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New Lower Bounds for the Number of ($\leq k$)-Edges and the Rectilinear Crossing Number of K_n*

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Abstract. We provide a new lower bound on the number of ($\leq k$)-edges of a set of n points in the plane in general position. We show that for $0 \leq k \leq \lfloor (n-2)/2 \rfloor$ the number of ($\leq k$)-edges is at least

$$E_k(S) \geq 3\binom{k+2}{2} + \sum_{j=\lfloor n/3 \rfloor}^{k} (3j - n + 3),$$

which, for $\lfloor n/3 \rfloor \leq k \leq 0.4864n$, improves the previous best lower bound in [12].

As a main consequence, we obtain a new lower bound on the rectilinear crossing number of the complete graph or, in other words, on the minimum number of convex quadrilaterals determined by n points in the plane in general position. We show that the crossing number

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is at least

$$\left(\frac{41}{108}+\varepsilon\right)\binom{n}{4}+O(n^3)\geq 0.379688\binom{n}{4}+O(n^3),$$

which improves the previous bound of $0.37533\binom{n}{4}+O(n^3)$ in [12] and approaches the best known upper bound $0.380559\binom{n}{4}+\Theta(n^3)$ in [4].

The proof is based on a result about the structure of sets attaining the rectilinear crossing number, for which we show that the convex hull is always a triangle.

Further implications include improved results for small values of n. We extend the range of known values for the rectilinear crossing number, namely by $\overline{\mathrm{cr}}(K_{19})=1318$ and $\overline{\mathrm{cr}}(K_{21})=2055$. Moreover, we provide improved upper bounds on the maximum number of halving edges a point set can have.

1. Introduction

Given a graph G, its *crossing number* is the minimum number of edge crossings over all possible drawings of G in the plane. Crossing number problems have both, a long history, and several applications to discrete geometry and computer science. We refrain from discussing crossing number problems in their generality, but instead refer the interested reader to the early works of Tutte [24] or Erdős and Guy [17], the recent survey by Pach and Tóth [23], or the extensive online bibliography by Vrt'o [25].

In 1960 Guy [20] started the search for the *rectilinear crossing number* of the complete graph, $\overline{\mathrm{cr}}(K_n)$, which considers only straight-edge drawings. The study of $\overline{\mathrm{cr}}(K_n)$ is commonly agreed to be a difficult task and has attracted a lot of interest in recent years, see, e.g., [3]–[6], [12], [14], and [22]. In particular, exact values of $\overline{\mathrm{cr}}(K_n)$ were only known up to $n=17$, see [7], and also the exact asymptotic behavior is still unknown. Furthermore, it has been shown independently by Ábrego and Fernández-Merchant [3] and Lovász et al. [22] that if we denote the number of $(\leq k)$-edges of S by $E_k(S)$ and the number of crossings that appear when the complete graph is drawn on top of S (equivalently, the number of convex quadrilaterals in S) by $\overline{\mathrm{cr}}(S)$, then

$$\overline{\mathrm{cr}}(S)=\sum_{k<(n-2)/2}(n-2k-3)\,E_k(S)+O(n^3). \tag{1}$$

It may be surprising that until very recently no results about the combinatorial properties of optimal sets were known. Motivated by this, we start our study considering structural properties of point sets minimizing the number of crossings, that is, attaining the rectilinear crossing number $\overline{\mathrm{cr}}(K_n)$. Relations are obtained by using basic techniques, like e.g. continuous motion and rotational sweeps. In particular, in Section 2 we investigate the changes of the order type of a point set when one of its points is moved. We define suitable moving directions which allow us to decrease $\overline{\mathrm{cr}}(K_n)$, concluding that point configurations attaining the rectilinear crossing number have a triangular convex hull. Independently, and using different techniques, this result has been extended to pseudolinear drawings by Balogh et al. [11].

