{T}$ contains at most one point of S, then S is a pebble set.*

Proof First we show that every triangle in $\mathcal{T}$ contains precisely one point of S. Let $T \in \mathcal{T}$. Then there exist an edge e and a vertex v such that $T = \Delta ev$. Triangulate P with all chords from v. Every triangle in this triangulation is in $\mathcal{T}$, and thus contains at most one point of S. Since there are $n-2$ triangles in this triangulation, and since no point of S lies on a chord of P, we have that T contains exactly one point of S.

Now we show that S is a pebble set. Let T be any triangle determined by three vertices of P. If $T \in \mathcal{T}$, then T contains precisely one point of S. So suppose $T \notin \mathcal{T}$. Then $T = \Delta v_iv_jv_k$, where no two of $\{v_i, v_j, v_k\}$ are adjacent. Assume that $0 \leq i < j < k \leq n-1$. Working counterclockwise, triangulate P with chords from v_i to all vertices v_{i+2} to v_j, chords from v_j to all vertices v_{j+2} to v_k, and chords from v_k to

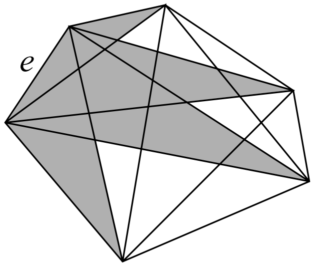


Fig. 1 Shaded region is the fan of edge e

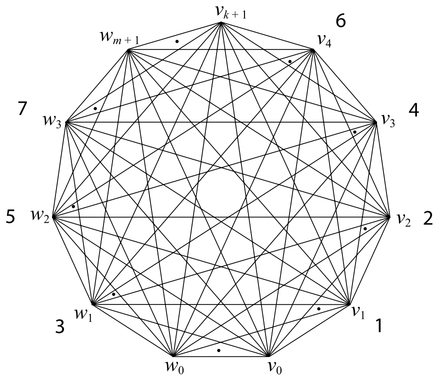


Fig. 2 A periphery pebble set for an 11-gon where $A = \{1, 2, 4, 6\}$

all vertices v_{k+2} to v_i. Every triangle except $\Delta v_i v_j v_k$ in this triangulation contains precisely one edge of P, and therefore contains precisely one point of S. Since there are $n - 3$ triangles in the triangulation other than $\Delta v_i v_j v_k$, and since no point of S lies on a chord of P, there must be precisely one point of S in $\Delta v_i v_j v_k$. □

Definition 2 For a given edge e of a polygon, the union of all triangles of the form Δev across all vertices v of P is called the *fan* of e. (See Fig. 1.)

Lemma 3 implies that it is only necessary to check that the triangles in the fans of the edges of a polygon contain at most one point of a set S to determine if S is a pebble set.

3 Characterizing Peripheral Pebble Sets

We say that a pebble set is *peripheral* provided all points in the pebble set lie in the periphery. We present a construction whereby all peripheral pebble sets of an n-gon can be obtained.

Construction 1 Given any subset of $W = \{1, 2, \ldots, n - 4\}$, we may determine a unique pebble set that contains a point close to edge $v_0 v_{n-1}$ in the following way.

First let A be a subset of W, and suppose A has k elements. (See Fig. 2.) Let $B = W - A$, so that B has $m = n - 4 - k$ elements. We begin by relabeling some of the vertices and edges of P for convenience. For $0 \le i \le m+1$, we let $w_i = v_{n-1-i}$. Label edges $f_i = w_i w_{i+1}$ for $0 \le i \le m$. For $0 \le i \le k$, let T_i denote the border triangle at v_i, and for $0 \le i \le m$, let U_i denote the border triangle at w_i.

Note that each T_i is divided into $n-2$ chambers by the chords emanating from v_i. We label these chambers $C_{i,j}$ $(0 \le j \le n-3)$ in the following way. For $0 \le i \le k$, we let $C_{i,0}$ be the chamber close to e_{i-1}, labeling consecutively up to $C_{i,n-3}$, the chamber close to e_i. Similarly for the U_i $(0 \le i \le n-3)$, we let $D_{i,0}$ be the chamber of U_i close to f_{i-1}, labeling up to $D_{i,n-3}$, the chamber close to f_i.

To construct a pebble set, begin by letting p_0 be a point in the chamber close to $v_0 w_0$, and p_{n-3} a point in the chamber close to $v_{k+1} w_{m+1}$. Writing the elements of A as $a_1 < a_2 < \cdots < a_k$, let p_i be a point in each C_{i,a_i}. Similarly, writing the elements of B as $b_1 < b_2 < \cdots < b_m$, let p_{i+k} be a point in each D_{i,b_i}. Let $S = \{p_0, \ldots, p_{n-3}\}$.

Theorem 4 *Let P be an n-gon $(n \ge 4)$ and e an edge of P. Then the construction described above determines a one-to-one correspondence between the peripheral pebble sets containing a point close to the edge $v_0 w_0$ and the power set of* $\{1, 2, \ldots, n-4\}$ $(n \ge 4)$.

Proof We use induction to show that S as constructed above is a pebble set. If $n = 4$, then $A = B = \emptyset$. Thus S consists of a point close to edge $v_0 w_0$ and a point close to $v_1 w_1$, which is clearly a pebble set. So suppose $n \ge 5$ and that this construction produces a pebble set for a polygon on fewer than n vertices. Note that we may assume that $1 \in A$. For otherwise, an argument similar to the one to follow can be applied to B.

First we let $n' = n - 1$ and create an n'-gon, which we will denote P', by deleting v_0 and all chords of P emanating from it, then including the segment $w_0 v_1$ as an edge of P'. (See Fig. 3 for an example.) We show that $S' = S - \{p_0\}$ is precisely the same construction in P' as the above construction is for P. To demonstrate this, we relabel some of the vertices, border triangles, and chambers of P', as well as the points in S' to reveal a subset A' of $W' = \{1, 2, \ldots, n' - 4\}$ that has $k' = k - 1$ elements, a set $B' = W' - A'$, and a selection of points in the chambers of the border triangles of P' analogous to that for P.

For $0 \le i \le k' + 1$, let $v'_i = v_{i+1}$ and $e'_i = e_{i+1}$. For $1 \le i \le n' - 3$, let $p'_i = p_{i+1}$. First note that S' contains $p'_0 = p_1$, which is close to edge $v'_0 w_0$, and it contains

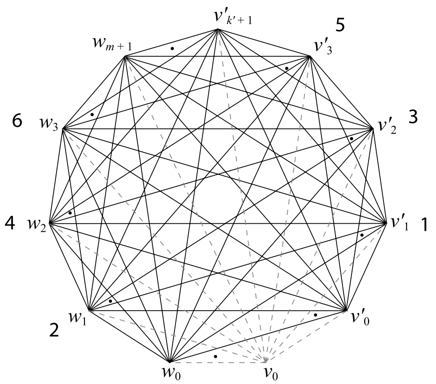


Fig. 3 Deletion of v_0 from P in the proof of Theorem 4

$p'_{n'-3} = p_{n-3}$, which is close to $v'_{k'+1}w_{m+1}$. Label the border triangles at the v'_i as T'_i, and note that $T_{i'} = T_{i+1}$. Label the chambers of T'_i as $C'_{i,j}$ as before, where $C'_{i,0}$ is the chamber in T'_i close to e'_{i-1}, labeling consecutively up to $C_{i,n'-3}$, the chamber of T'_i close to e'_i. With this labeling, we have that $C'_{i,j} = C_{i+1,j+1}$ for $i \le j \le n' - 3$. Let $A' = \{a - 1 : a \in A, a \neq 1\}$ and label the elements of A' as $a'_i = a_{i+1} - 1$ for $1 \le i \le k'$. Note that A' is a subset of $W' = \{1, 2, \dots, n' - 4\}$ that contains k' elements. Furthermore, since $a_i \ge i$, we have that $a'_i \ge i$. Also, $C'_{i,a'_i} = C_{i+1,a'_i+1} = C_{i+1,a_{i+1}}$, so that C'_{i,a'_i} contains $p'_i = p_{i+1}$.