In Section 3, and using the same technique of continuous motion, we show that when proving a lower bound for $(\leq k)$-edges it can be assumed that the set has a triangular convex hull. Based on this, we first give a really simple proof of the known bound

$3\binom{k+2}{2}$, tight in the range $k \leq \lfloor n/3 \rfloor - 1$ but no longer tight for $k \geq \lfloor n/3 \rfloor$ as shown in [1]. Finally, we obtain a new bound for $k \geq \lfloor n/3 \rfloor$ which generally improves the previous best lower bound obtained by Balogh and Salazar [12]: We show that, for $0 \leq k < \lfloor (n-2)/2 \rfloor$, the number of ($\leq k$)-edges of a set of n points in the plane in general position is at least

$$3\binom{k+2}{2} + \sum_{j=\lfloor n/3 \rfloor}^{k} (3j - n + 3).$$

According to whether n is divisible by 3 or not, for $k \geq \lfloor n/3 \rfloor$ this bound can be written as follows:

$$3\binom{k+2}{2} + 3\binom{k - \frac{n}{3} + 2}{2} \qquad \text{if} \quad \frac{n}{3} \in \mathbb{N},$$

$$3\binom{k+2}{2} + \frac{1}{3}\binom{3k - n + 5}{2} \qquad \text{if} \quad \frac{n}{3} \notin \mathbb{N}.$$

If we plug our new lower bound for ($\leq k$)-edges into (1), we get

$$\overline{\text{cr}}(K_n) \geq \left(\frac{41}{108} + \varepsilon\right)\binom{n}{4} + O(n^3) \geq 0.379688\binom{n}{4} + O(n^3),$$

that improves the best previous lower bound of $0.37533\binom{n}{4} + O(n^3)$ obtained by Balogh and Salazar [12] and approaches the best known upper bound of $0.380559\binom{n}{4} + \Theta(n^3)$ by Ábrego and Fernández-Merchant [4]. For another implication of our result, observe that a bound of $\overline{\text{cr}}(K_n) \geq \frac{41}{108}\binom{n}{4} + O(n^3)$ is proved in [1] for drawings of K_n with a certain type of three-fold symmetry called 3-decomposability, posing the conjecture that all optimal drawings have this property. Our result proves this bound without any restrictions on the point sets.

For small values of n the rectilinear crossing number is known for $n \leq 17$, see [7] and references therein. Our results imply that some known configurations of [5] are optimal. We thus extend the range of known values for the rectilinear crossing number by $\overline{\text{cr}}(K_{19}) = 1318$ and $\overline{\text{cr}}(K_{21}) = 2055$. Moreover, our results confirm the values for smaller n, especially $\overline{\text{cr}}(K_{17}) = 798$, which have been numerically obtained in [7]. Finally we provide improved upper bounds on the maximum number of halving lines that a set of n points can have for some small values of n.

2. Minimizing the Number of Rectilinear Crossings

Let $S = \{p_1, \ldots, p_n\}$ be a set of n points in the plane in general position, that is, no three points lie on a common line. It is well known that crossing properties of edges spanned by points from S are exactly reflected by the order type of S, introduced by Goodman and Pollack in 1983 [19]. The *order type* of S is a mapping that assigns to each ordered triple i, j, k in $\{1, \ldots, n\}$ the orientation (either clockwise or counterclockwise) of the point triple p_i, p_j, p_k.

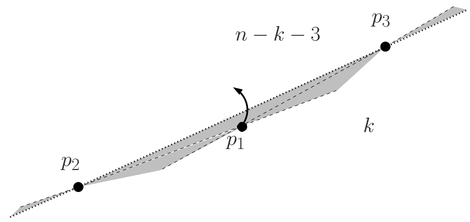

Fig. 1. The point p_1 crosses over the segment p_2p_3, changing the orientation of the triple p_1, p_2, p_3.

Consider a point $p_1 \in S$ and move it in the plane in a continuous way. A change in the order type of S occurs if, and only if, the orientation of a triple of points of S is reversed during this process. This is the case precisely if p_1 crosses the line spanned by two other points, say p_2 and p_3, of S. This event has been considered previously in [9] and [10] in the context of studying the change in the number of j-facets under continuous motion of the points, and is called a *mutation*.