In a somewhat similar fashion, we relabel the chambers in P' of the border triangles at the w_i $(0 \le i \le m)$, where $D_{i,0}$ is the chamber of U_i close to f_{i-1}, up to $D_{i,n'-3}$, the chamber of U_i close to f_i. With this relabeling, $D'_{i,j} = D_{i,j+1}$ for $i \le j \le n' - 3$. Let $B' = \{b - 1 : b \in B\}$, and label the elements of B' as $b'_i = b_i - 1$ for $1 \le i \le m$. Note that $B' = W' - A'$. Since $1 \in A$, we have that $b_i \ge i$ for $1 \le i \le m$. Also, $D'_{i,b'_i} = D_{i,b'_i+1} = D_{i,b_i}$, so that D'_{i,b'_i} contains $p'_{i+k'} = p_{i+k}$.

With this, we have that the removal of v_0 from P leaves an n'-gon whose set of points S' is of the same construction as that for P. We now show that S is a pebble set in P, for $n \ge 5$ by applying Lemma 3.

Let e be any edge of P and v any vertex. We show Δev contains precisely one point of S. If $e \notin \{v_0w_0, e_0\}$, and $v \neq v_0$, then Δev is a triangle in P'. By the inductive assumption, Δev contains precisely one point of S', and therefore of S.

Now suppose $v = v_0$. If $e = f_0$, then by the way S was constructed, Δev contains the unique point p_0. Also, if $e = v_{k+1}w_{m+1}$, then Δev contains the unique point p_{n-3}. If $e = e_i$ for some $1 \le i \le k$, then $\Delta ev \cap T_i$ contains all chambers $C_{i,j}$ for $i - 1 \le j \le n - 3$. Since $a_i \ge i$, Δev contains p_i. Also, $\Delta ev \cap T_{i+1}$ contains the chambers $C_{i+1,j}$ for $0 \le j \le i - 1$. Since $a_{i+1} \ge i + 1$, p_{i+1} does not lie in Δev.

Reasoning in a similar fashion, if $1 \le i \le m$ and $e = f_i$, then $\Delta ev \cap U_i$ contains chambers $D_{i,i}$ to $D_{i,n-3}$, and $\Delta ev \cap U_{i+1}$ contains chambers $D_{i+1,0}$ to $D_{i+1,i}$. Since $b_i \ge i$, S_{i,b_i} is one of the chambers in Δev, and thus Δev contains p_{i+k}. Furthermore, Δev does not contain $D_{i+1,b_{i+1}}$, so that Δev does not contain p_{i+k+1}.

Finally, suppose $e \in \{v_0w_0, e_0\}$. We may assume $v \notin \{v_0, v_1, w_0\}$, for otherwise Δev contains p_0. By the inductive assumption, Δv_1w_0v contains precisely one point of S'. In particular, Δv_1w_0v contains p_1, and no other points of S'. By the way S was constructed, if $e = v_0w_0$, then Δev contains p_0. Furthermore, since Δv_1w_0v contains only p_1, there are no other points of S in Δev. On the other hand, if $e = e_0$, then Δev contains p_1. Since Δv_1w_0v contains no other points of S, then neither does Δev.

Thus every triangle of the form Δev contains precisely one point of S. By Lemma 3, S is a pebble set.

Now we show that all peripheral pebble sets are derivable from the construction described above. Let S be any pebble set, all of whose points lie in the periphery of P. Suppose the vertices of P are labeled so that S contains a point p_0 close to e_{n-1}. Since all points of S lie in the periphery, there exists precisely one other point p_{n-3} close to some other edge of P. Relabel vertices, edges, border triangles, and chambers of the border triangles as was done above, so that p_0 is close to edge v_0w_0, and p_{n-3} is close to $v_{k+1}w_{m+1}$ for the appropriate k and m such that $k + m = n - 4$. Since each border triangle contains precisely one point of S, we may let A be the set of all positive integers a_i for which S contains a point in C_{i,a_i} $(1 \le i \le k)$, and B the set of all positive integers for which S contains a point in D_{i,b_i} $(1 \le i \le m)$. We show that A is a uniquely determined subset of $W = \{1, 2, \dots, n-4\}$, and that $B = W - A$.

Since p_0 lies close to v_0w_0, no other points of S can lie in T_0 or U_0. But since $\Delta e_0 w_1$ and $\Delta f_0 v_1$ must contain precisely one point of S, then there exists a unique point $p_1 \in S$ in either $C_{1,1}$ or $D_{1,1}$. Thus either $a_1 = 1 \in A$ or $b_1 = 1 \in B$, but not both.

Now suppose that $s \ge 1$ and that for $1 \le t \le s$, either $t \in A$ or $t \in B$, but not both. We may assume that $s \in A$, for a similar argument would work in the event $s \in B$. Let i be the number of elements in $A \cap \{1, 2, \dots, s\}$, so that $a_i = s$. Let j be the number of elements in $B \cap \{1, 2, \dots, s\}$, and let p_s be the point of S that lies in C_{i,a_i}.

Notice that for $1 \le h \le n-2$, $\Delta e_i v_{i-h}$ (where $i-h$ is taken modulo n) when intersected with T_i, yields all chambers $C_{i,h-1}$ to $C_{i,n-3}$. Since $w_j = v_{n-1-i} = v_{i-(i+j+1-n)}$, we may let $h = i+j+1$ to have that $\Delta e_i w_j$ intersects T_i to yield the chambers from $C_{i,i+j} = C_{i,s}$ to $C_{i,n-3}$. Thus p_s is the unique point of S in $\Delta e_i w_j$. Similarly, $\Delta e_{i+1} w_j$ intersects T_{i+1} to yield $C_{i+1,0}$ through $C_{i+1,i+j+1}$. Thus $\Delta e_i w_j$ intersects T_{i+1} to yield $C_{i+1,0}$ through $C_{i+1,i+j}$. Thus in T_{i+1} there are no points of S in any of the $C_{i+1,h}$ for $0 \le h \le i+j$. But $\Delta e_i v_{j+1}$ contains a point of S. Thus there must exist a point p_{s+1} either in $C_{i+1,i+j+1}$ or in $D_{j+1,i+j+1}$, but not both. Since $i+j = s$, there exists a point of S in either $C_{i+1,s+1}$ or $D_{j+1,s+1}$. If $p_{s+1} \in C_{i+1,s+1}$, then $s+1 \in A$, and we may write $a_{i+1} = s+1$. On the other hand, if $p_{s+1} \in D_{j+1,s+1}$, then $s+1 \in B$, and we may write $b_{j+1} = s+1$.

By induction $(1 \le s \le n-4)$, A and B are uniquely determined. □

If $n = 3$, then an n-gon has a unique pebble set. If $n \ge 4$, then the power set of $\{1, 2, \dots, n-4\}$ contains 2^{n-4} elements. Thus we have the following.

Corollary 5 *Let P be a convex n-gon, and e an edge of P. Let $d(P) = 1$ if $n = 3$, and $d(P) = 0$ otherwise. Then the number of peripheral pebble sets in P that contain a point close to e is $2^{n-4+d(P)}$.*

If $n \ge 4$, we may sum the numbers of pebble sets from Corollary 5 across all edges of an n-gon. Since a peripheral pebble set contains precisely two points close to an edge, we count each pebble set exactly twice. Thus we arrive at the following.

Corollary 6 *If P is an n-gon $(n \ge 4)$, then the number of peripheral pebble sets is $n2^{n-5}$.*

4 Characterizing Non-Peripheral Pebble Sets

For a given n-gon, pebble sets other than peripheral pebble sets might exist. In this section, we want to characterize chambers for which a pebble set exists that contains a point in the chamber. An important example of such a chamber lies in the hexagon in Fig. 4.