Assume that at some instant t_0 during the mutation the three points p_1, p_2, p_3 are collinear and that the orientation of the triple at time $t_0 + \varepsilon$ is inverse to its orientation at time $t_0 - \varepsilon$ for some $\varepsilon > 0$, which can be chosen small enough to guarantee that the orientation of the rest of triples does not change in the interval $[t_0 - \varepsilon, t_0 + \varepsilon]$. Let us assume that during the mutation p_1 crosses the line segment p_2p_3 as indicated in Fig. 1; otherwise we can interchange the role of p_1 and p_2 (or p_3, respectively). We say that p_1 plays the *center role* of the mutation.

We call the above defined mutation a *k-mutation* if there are k points on the same side of the line through p_2 and p_3 as p_1, excluding p_1. Our first goal is to study how mutations affect the number of crossings of S, that is, the number of crossings of a straight-line embedding of K_n on S. Note that we are considering only rectilinear crossings.

Lemma 1. *A k-mutation increases the number of crossings of S by* $2k - n + 3$.

Proof. By definition of the k-mutation, the only triple of points changing its orientation is p_1, p_2, p_3. Thus precisely the $n - 3$ quadruples of points of S including this triple inverse their crossing properties. Observe that at time $t_0 - \varepsilon$ the shaded region in Fig. 1 has to be free of points of S. Therefore, any of the $n - k - 3$ points opposite to p_1 with respect to the line through p_2 and p_3 produced a crossing together with p_1 and the segment p_2p_3. On the other hand, none of the k points on the same side as p_1 does. This situation is precisely inverted after the flip and hence we eliminate $n - k - 3$ crossings, but generate k new crossings. □

Since we know how mutations affect the number of crossings, we are now interested in good moving directions. A point $p \in S$ is called *extreme* if it is a vertex of the convex hull of S. Two extreme points $p, q \in S$ are called non-consecutive if they do not share a common edge of the convex hull of S. We define a *halving ray* ℓ to be an oriented line passing through one extreme point $p \in S$, avoiding $S \backslash \{p\}$ and splitting $S \backslash \{p\}$ into two subsets of cardinality $n/2$ and $(n-2)/2$, for n even and $(n-1)/2$ each for n

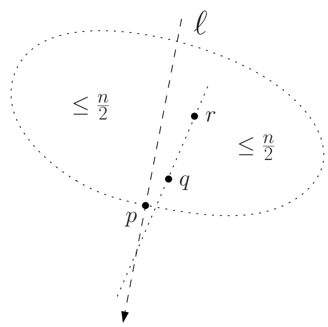

Fig. 2. Moving p along a halving ray ℓ decreases the number of crossings.

odd, respectively. Furthermore, we orient ℓ away from S: for H a half-plane through p containing S, the "head" of ℓ lies in the complement of H and the "tail" of ℓ splits S.

Lemma 2. *Let p be an extreme point of S and ℓ be a halving ray for p. If p is moved along ℓ in the given orientation, every mutation decreases the number of crossings of S.*

Proof. For the whole proof refer to Fig. 2. First, we observe that p has to be involved in any mutation and that the center role is played by another point $q \in S$, since p is extreme. Let $r \in S$ be the third point involved in the mutation, so that q crosses over the segment pr. As ℓ is a halving ray and p is an extreme point for the k-mutation which takes place when p crosses the line defined by q and r, we have that $k \leq n/2 - 2$. Therefore, from Lemma 1 it follows that the number of crossings of S decreases. □

Lemma 3. *For every pair of non-consecutive extreme points p and q of S, we can choose a pair of halving rays containing p and q (one each) that cross in the interior of the convex hull of S.*

Proof. Let h be the line through p and q. Then there is at least one open half-plane H defined by h which contains at least $\lceil (n-2)/2 \rceil$ points of S. So we can choose the two halving rays in such a way that their tails lie in H. Now suppose that the two halving rays do not cross in the interior of the convex hull of S. Then they split S into three regions, two outer regions and one central region. In each outer region there are at least $\lfloor (n-1)/2 \rfloor$ points, since they are supported by halving rays. In the central region there is at least the point which lies on the convex hull of S between p and q and not in H. Finally, there are p and q themselves. All together we have $2 \cdot \lfloor (n-1)/2 \rfloor + 1 + 2 \geq n + 1$ points, a contradiction, hence the lemma follows. □

Observation 1. Using order type preserving projective transformations it can also be seen that a triangular convex hull can be obtained by projection along the halving ray. This is a rather common tool when working with order types, see, e.g., [21]. However, we have decided to use a self-contained, planar approach.

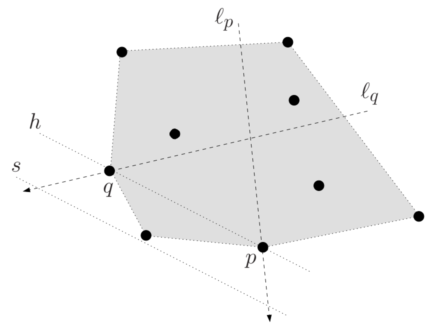

Fig. 3. Decreasing the number of crossings and the number of extreme points.

We now have the ingredients to go for our first result, which seems to have been a common belief (see, e.g., [14]) and for which evidence was provided by all configurations attaining $\overline{\text{cr}}(K_n)$ for $n \leq 17$ [5], [7]:

Theorem 4. *Any set S of $n \geq 3$ points in the plane in general position attaining the rectilinear crossing number has precisely three extreme points*, *that is*, *a triangular convex hull*.

Proof. For the sake of a contradiction, assume that S is a set of points attaining the rectilinear crossing number and having more than three extreme points. Let p and q be two non-consecutive extreme points of S and let ℓ_p and ℓ_q be their halving rays chosen according to Lemma 3. Let s be a line parallel to the line h through p and q, such that S entirely lies on one side of s and ℓ_p, ℓ_q are oriented towards s (see Fig. 3). Furthermore, let s be placed arbitrarily close to S.

Now move p along ℓ_p until it reaches the intersection of ℓ_p and s: If a mutation occurs, we already reduce the number of crossings by Lemma 2. Then move q along ℓ_q up to the intersection of ℓ_q and s. Note that, by Lemma 3, ℓ_p and ℓ_q do not cross in their heads; hence the movements of p along ℓ_p and q along ℓ_q do not interfere with each other.

After moving p and q, all extreme points of S between them changed to interior points. Hence at least one mutation happened, since p and q are non-consecutive, and therefore the number of crossings of S decreased, which contradicts the optimality of S. □

Observation 2. If S has three extreme points, from the proof of Theorem 4 it follows that we can keep moving the three extreme points of S along their respective halving rays: if mutations occur, this further reduces the rectilinear crossing number. Thus, for an optimal set S the three extreme points have to be "far away" in the following sense: For every extreme point p of S, the cyclic sorted order of $S \setminus \{p\}$ around p has to be the same as its sorted order in the direction orthogonal to the halving ray of p. (Otherwise another mutation would occur when we keep on moving p.)

3. Lower Bound for ($\leq k$)-Edges

A *j-edge*, $0 \leq j \leq \lfloor (n-2)/2 \rfloor$, is a segment spanned by the points $p, q \in S$ such that precisely j points of S lie in one open half-space defined by the line through p and q. In other words, a j-edge splits $S \backslash \{p, q\}$ into two subsets of cardinality j and $n-2-j$, respectively. Note that here we consider non-oriented j-edges, i.e., the edge pq equals the edge qp. We say that a j-edge is a *halving edge* if it splits the set as equally as possible, i.e., if $j = (n-2)/2$ when n is even and if $j = (n-3)/2$ when n is odd.

A k *($\leq k$)-edge* has at most k points in a half-space, that is, it is a j-edge for $0 \leq j \leq k$. We denote the number of ($\leq k$)-edges of S by $E_k(S)$ and omit the set when it is clear from the context. Finally, $(E_0, \ldots, E_{\lfloor (n-2)/2 \rfloor})$ is the *($\leq k$)-edge vector* of S.