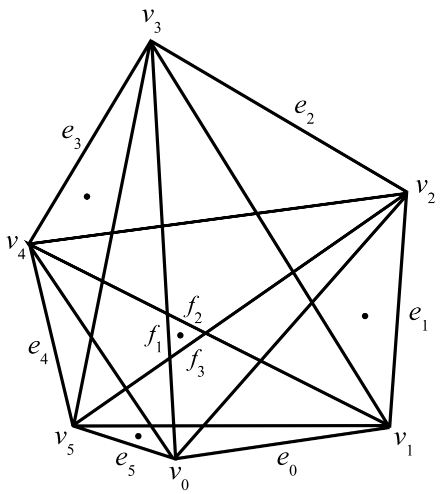


Fig. 4 Example of a non-peripheral pebble set in a hexagon

In any hexagon H, the chords $f_1 = v_0v_3$, $f_2 = v_1v_4$, and $f_3 = v_2v_5$ intersect pairwise. If these three points of intersection are distinct, they determine a triangular region R in H. For any given border triangle T, none of these three points lies in T, so that R lies in the core of H. Furthermore, all chords of H other than $\{f_1, f_2, f_3\}$ are boundary segments of the border triangles, and thus do not intersect the core of H. Therefore, R is a triangular chamber in the core of H.

4.1 Edge Supports and Viability

Definition 3 We say that a chamber is *viable* if there exists a pebble set containing a point in the chamber.

By Theorem 4, we see that every chamber in the periphery of a polygon is viable.

Definition 4 Given a chamber C of a polygon P, the *edge support* of C, written $\mathrm{ES}(C)$, is the set of edges e of P for which C is contained in the fan of e.

As an example, note that in Fig. 4, the edge support of the triangular chamber in the core is $\{e_0, e_2, e_4\}$. By the *support size* of C, we mean the cardinality of $\mathrm{ES}(C)$, which we denote $|\mathrm{ES}(C)|$. A major result of this section is that a chamber C is viable if and only if $|\mathrm{ES}(C)| = 3$ (Theorem 17 and Corollary 22). Along the way, we note other necessary conditions of viability that are helpful in practice. In particular, if the edge support of a viable chamber contains two incident edges, the chamber must lie in the periphery (Proposition 15). In addition, the only pebble sets in a regular polygon are peripheral (Theorem 16).

Proposition 7 *If C is a chamber in a polygon and v_k is any vertex, then there exists a unique edge e_j $(j \notin \{k-1, k\})$ such that C is contained in $\Delta e_j v_k$.*

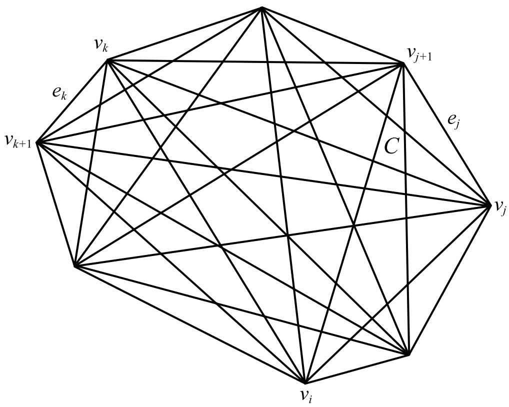


Fig. 5 A possible chamber from Proposition 9

Proof The chords of the polygon emanating from v_k form a triangulation of the polygon. Thus C must lie within precisely one of these triangles, which are of the form $\Delta e_j v_k$, where $j \notin \{k-1, k\}$. □

Proposition 8 *If C is a chamber in the periphery of a polygon, then $|\mathrm{ES}(C)| = 3$, and at least two edges of* $\mathrm{ES}(C)$ *are incident.*

Proof Suppose C is in the periphery. Then it is contained in the border triangle at some v_k, so that $e_{k-1}, e_k \in \mathrm{ES}(C)$. Applying Proposition 7 to v_k, we see that there is an additional edge in $e_j \in \mathrm{ES}(C)$. Now any triangle containing C must include v_k as a vertex. Since e_j is the unique edge such that C is contained in $\Delta e_j v_k$, there are no other edges in $\mathrm{ES}(C)$. □

For convenience in the next proof, we will say that a vertex v_j is *between* vertices v_i and v_k if the counterclockwise ordering of the three vertices is $\langle v_i, v_j, v_k \rangle$.

Proposition 9 *Suppose v_k, v_{k+1}, v_j and v_{j+1} are distinct vertices in a polygon, and that C is a chamber in $\Delta e_j v_k$. Let v_i be a vertex between v_k and v_j, and suppose also that C lies in $\Delta e_j v_i$. Then C is not in the fan of e_k. (See Fig. 5 for an illustration.)*

Proof Let v_m be any vertex ($m \notin \{k, k+1\}$). We show that C is not in $\Delta e_k v_m$ to have that C is not in the fan of e_k. If v_m is between v_{k+1} and v_{j+1}, then we may triangulate the polygon by including the chord from v_{k+1} to v_m, all chords from v_{k+1} to the vertices between v_{k+1} and v_m, the chord from v_k to v_m, and all chords from v_k to the remaining vertices. This triangulation contains $\Delta e_k v_m$ and $\Delta e_j v_k$, and are therefore disjoint. Since C is contained in $\Delta e_j v_k$ it is not in $\Delta e_k v_m$.

If v_m is between v_j and v_k, triangulate the polygon by including chords from v_i to all vertices between v_i and v_m, and chords from v_m to all remaining vertices. This triangulation contains $\Delta e_j v_i$ and $\Delta e_k v_m$. Since C is in $\Delta e_j v_i$ it is not in $\Delta e_k v_m$.

In either case, C is not contained in $\Delta e_k v_m$. Since this is true for all m, we have that C is not in the fan of e_k. □

To characterize viable chambers in the core of a polygon, we will need to relate a given n-gon to an $(n-1)$-gon formed by removing a vertex, as in the proof of Theorem 4.

Let v_k be a vertex in an n-gon P, and let C be a chamber of P not in the border triangle at v_k. Let P' be the $(n-1)$-gon formed by deleting v_k and including the edge $v_{k-1}v_{k+1}$, and let C' be the chamber of P' that contains C. Since chambers do not intersect any chord in a polygon, we may make the following observation.

Proposition 10 *Suppose T is a triangle determined by three vertices of P', and suppose T contains C. Then T also contains C'.*

Proposition 10 allows us to show that an edge e is in $\mathrm{ES}(C')$ by showing merely that C is contained in Δev for some vertex v in P'. The following proposition relates the edge support of a chamber in P to the edge support of the chamber that contains it in P'.

Proposition 11 *Let v_k be a vertex in an n-gon P, let C be a chamber of P not in the border triangle at v_k, and let C have edge support $\mathrm{ES}(C)$. Let P' be the $(n-1)$-gon formed by deleting v_k and including the edge $f = v_{k-1}v_{k+1}$. Let C' be the chamber of P' that contains C, and let $\mathrm{ES}(C')$ be the edge support of C' in P'. Then the following are true*:

1. *If neither e_{k-1} and e_k is in $\mathrm{ES}(C)$, then $\mathrm{ES}(C') = \mathrm{ES}(C)$.*
2. *If precisely one of e_{k-1} and e_k is in $\mathrm{ES}(C)$ (say e_k), then $\mathrm{ES}(C') = [\mathrm{ES}(C) - \{e_k\}] \cup \{f\}$.*
3. *If $e_{k-1}, e_k \in \mathrm{ES}(C)$, then there exists an edge $e_j \in \mathrm{ES}(C)$ $(j \notin \{k-1, k\})$ such that $\mathrm{ES}(C') = [\mathrm{ES}(C) - \{e_{k-1}, e_k, e_j\}] \cup \{f\}$.*

Proof We prove the result by showing the following.

1. $f \in \mathrm{ES}(C')$ if and only if $e_{k-1} \in \mathrm{ES}(C)$ or $e_k \in \mathrm{ES}(C)$.
2. For every other edge e_j in P' (so that $j \notin \{k-1, k\}$), the following hold:
   (a) If e_{k-1} and e_k are not both in $\mathrm{ES}(C)$, then $e_j \in \mathrm{ES}(C)$ if and only if $e_j \in \mathrm{ES}(C')$; and
   (b) If $e_{k-1}, e_k \in \mathrm{ES}(C)$, and if (by Proposition 7) e_j is the unique edge for which C is contained in $\Delta e_j v_k$, then $e_j \in \mathrm{ES}(C) - \mathrm{ES}(C')$, and for every other edge e_i $(i \neq j)$, $e_i \in \mathrm{ES}(C')$ if and only if $e_i \in \mathrm{ES}(C)$.

For claim 1, if $f \in \mathrm{ES}(C')$, then there exists a vertex v_j $(j \notin \{k-1, k, k+1\})$ such that C' is contained in $\Delta f v_j$. This triangle is itself contained in the union of $\Delta e_{k-1} v_j$ and $\Delta e_k v_j$. Thus C is contained in one of these triangles, so that either e_{k-1} or e_k is in $\mathrm{ES}(C)$. Conversely, if either e_{k-1} or e_k is in $\mathrm{ES}(C)$, then we may consider that $e_k \in \mathrm{ES}(C)$ (the argument for e_{k-1} would be similar). Thus there exists a vertex v_j $(j \notin \{k, k+1\})$ such that C is contained in $\Delta e_k v_j$. But $j \neq k-1$ also, because C is not in the border triangle at v_k. Thus C is contained in $\Delta f v_j$. By Proposition 10, C' is also contained in $\Delta f v_j$, so that $f \in \mathrm{ES}(C')$.

For claim 2a, suppose $j \notin \{k-1, k\}$ and $e_k \notin \mathrm{ES}(C)$. (The arguments for the case of $e_{k-1} \notin \mathrm{ES}(C)$ would be similar to the ones to follow.) Suppose $e_j \in \mathrm{ES}(C)$ and that $e_j \notin \mathrm{ES}(C')$. Then there exists a vertex v_i such that C is contained in $\Delta e_j v_i$. If $i \neq k$, then $\Delta e_j v_i$ contains C, and therefore contains C' by Proposition 10. This contradicts the fact that $e_j \notin \mathrm{ES}(C')$. Therefore, v_k is the only vertex such that C is contained in $\Delta e_j v_k$. Since $e_k \notin \mathrm{ES}(C)$, it is impossible that $j = k+1$. Thus the edges e_j and e_k determine a quadrilateral in P, which is the union of triangles $\Delta e_k v_{j+1}$ and $\Delta e_j v_{k+1}$. Furthermore, C must lie in this quadrilateral, because it lies in $\Delta e_j v_k$. But C does not lie in $\Delta e_k v_j$ because $e_k \notin \mathrm{ES}(C)$. Also, C' does not lie in $\Delta e_j v_{k+1}$ by supposition, so that C does not lie in $\Delta e_j v_{k+1}$ either. This is a contradiction, so it must be that $e_j \in \mathrm{ES}(C')$.

Conversely, suppose $e_j \in \mathrm{ES}(C')$. Then there exists vertex v_i ($i \neq k$) such that C' is contained in $\Delta e_j v_i$. Since $C \subseteq C'$, C is also contained in $\Delta e_j v_i$, so that $e_j \in \mathrm{ES}(C)$.

For claim 2b, suppose $e_{k-1}, e_k \in \mathrm{ES}(C)$ and that e_j is the unique edge such that C is contained in $\Delta e_j v_k$. Then clearly $e_j \in \mathrm{ES}(C)$, so we must show that $e_j \notin \mathrm{ES}(C')$. First suppose that $j = k+1$. Then C is in the border triangle at v_{k+1}. By Proposition 8, $\mathrm{ES}(C) = \{e_{k-1}, e_k, e_{k+1}\}$, and by Proposition 7 applied to v_{k+1}, C must lie in the border triangle at v_k. This is impossible, since C is assumed not to be in this border triangle. By similar reasoning, $j \neq k-2$. Thus e_{k-1}, e_k and e_j determine a pentagon in P, and we may apply Proposition 9. If C is contained in $\Delta e_j v_i$ for some vertex v_i between v_k and v_j, then C cannot be in the fan of e_k. But $e_k \in \mathrm{ES}(C)$ by assumption. Reasoning similarly from Proposition 9, if C is contained in $\Delta e_j v_i$ for some v_i between v_{j+1} and v_{k+1}, then C cannot be in the fan of e_{k-1}. But $e_{k-1} \in \mathrm{ES}(C)$. Thus for any $i \neq k$, C does not lie in $\Delta e_j v_i$, and neither is C' contained in $\Delta e_j v_i$. Therefore, $e_j \notin \mathrm{ES}(C')$.

Finally, if e_i is an edge, $i \neq j$, we show that $e_i \in \mathrm{ES}(C')$ if and only if $e_i \in \mathrm{ES}(C)$. As before, if $e_i \in \mathrm{ES}(C')$, then there exists a vertex v_m so that $C' \subset \Delta e_i v_m$. Since $C' \supset C$, we have that $e_i \in \mathrm{ES}(C)$. Conversely, suppose $e_i \in \mathrm{ES}(C)$. Then there exists a vertex v_m so that $C \subset \Delta e_i v_m$. Now m cannot equal k, since $i \neq j$ and $\Delta e_j v_k$ is the unique triangle given by Proposition 7 based at v_k containing C. Therefore, $\Delta e_i v_m$ is in P', and so $e_i \in \mathrm{ES}(C')$. □

Proposition 11 implies the following about the support size of C, and its relationship to the support size of C'.

Corollary 12 *Let C be a chamber in an n-gon P. Then $|\mathrm{ES}(C)|$ is odd and at least three. In fact, If v_k is a vertex whose border triangle does not contain C, and C' is the chamber in the $(n-1)$-gon created from P by deleting v_k, then $|\mathrm{ES}(C)| = |\mathrm{ES}(C')| + 2$ whenever $e_{k-1}, e_k \in \mathrm{ES}(C)$. Otherwise, $|\mathrm{ES}(C)| = |\mathrm{ES}(C')|$.*

Proof If $n = 3$ then $|\mathrm{ES}(C)| = 3$. So suppose $n \geq 4$ and that the result is true for all polygons on fewer than n vertices. Now C cannot be in every border triangle. We may therefore remove some vertex v_k whose border triangle does not contain C to create the n-gon P' with a chamber $C' \supseteq C$, where by the inductive assumption, $|\mathrm{ES}(C')|$ is odd and at least three. If $e_{k-1}, e_k \in \mathrm{ES}(C)$, then by Proposition 11, $|\mathrm{ES}(C)| = |\mathrm{ES}(C')| + 2$. Otherwise, $|\mathrm{ES}(C)| = |\mathrm{ES}(C')|$. □

We will show that a support size of three is necessary and sufficient for the viability of a chamber. First we show the following.

Proposition 13 *Suppose S is a pebble set in an n-gon P $(n \geq 4)$, and v is any vertex of P. Let p be the point of S that lies in the border triangle at v, and let P' be the polygon formed from P by removing v. Then $S - \{p\}$ is a pebble set in P'.*

Proof If T is any triangle in P', it is also a triangle in P. Thus T contains a unique point of S. Now every triangle that contains p includes v as a vertex, so that p is not the point of S that lies in T. Therefore, $S - \{p\}$ is a pebble set in P'. □

Corollary 14 *Suppose C is a viable chamber in an n-gon $(n \geq 4)$, and that C is not in the border triangle at vertex v. Then $C' \supseteq C$ is viable in the $(n-1)$-gon created by removing v.*

4.2 Proof of Necessity

We are now ready to show that every viable chamber has support size three. In Sect. 4.3 we will prove the converse. In preparation for the proof of necessity, we first demonstrate another necessary condition for viability that has practical application. Proposition 15 implies that if a chamber lies in the core of a polygon and its edge support contains two incident edges, then it is not viable.

Proposition 15 *If C is a viable chamber and* $\mathrm{ES}(C)$ *contains two incident edges, then C is in the periphery.*

Proof Suppose C is viable and $ES(C)$ contains e_k and e_{k+1}. Then there exist vertices v_i, v_j $(i, j \neq k+1)$ such that $\Delta e_k v_i$ and $\Delta e_{k+1} v_j$ contain C. Let T be the border triangle at v_{k+1}. Now T consists of $n-2$ chambers, which we label $C_0, \ldots, C_{n-3}$, where $C_0 = T \cap \Delta e_{k-1} v_{k+1}$, and the others are numbered consecutively up to $C_{n-3} = T \cap \Delta e_{k+2} v_{k+1}$.