A *k-set* of S is a set $S' \subset S$ of k points that can be separated from $S \backslash S'$ by a line (hyperplane in general dimension). In dimension 2 there is a one-to-one relation between the numbers of k-sets and $(k-1)$-edges [8]. Thus, in this paper we solely use the notion of j-edges, although all the results can also be stated in terms of k-sets.

In the next lemma we study how the number of j-edges changes during a mutation.

Lemma 5. *Let $k \leq (n-3)/2$. During a k-mutation, the number of j-edges changes in the following way: For $k < (n-3)/2$, the number of k-edges decreases by one and the number of $(k+1)$-edges increases by one. For $k = (n-3)/2$ everything remains unchanged.*

Proof. We use the same notation as for the proof of Lemma 1. First observe that the only edges that change their property are the edges spanned by points p_1, p_2, and p_3. Let $k < (n-3)/2$: Before the mutation, $p_1 p_2$ and $p_1 p_3$ are k-edges, while $p_2 p_3$ is a $(k+1)$-edge. After the mutation, the situation is reversed: $p_1 p_2$ and $p_1 p_3$ are $(k+1)$-edges while $p_2 p_3$ is a k-edge. So in total we get one more $(k+1)$-edge and one less k-edge. For $k = (n-3)/2$ the two types of edges considered are halving edges before and after the mutation, that is, the number of halving edges does not change. □

From Lemmas 1 and 5 we get a relation between the number $\overline{\text{cr}}(S)$ of rectilinear crossings of S and the number of j-edges of S, denoted by e_j. An equivalent relation can be found in [3] and [22].

Lemma 6.

$$\overline{\text{cr}}(S) + \sum_{j=0}^{\lfloor (n-2)/2 \rfloor} j \cdot (n-j-2) \cdot e_j = \tfrac{1}{8} \cdot (n^4 - 6n^3 + 11n^2 - 6n).$$

Proof. Looking for an expression of the form $\sum_j \alpha_j e_j$ that cancels the variation in the number of crossings during a mutation, we get the relation $\alpha_{j+1} = \alpha_j + n - 2j - 3$. The result corresponds to choosing $\alpha_0 = 0$. The right-hand side of the equation can be easily derived from the convex set. □

For the extremal case of j-edges, that is, halving edges, we can state a result similar to Theorem 4:

Theorem 7. *For any fixed $n \geq 3$, there exist point sets with a triangular convex hull that maximize the number of halving edges.*

Proof. As observed in the proof of Lemma 2, when an extreme point p is moved along a halving ray, only k-mutations with $k \leq n/2 - 2$ can occur. Therefore, from Lemma 5 it follows that the number of halving lines cannot decrease. Then we can proceed as in the proof of Theorem 4. □

One might wonder whether we can obtain a stronger result similar to Theorem 4 stating that any point set maximizing the number of halving edges has to have a triangular convex hull. However, there exist sets of eight points with four extreme points bearing the maximum of nine halving edges, see [9] and similar examples exist for larger n. Hence, the stated relation is tight in this sense. We leave as an open problem the existence of a constant h such that any point set maximizing the number of halving edges has at most $\sim h$ extreme points. We conjecture that such a constant exists, and the results for $n \leq 11$ suggest that $h = 4$ could be the tight bound.

Similar arguments as above can be used to prove the next result, which is our starting point for the lower bound of $(\leq k)$-edges:

Lemma 8. *Let S be a set of n points with $h > 3$ extreme points and $(\leq k)$-edge vector $(E_0, \dots, E_{\lfloor (n-2)/2 \rfloor})$. Then there exists a set S' of n points with triangular convex hull and $(\leq k)$-edge vector $(E'_0, \dots, E'_{\lfloor (n-2)/2 \rfloor})$ with $E'_i \leq E_i$ for all $i = 0, \dots, \lfloor (n-2)/2 \rfloor$ (where at least one inequality is strict).*