By Proposition 7, there is an edge e_m so that $\Delta e_m v_{k+1}$ contains C. Thus $T \cap \Delta e_m v_{k+1}$ is a single chamber, which we call C_m. By convexity, any triangle containing C and v_{k+1} must intersect (and therefore contain) C_m.

Since the triangles $\Delta e_k v_i$ and $\Delta e_{k+1} v_j$ both contain C, they both contain C_m. Now $\Delta e_{k+1} v_j$ contains C_0, so by convexity, it contains chambers C_0 through C_m. Similarly, $\Delta e_k v_i$ contains C_m and C_{n-3}, and therefore, all the chambers C_m through C_{n-3}. We can then conclude that $\Delta e_k v_i \cup \Delta e_{k+1} v_j$ contains T.

Now since $\Delta e_k v_i$ is a triangle, any pebble set containing a point in C cannot contain another point in $\Delta e_k v_i$. Similarly for $\Delta e_{k+1} v_j$. Thus a pebble set containing a point in C can have no other point in T. But T is a triangle, and so the pebble set must have a point in it. Therefore C must be in T, and hence in the periphery. □

An interesting implication of Proposition 15 is the following.

Theorem 16 *If P is a regular n-gon, then all pebble sets are peripheral.*

Proof If $n \in \{3, 4\}$ then every chamber is in the periphery, so let $n \geq 5$, and let P be the regular polygon on n vertices. We show that every point in the polygon that does not lie on a chord is in a chamber whose edge support contains two incident

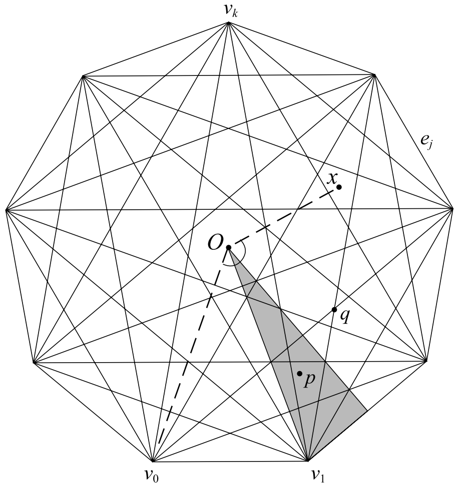


Fig. 6 A regular polygon from the proof of Theorem 16

edges. Let O be the center point of P. For a point $x \neq O$ anywhere in P, define $\theta(x)$ to be the counterclockwise angle between segments Ov_0 and Ox ($0 \le \theta < 2\pi$). Let $k = \lceil n/2 \rceil$. Then k is the smallest positive integer for which $\theta(v_k) \ge \pi$.

Suppose p is any point in a viable chamber C of P. Since chambers are open and nonempty, we may assume $p \neq O$. By rotating the indices of P and exploiting reflective symmetries of P, we may assume that $\frac{2\pi}{n} \le \theta(p) \le \frac{3}{2}\frac{2\pi}{n}$. In other words, we may assume that p lies in a closed triangular $B \subseteq P$ defined by O, v_1, and the midpoint of e_1.

If p lies in $\Delta e_1 v_0$, then it is in the periphery. So we may assume otherwise. By the way k was chosen, we may consider a triangulation of chords emanating from v_0 and have that there is some e_j ($2 \le j \le k-1$) such that p lies in $\Delta e_j v_0$. This triangle is a subset of the quadrilateral determined by e_0 and e_j. We show that p is in the fans of both e_0 and e_1.

First we show that p is in the fan of e_1. If n is odd, then $\Delta e_1 v_{k+1}$ contains O, so that it contains all of B. Thus p lies in $\Delta e_1 v_{k+1}$. If n is even, then chord $v_1 v_{k+1}$ passes through O. Since p does not lie on a chord of P, then p must lie in $\Delta e_1 v_{k+1}$. In either case, p, and therefore all of C, lies in the fan of e_1.

Now we show that p is in the fan of e_0. Suppose that p does not lie in either $\Delta e_0 v_j$ or $\Delta e_0 v_{j+1}$. Then it must lie in $\Delta e_j v_1 \cap \Delta e_j v_0$. Now the chords $v_0 v_j$ and $v_1 v_{j+1}$ intersect at a point q where $\theta(q)$ is the average of $\theta(v_1) = 2\pi/n$ and $\theta(v_j) = 2j\pi/n$. Thus $\theta(q) = (j+1)\pi/n$, so that $\theta(p) > \pi(j+1)/n$. But since $j \ge 2$, we have that $\theta(p) \ge 3\pi/n$. This is a contradiction, so p must lie in either $\Delta e_0 v_j$ or $\Delta e_0 v_{j+1}$, Thus p, and therefore C, lies in the fan of e_0.

Therefore e_0 and e_1, which are incident edges, are in $ES(C)$, and by Proposition 15, we can conclude C is in the periphery. □

Theorem 16 can actually be stated in a slightly stronger form. If P is an n-gon for which the intersections of the chords generates the same internal topology as the regular n-gon, then all pebble sets are peripheral. Given that we have characterized all peripheral pebble sets in general in Sect. 3, we therefore have solved the problem of characterizing all pebble sets for polygons whose internal topology is equivalent to that of a regular polygon.

We are now ready to prove one direction of the viability question.

Theorem 17 *If C is a viable chamber in an n-gon, then* $|\mathrm{ES}(C)| = 3$.

Proof Suppose C is viable. If $n = 3$, the result it clear, so suppose $n \geq 4$, and that the result is true for all polygons on fewer than n vertices. If C is in the periphery, then $|\mathrm{ES}(C)| = 3$ by Proposition 8. Thus we may assume C is in the core, so that $\mathrm{ES}(C)$ does not contain two incident edges, by Proposition 15. Let v_k be a vertex for which e_{k-1} and e_k are not both in $\mathrm{ES}(C)$. Remove v_k to form the $(n-1)$-gon as in Proposition 11. Since at least one of e_k and e_{k-1} is not in $\mathrm{ES}(C)$, then by Corollary 12, $|\mathrm{ES}(C)| = |\mathrm{ES}(C')|$. By Corollary 14, C' is viable, so that by the inductive assumption, $|\mathrm{ES}(C')| = 3$. Thus $|\mathrm{ES}(C)| = 3$. □

An important consequence of Theorem 17 is the following.

Proposition 18 *A chamber with support size three is triangular.*

Proof Suppose $|\mathrm{ES}(C)| = 3$. Since all chambers in the periphery are triangular, we assume C is in the core. We demonstrate a one-to-one correspondence between the chords that determine the boundary of C and the pairs of edges chosen from $\mathrm{ES}(C)$. Let $v_i v_j$ be a chord that determines part of the boundary of C. The line $\overleftrightarrow{v_i v_j}$ cuts the plane into two half planes. We may assume that C lies in the half plane containing edges e_{i-1} and e_j. Since C is a chamber, it lies in both $\Delta e_{i-1} v_j$ and $\Delta e_j v_i$. Thus $e_{i-1}, e_j \in \mathrm{ES}(C)$, and we have that every chord $v_i v_j$ of P determines a unique pair of edges in $\mathrm{ES}(C)$ that lie in the same half plane as C upon cutting the plane with $v_i v_j$.

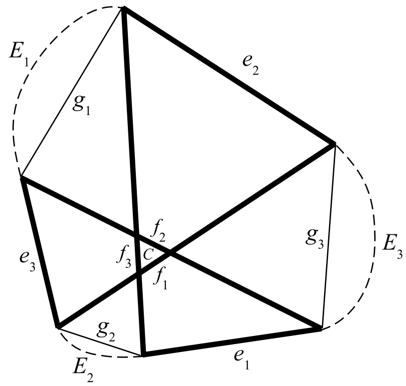
Now let $v_s v_t$ be a chord of P that determines part of the boundary of C. We show that the only way the above algorithm can yield e_{i-1} and e_j is if $v_s v_t = v_i v_j$. Since C lies in the core, there are four distinct chords incident with e_{i-1} and e_j, defined by the four distinct endpoints of e_{i-1} and e_j. Since the chord $v_{i-1} v_j$ cuts the plane so that e_{i-1} and e_j lie in different half planes, it is impossible that $v_s v_t = v_{i-1} v_j$. By similar reasoning $v_s v_t \neq v_i v_{j+1}$. Furthermore, since C is a chamber, it is impossible that the chord $v_{i-1} v_{j+1}$ form part of the boundary of C, since $v_i v_j$ and $v_{i-1} v_{j+1}$ lie on opposite sides of $v_i v_{j+1}$. Thus the only chord for which the above algorithm can determine e_{i-1} and e_j is $v_i v_j$. □

4.3 Proof of Sufficiency

Now we address the converse of Theorem 17 by showing that if the support size of a chamber is three, then the chamber is viable. We will use the three edges in its edge support to demonstrate a construction that produces all pebble sets containing a point in the chamber. We begin by describing a construction whereby such a chamber can be used to determine three polygonal subsets of a given n-gon.

Construction 2 Let C be a chamber in an n-gon P with support size three, and let the edges in $\mathrm{ES}(C)$ be labeled counterclockwise as e_1, e_2, and e_3. By Proposition 18,

Fig. 7 A triangular chamber with support size three



C is triangular. (See Fig. 7.) Label the three chords that define the boundary of C as f_1, f_2, and f_3, where f_i is incident with e_j and e_k (i, j, and k all distinct). Similarly, for $1 \le i \le 3$, let E_i be the set of edges of P between e_j and e_k (i, j, and k all distinct). Note that some of the E_i might be empty. Let Q_1 be the polygonal subset of P whose perimeter is defined by e_3, f_1, e_2, and all edges in E_1. Let Q_2 and Q_3 be analogously defined polygonal subsets of P. Finally, let p be any point in C.

Since C is not in the fan (in P) of any edge in E_1, it is also not in the fan in Q_1 of any edge in E_1. Thus the edge support of C in Q_1 is precisely $\{f_1, e_2, e_3\}$, and p lies close to f_1 in Q_1. By Theorem 4, there exists a pebble set S_1 in Q_1 that includes p. By similar reasoning, there exist pebble sets S_2 and S_3 in Q_2 and Q_3, respectively, that include p. Let $S = S_1 \cup S_2 \cup S_3$.

We now show that S as described in Construction 2 is a pebble set in P that contains p, and that all pebble sets in P that contain p are uniquely described in this way. We construct a bijection between the set of all possible ordered triples (S_1, S_2, S_3) of pebble sets in the Q_i, where each S_i contains p, and the set of all pebble sets in P that contain p. For a polygon P and a point p in a chamber of P, we write $\mathcal{S}(P, p)$ to mean the set of all pebble sets in P that contain the point p. Thus the construction we desire is a one-to-one correspondence between $\mathcal{S}(P, p)$ and $\prod_{i=1}^{3} \mathcal{S}(Q_i, p)$.

Let P be a polygon and $S \in \mathcal{S}(P, p)$. Define $\pi : \mathcal{S}(P, p) \to \prod_{i=1}^{3} \mathcal{S}(Q_i, p)$ to map S to the ordered triple $(S \cap Q_1, S \cap Q_2, S \cap Q_3)$. To show that π is a bijection, the following lemmas will be helpful. For convenience, we state them with specific reference to Q_1, though analogous results clearly hold for Q_2 and Q_3.

Lemma 19 *Let P be a polygon and C a chamber of P. Let Q_1, E_2 and E_3 be defined as in Construction 2. Let S_1 be a pebble set in Q_1 that contains p, and let D be any chamber of Q_1 that contains a point of $S_1 - \{p\}$. Then no point in $S_1 - \{p\}$ lies in the fan of any edge of $E_2 \cup E_3 \cup \{e_1\}$.*

Proof Let q be any point in $S_1 - \{p\}$, let e be any edge in $E_2 \cup E_3 \cup \{e_1\}$, and let v be any vertex of P. If v is not an endpoint of an edge in E_1, then Δev does not lie in Q_1, so that $q \notin \Delta ev$. So suppose v is an endpoint of an edge in E_1. Now $\Delta ev \cap Q_1 \subset \Delta f_1 v$, and $q \in Q_1$, so if $q \in \Delta ev$, then we would have $q \in \Delta f_1 v$. Yet, $p \in \Delta f_1 v$, and S_1 is a pebble set in Q_1, and so $q \notin \Delta ev$. □

Lemma 20 *Let P be a polygon and C a chamber of P. Let Q_1, E_2 and E_3 be defined as in Construction 2. Let S_1 be a pebble set in Q_1 that contains p, and let D be any chamber of Q_1 that contains a point of $S_1 - \{p\}$. Then D is also a chamber of P.*

Proof Suppose D is a chamber of Q_1 that contains some $q \in S_1 - \{p\}$, and which is not a chamber of P. Then there is a chord of P that intersects D. Such a chord is clearly not a chord of Q_1, and is thus of the form uv, where u is a vertex of Q_1 and v is not. Of all such chords, let uv be closest to q. If q does not lie on uv, then there is an edge $e \in E_2 \cup E_3 \cup \{e_1\}$ that is incident with v such that $q \in \Delta eu$. But then q lies in the fan of e, which contradicts Lemma 19. If, on the other hand, q lies on uv, then the fact that uv intersects f_1 reveals that $q \in \Delta f_1 u$, which is impossible. Thus the chords that determine the chamber in P that contains q are also chords of Q_1, and D is a chamber of P. □

Theorem 21 *The mapping $\pi : \mathcal{S}(P, p) \to \prod_{i=1}^{3} \mathcal{S}(Q_i, p)$ defined above is a bijection.*

Proof First we show that π is a function. If $S \in \mathcal{S}(P, p)$, then clearly for $1 \le i \le 3$, $S \cap Q_i$ is a pebble set in Q_i that contains p. Thus $\pi(S) \in \prod_{i=1}^{3} \mathcal{S}(Q_i, p)$. To show that π is well defined, suppose $S, T \in \mathcal{S}(P, p)$ are equivalent pebble sets. Then the chambers of P that contain the points of S are precisely those that contain the points of T. Let C be any such chamber, and suppose $q \in S \cap C$ and $r \in T \cap C$. Suppose also that $C \subseteq Q_i$. Since every chamber of P is a subset of some chamber of Q_i, the chamber in Q_i that contains q also contains r, so that $S \cap Q_i$ and $T \cap Q_i$ are equivalent pebble sets for Q_i. Since this is true for all $1 \le i \le 3$, we have that $S \cap Q_i$ and $T \cap Q_i$ are equivalent pebble sets for all i. Thus π is well defined.

To show that π is one-to-one, suppose that $S, T \in \mathcal{S}(P, p)$ are two pebble sets such that $S \cap Q_i$ and $T \cap Q_i$ are equivalent pebble sets in all the Q_i $(1 \le i \le 3)$. Let $q \in S$ be any point. Since S and T both contain p, we may assume that $q \neq p$. Since the Q_i cover P, there exists some i such that $p \in Q_i$. Since $S \cap Q_i$ and $T \cap Q_i$ are equivalent pebble sets in Q_i, then there exists a point $r \in T \cap Q_i$ that lies in the same chamber of Q_i as q. But by Lemma 20, the chamber of Q_i that contains q is also a chamber of P. Thus q and r lie in the same chamber of P. Since q was chosen arbitrarily, we have that S and T are equivalent pebble sets in P.

To show that π is onto, let S_1, S_2, and S_3 be, respectively, pebble sets in Q_1, Q_2, and Q_3 that contain p. We show that $S = S_1 \cup S_2 \cup S_3$ is a pebble set in P by applying Lemma 3. Let e be any edge and v any vertex of P. We may assume that v is an endpoint of an edge in E_1, for an identical argument applies if v is an endpoint of an edge in $E_2 \cup E_3$. If e is an edge of Q_1, then since S_1 is a pebble set in Q_1, Δev contains precisely one point of S_1. Furthermore, Δev cannot contain a point of S_2. For the only way that Δev intersects Q_2 is if $e = e_3$, in which case the facts that $p \in \Delta e_3 f_2$ and that S_2 is a pebble set in Q_2 reveal that $\Delta ev \cap Q_2$ contains no point of S_2. By similar reasoning, Δev contains no point of S_3.