Proof. The proof follows the lines of the proof for Theorem 4 to obtain a set with only three extreme vertices. Observe that for all k-mutations which occur during this process it holds that $k \leq n/2 - 2$ because we are moving along halving rays. Thus by Lemma 5 every mutation decreases the number of k-edges by one and increases the number of $(k+1)$-edges by one. For the $(\leq k)$-edge vector this means that E_k is decreased by one and the rest of the vector remains unchanged. The statement follows. □

As a warm-up, we start with a really simple and geometric proof of the following bound, which has been independently shown in [3] and [22] using circular sequences:

Theorem 9. *Let S be a set of n points in the plane. The number of $(\leq k)$-edges of S is at least $3\binom{k+2}{2}$ for $0 \leq k < (n-2)/2$. This bound is tight for $k \leq \lfloor n/3 \rfloor - 1$.*

Proof. By Lemma 8 we can assume that S has a triangular convex hull, as otherwise we can find a point set with a strictly smaller $(\leq k)$-edge vector for which the theorem still has to hold. Let p, q, r be the three extreme points of S.

By rotating a ray around each extreme point of S, we get exactly three 0-edges and six j-edges for every $1 \leq j < (n-2)/2$, all of them incident to p, q, or r. This gives a total of $3 + 6k$ $(\leq k)$-edges, which already proves the lower bound for $k = 1$.

For $2 \leq k < (n-2)/2$ we will prove the lower bound by induction on n. The cases $n \leq 3$ are obvious and serve as an induction base. So for $n \geq 4$ consider $S_1 = S \backslash \{p, q, r\}$.

Observe that, since the convex hull of S is a triangle, a j-edge of S_1 is either a $(j+1)$-edge or a $(j+2)$-edge of S. Therefore, if $2 \leq k < (n-2)/2$ we get

$$E_k(S) \geq E_{k-2}(S_1) + 3 + 6k \geq 3\binom{k}{2} + 3 + 6k = 3\binom{k+2}{2}.$$

Finally, the example in [15] shows that the bound $3\binom{k+2}{2}$ is tight for $k \leq \lfloor n/3 \rfloor - 1$. □

In view of the preceding proof, it is clear that in order to improve the bound for $k \geq \lfloor \frac{n}{3} \rfloor$ we need to show that a number of j-edges of S_1 are $(j+1)$-edges of S. This is going to be our next result, but first we need some preparation.

It is more convenient now to consider *oriented* j-edges: an oriented segment pq is a j-edge of S if there are exactly j points of S in the open half-plane to the right of pq. Following [18], we rotate a directed line ℓ around points of the set S, counterclockwise, and in such a way that, when ℓ contains only one point, it has exactly k points of S on its right. We refer to this movement as a *k-rotation*. If the line rotates around a point p, the half-lines into which p divides ℓ are the *head* and the *tail* of the line. Observe that, if during a k-rotation the line ℓ reaches a new point q on its tail, then qp is a $(k-1)$-edge and the k-rotation continues around q. On the other hand, if the new point q appears on the head of the ray, then pq is a k-edge and the k-rotation continues also around q. We recall that when a k-rotation of 2π is completed, all k-edges of S have been found.

We denote by ℓ^+ and $\overline{\ell^+}$, respectively, the open and closed half-planes to the right of ℓ and, similarly, ℓ^- and $\overline{\ell^-}$ will be the half-planes to the left of ℓ.

Theorem 10. *Let S be a set of n points in the plane in general position and let T be a triangle containing S. If $\lfloor n/3 \rfloor \leq k \leq n/2 - 1$, then there exist at least $3k - n + 3$ k-edges of S having to the right only one vertex of T.*

Proof. Let p, q, and r be the vertices of T in counterclockwise order. Throughout this proof, we refer to a k-edge and its supporting line synonymously. Moreover, edges having one vertex of T on its right will be called *good* edges, and the rest will be said to be *bad*.

We start with the case of halving lines for n even, which is straightforward. There are at least n halving lines and exactly half of them are good (because each edge is a halving line in both orientations). Therefore, because $k = n/2 - 1$, we have that $n/2 = 3k - n + 3$. In the following, $k < n/2 - 1$.