Now if e is not an edge of Q_1, then it is an edge in $E_2 \cup E_3 \cup \{e_1\}$. First consider that $e = e_1$. Then the interior of $\Delta ev \cap Q_1$ is a subset of $\Delta f_1 v$. But $p \in \Delta f_1 v$, and S_1 is a pebble set in Q_1. Thus Δev contains no point of $S_1 - \{p\}$. Furthermore, since $p \in \Delta e_1 f_2$ and S_2 is a pebble set in Q_2, then Δev contains no point of $S_2 - \{p\}$.

Similarly, since $p \in \Delta e_1 f_3$ and S_3 is a pebble set in Q_3, then Δev contains no point of $S_3 - \{p\}$. Thus Δev contains at most one point of S.

Finally, consider that $e \in E_2 \cup E_3$. We may assume $e \in E_2$, for an identical argument will apply if $e \in E_3$. By Lemma 19, Δev contains no point of S_1. Since Δev does not intersect Q_3, then Δev contains no point of S_3. Finally, consider the quadrilateral determined by f_2 and e. This quadrilateral contains precisely two points of S_2, one of which is p. Since $e \notin \mathrm{ES}(C)$, we have that $p \notin \Delta ev$. Thus Δev contains at most one point of S_2.

We therefore have that every triangle of the form Δev contains at most one point of S, so that by Lemma 3, S is a pebble set in P. Thus π is onto. □

We are now ready to prove the converse of Theorem 17. If C is a chamber with support size three, and we let p be any point in C, then p is close to an edge in each of the Q_i. Theorem 4 assures us that there are pebble sets in each of the Q_i that contain p. By the proof of Theorem 21, the union of these pebble sets is a pebble set in the polygon. Thus we arrive at the following.

Corollary 22 *A chamber with support size three is viable.*

With the help of the next proposition, we can count the number of pebble sets that contain a point in a given chamber, where all other points lie in the periphery of the polygon.

Proposition 23 *Let C be a viable chamber in a polygon P, and let S be a pebble set in P that contains a point $p \in C$. Then the $S \cap Q_i$ are peripheral pebble sets in Q_i if and only if all points of $S - \{p\}$ lie in the periphery of P.*

Proof First suppose all the $S \cap Q_i$ are peripheral pebble sets in the Q_i. Now the only border triangles of Q_i that are not also border triangles of P must be defined by f_i and an edge of P incident with f_i. Such a triangle contains p. Thus if $q \in S - \{p\}$, q lies in a border triangle of some Q_i that is also a border triangle of P. Conversely, if every point $q \in S - \{p\}$ lies in the periphery of P, then for any i, either q lies in the periphery of Q_i, or it lies outside Q_i. □

By Proposition 23, the bijection in Theorem 21 can be restricted to a bijection between pebble sets in P that contain a point in C and where all other points lie in the periphery, and ordered triples of peripheral pebble sets in the Q_i that contain a point close to the edge f_i. With Theorem 4, we now have a way to construct all pebble sets in P that have a point in C and where all other points lie in the periphery. The following corollary allows us to determine the number of such pebble sets.

Corollary 24 *Let C be a chamber of a polygon P such that $|\mathrm{ES}(C)| = 3$. Define E_i and Q_i as in Construction 2, and let n_i be the number of distinct endpoints of the edges of E_i. Let $D = \sum_{i=1}^{3} d(Q_i)$, where $d(Q_i)$ is defined as in Corollary 5. Then the number of pebble sets in P containing p and with all other points in the periphery is 2^{n-6+D}.*

Proof Each Q_i contains $n_i + 2$ vertices, and $n_1 + n_2 + n_3 = n$. By Theorem 21 and Corollary 5, the number of such pebble sets is $\prod_{i=1}^{3} 2^{n_i-2+d(Q_i)} = 2^{n-6+D}$. □

5 Finding Viable Chambers

In this section we approach the task of finding the viable chambers of a polygon in two slightly different ways. First, given a particular chamber in a polygon, we show that determining whether its support size is three is surprisingly easy. Second, given an arbitrary n-gon, we describe an algorithm for finding all viable chambers that is $O(n^3)$. This represents an improvement over checking every chamber, for as we will show, the number of chambers in an n-gon is $O(n^4)$.

5.1 A Second Characterization of Viability

According to Proposition 18, only triangular chambers are viable. The converse, however, is not true in general. But if C is a triangular chamber, and the chords and edges that determine the boundary of C are incident with edges of the polygon in a particular way we will describe, then we can conclude that $|\mathrm{ES}(C)| = 3$ and that therefore C is viable.

One direction of claim 1 from the proof of Theorem 11 says that if either $e_k \in \mathrm{ES}(C)$ or $e_{k-1} \in \mathrm{ES}(C)$, then $f \in \mathrm{ES}(C')$. Equivalently, if $f \notin \mathrm{ES}(C')$, then neither e_k nor e_{k-1} is in $\mathrm{ES}(C)$; and furthermore, by claim 2a in the proof of Proposition 11, every other edge of P is in $\mathrm{ES}(C)$ if and only if it is in $\mathrm{ES}(C')$. The important point to be made here is that if $f \notin \mathrm{ES}(C')$, then $\mathrm{ES}(C') = \mathrm{ES}(C)$.

We may apply this result inductively on a set of consecutive edges in a polygon to have the following.

Lemma 25 *Let $E = \{e_i, e_{i+1}, \dots e_j\}$ be any set of consecutive edges of a polygon P, where $2 \leq |E| \leq n-2$. Let g be the chord $v_i v_{j+1}$, and C be any chamber of P that does not lie in the polygon determined by E and g. Let P' be the polygon determined by g and the edges of P not in E. Let C' be the chamber of P' that contains C. Finally, suppose that $g \notin \mathrm{ES}(C')$. Then $ES(C') = \mathrm{ES}(C)$.*

Now we demonstrate how to determine if a triangular chamber has support size three. Let C be a triangular chamber in a polygon P, and suppose f is an edge or chord that determines part of the boundary of C. In the proof of Proposition 18, we noted that the two edges of P that are incident with f and lie on the same side of f as C are edges in $\mathrm{ES}(C)$.

So consider all the edges of P with the property that they are incident with an edge or chord that bounds C and lie on the same side of the chord as C. Call this set of edges the *boundary support of* C, which we will denote $\mathrm{BS}(C)$. Then by the observation in the previous paragraph, we have that $\mathrm{BS}(C) \subseteq \mathrm{ES}(C)$.

Proposition 26 *Suppose C is a triangular chamber in an n-gon P, and suppose that* $\mathrm{BS}(C)$ *contains precisely three edges. Then* $\mathrm{BS}(C) = \mathrm{ES}(C)$.

Proof Let $\{f_1, f_2, f_3\}$ be the chords that determine the boundary of C, and write $\mathrm{BS}(C) = \{e_1, e_2, e_3\}$, where f_i is incident with e_j and e_k (i, j, k distinct). Let V be the set of endpoints of the f_i. Note that $3 \le |V| \le 6$. First consider that $|V| \le 5$, so that at least two of the f_i are incident. Now the intersections of pairs of the f_i determine the vertices of C. We thus have that a vertex of P is also a vertex of C, so that C lies in the periphery. In this case, $\mathrm{ES}(C)$ contains precisely three edges, so that $\mathrm{BS}(C) = \mathrm{ES}(C)$.

We may assume, therefore, that $|V| = 6$. Let P' be the convex hull of $\mathrm{BS}(C)$. Since $|V| = 6$, we have that P' is a hexagon, and C is a chamber in the core of P'. (See Fig. 7.) Each f_i is a chord of P, and the perimeter of P' consists of $\{e_1, e_2, e_3, g_1, g_2, g_3\}$, where the g_i are chords of P joining e_j and e_k (i, j, and k all distinct). The perimeter of P' naturally partitions the edges of P into six subsets: $\{e_1\}$, $\{e_2\}$, $\{e_3\}$, E_1, E_2, and E_3, as illustrated in Fig. 7. Note that the E_i are all nonempty. We show that no edge of the E_i is in the edge support of C.

Let $1 \le i \le 3$, and let P_i be the polygon formed from P by removing the edges of E_i and including g_i. Now C is a chamber in both P and P_i. We claim $g_i \notin \mathrm{ES}(C)$ in P_i. If $j \ne i$, then C lies in the quadrilateral defined by e_i and e_j. Thus C lies in no triangle determined by g_i and an endpoint of any edge in E_j, so that $g_i \notin \mathrm{ES}(C)$ (in P_i). Thus by Lemma 25, no edge in E_i is in $\mathrm{ES}(C)$ (in P). Since this is true for $1 \le i \le 3$, we have that $\mathrm{ES}(C) = \{e_1, e_2, e_3\}$. □

If a chamber is not triangular, its boundary support must contain more than three edges. With this and Propositions 18 and 26, we have the following.

Theorem 27 *A chamber is viable if and only if its boundary support contains precisely three edges.*

5.2 An Algorithm for Determining all Viable Chambers

To determine all viable chambers of a given n-gon, one approach would be to check every chamber to see if satisfies the criterion of Theorem 27. For an n-gon in general position (so that no three chords intersect at a single point), the number of chambers is $\binom{n-1}{2} + \binom{n}{4}$.[1] Therefore, any algorithm that checks every chamber for viability will generally run in $O(n^4)$ time.

This can be improved in the following way. Select three edges of the n-gon, and denote them $E = \{e_i, e_j, e_k\}$. In this subsection, we will describe a decision procedure for determining when E is the edge support for some chamber of P.

Let P' be the convex hull of E. If P' is a quadrilateral or pentagon, then E is the edge support of a peripheral chamber of P, which is viable. If P' is a hexagon such

[1] The number of chambers in P can be determined by considering the graph G whose vertices are the intersections of all the chords and edges of P, and whose edges are the edges of P and segments of the chords joining two points of intersection. We note that there is precisely one intersection of chords for every choice of four vertices of P, so that the number of vertices of G is $n + \binom{n}{4}$. Every vertex of P has degree $n - 1$ and the remaining vertices of G have degree four. Since the sum of the degrees of the vertices of G is twice the number of edges, we have that G contains $\binom{n}{2} + 2\binom{n}{4}$ edges. Applying Euler's formula to this planar graph, the number of chambers in P is therefore $\binom{n-1}{2} + \binom{n}{4}$.

that the chords that connect opposite vertices all meet at one point, then no chamber of P' (and therefore no chamber of P) has E as its edge support (as can be seen by considering boundary support instead).

If P' is a hexagon in general position, then P' contains a triangular chamber C in the core. If $\mathrm{BS}(C) = E$, then C is a viable chamber of P', as illustrated in Fig. 4. Furthermore, C is also a viable chamber in P, as the following theorem demonstrates. For notational simplicity, we let $\{e_1, e_2, e_3\}$ represent any choice of three edges of the n-gon.

Theorem 28 *Let $E = \{e_1, e_2, e_3\}$ be any three edges of an n-gon P, and let P' be the convex hull of E. Suppose P' is a hexagon in general position, so that it contains a triangular chamber C in the core. Finally, suppose that $\mathrm{BS}(C) = E$. Then C is a viable chamber of P.*

Proof Let $\{g_1, g_2, g_3\}$ be the other three edges of P', and let E_1, E_2, and E_3 be the edge subsets of P as labeled in Fig. 7. First we show that C is a chamber of P by supposing that D is a chamber of P that is a proper subset of C. Then there is a chord f of P that is not a chord of P' that intersects C. Clearly at least one endpoint of f is not a vertex of P'. Thus this endpoint would determine an edge of P in one of the E_i that is in $\mathrm{BS}(D)$, hence in $\mathrm{ES}(D)$ in P. But then by Lemma 25, $g_i \in \mathrm{ES}(C)$ in P', which is false. Thus C is a chamber of P.

Let P_1 be the convex hull of $E \cup E_1$. By Lemma 25, since $g_1 \notin \mathrm{ES}(C)$ in P', then $\mathrm{ES}(C)$ in P_1 is precisely the same as $\mathrm{ES}(C)$ in P'. Now let P_2 be the convex hull of $E \cup E_1 \cup E_2$. Again by Lemma 25, since $g_2 \notin \mathrm{ES}(C)$ in P_1, then $\mathrm{ES}(C)$ in P_2 is precisely the same as $\mathrm{ES}(C)$ in P_1. Finally, since $g_3 \notin \mathrm{ES}(C)$ in P_2, applying Lemma 25 a third time reveals that $\mathrm{ES}(C)$ in P is precisely the same as $\mathrm{ES}(C)$ in P_2. Thus in P, $\mathrm{ES}(C) = E$ and by Corollary 22, C is a viable chamber of P. □

Since the number of such edge triples is $\binom{n}{3}$, any algorithm that enumerates all viable chambers of P using Theorem 28 will run in $O(n^3)$ time.

6 Further Questions

In this paper, we have made some progress in characterizing all pebble sets in a given polygon. First, we have a lower bound on the number of pebble sets, given by the $n2^{n-5}$ peripheral pebble sets. This lower bound is actually achieved when there are no chambers outside the periphery whose support size is three—for instance, when the polygon is regular.

Stronger still, a point can be in a pebble set if and only if the support size of its chamber is three. And given such a chamber, we can use the construction in Sect. 4.3 to decompose the polygon into three subpolygons Q_1, Q_2 and Q_3. Theorem 21 assures us that finding pebble sets in each of the Q_i is equivalent to finding a pebble set in the original polygon. This suggests an inductive approach to characterizing pebble sets. This inductive approach is algorithmic, but leaves much to be desired. It seems inefficient, but worse, it does not seem to capture the entire structure of the problem.

The set of chambers whose support sizes are all three does seem to relate to the corresponding sets for the Q_i, and it might be possible to use this fact to extract a more refined object that makes counting pebble sets easier.

For instance, we showed that if p lies in a viable chamber in the core, then we can easily count the number of pebble sets whose only core point is p. What is the corresponding description for two core points? Or more generally?

Beyond this main question, we could seek solutions to related problems. First, the statement of the problem by Lukácz and András specified that there should be $n-2$ points in the required set and that a point on the boundary of a triangle does not count as being in that triangle. We showed in Proposition 1 that as a result, points in a pebble set may not lie on chords of the polygon. But if we allow more points, or if we change how we count points on the boundary of a triangle, this proposition no longer holds. For instance, if the polygon is a square, we can augment any pebble set by adding a point in the center of the square, which does not count toward any total since it does not lie in any triangle. On the other hand, if we decide that points on the boundary of the triangle count as lying in the triangle (so that the triangles are closed), we could have a set with one point (the center of the square) with the required property. If we count points on the boundary of a triangle as providing $1/2$ a point to that triangle, we could have sets with two points, both straddling the same chord but from opposite sides of the center. The structure of these sets seems less clear.

Another obvious generalization would be to consider polytopes in higher dimensions, as in [3], replacing the notion of triangle with that of simplices. But there are many cases where the problem analogous to the polygon question has no solution. For instance, in three dimensions, consider the triangular bipyramid—that is, the union of two tetrahedra that share a face. This has a triangulation into two simplices (the two tetrahedra used to define it), and another triangulation into three simplices, where the simplices all surround the edge between the apexes of the tetrahedra. As a result, if each tetrahedron contains a point, then we are left with requiring a set with two points and three points simultaneously, which is impossible. Thus, the structure of this problem seems very different.

One way around this is to insist that every simplex has *at most* one point, which is precisely the definition of pebble set in [3]. It is likely that any generalization to higher dimensions should use this notion.

Another generalization in the plane would be to insist that every triangle contain k points, for a given integer $k \geq 1$. Some preliminary investigations have indicated that this problem has some interesting structure, though this problem seems to have a character very different from the problem posed in this paper.

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