Since the number of k-edges is always at least $2k + 3$ (see [22]), if all k-edges are good the result is true. Therefore, without loss of generality, we can assume that there are bad edges having q and r on its right. Among them, let ℓ_1 be the bad k-edge which intersects pq closest to q. Now, we make a k-rotation of ℓ_1 and distinguish two cases. For the whole proof refer to Fig. 4.

Case 1. If a k-edge having p and r on its right is not found, then a good k-edge can be found for each of the k points to the right of ℓ_1: if $a \in \ell_1^+$, consider a directed line

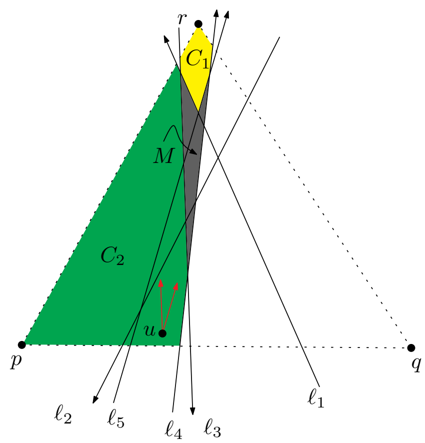

Fig. 4. Proving Theorem 10: k-edges and their relation to the triangle T.

through a and parallel to ℓ_1 and rotate it around a. Before rotating 180°, a k-edge is found and it has to be good because there is no k-edge having p and r on its right. Therefore, because $k \leq (n-3)/2$, it holds that $k \geq 3k - n + 3$ and the result follows.

Case 2. Let ℓ_2 be the first k-edge obtained from the k-rotation of ℓ_1 for which p and r lie on its right. Let $H = S \cap \ell_2^+ \cap \ell_1^-$ and denote by h the cardinality of H. Observe that all k-edges between ℓ_1 and ℓ_2 we get during the rotation are good edges. Since points in H are necessarily encountered in the head of the ray during the k-rotation at least once, there is a good k-edge incident to each of them. Consider $C_1 = S \cap \overline{\ell_1^+} \cap \overline{\ell_2^+}$ and denote by c_1 its cardinality. If $c_1 = 0$, then $h = k$ and the result follows as in Case 1. Let ℓ_3 be a k-edge tangent to C_1 at only one point and leaving C_1 on its left. Observe that ℓ_3 can be found by considering tangents to C_1 parallel to a rotating direction: We start with the tangent parallel to ℓ_2 and rotate the direction counterclockwise. If a k-edge uv defined by two points of C_1 is found, we proceed with the rotation (the new tangent will touch $\sim C_1$ at v). Also note that ℓ_3 could have r to its right. Finally, let $C_2 = S \cap \ell_1^- \cap \overline{\ell_3^+}$ and ℓ_4 be the common tangent to C_1 and C_2 leaving both sets to its left.

Now we want to bound the number of points in ℓ_4^+. To this end, observe that $k - h \leq c_1 \leq k - h + 2$, depending on the number of points defining ℓ_2 that belong to C_1. Therefore, if we denote by m the number of points in $M = (S \cap \ell_4^-) \backslash (C_1 \cup C_2)$, then $|S \cap \overline{\ell_4^-}| \geq k + m + c_1 + 1$ (start from ℓ_3: It has k points to the right. In addition we have $m + c_1$ points plus a point not in C_1 defining ℓ_3). Therefore, $|S \cap \ell_4^+| \leq n - k - m - c_1 - 1$. Again we have to distinguish two cases:

Case 2a: $c_1 \geq k - h + 1$. In this case, $|S \cap \ell_4^+| \leq n - 2k + h - m - 2$. Therefore, if $|S \cap \ell_4^+| > k$, then $h > 3k - n + 2 + m$, implying $h \geq 3k - n + 3$ and thus the h good k-edges incident to points in H (see above) are sufficient to guarantee the result.

On the other hand, if there are at most k points to the right of ℓ_4, we can rotate ℓ_4 around C_1, clockwise, and find a k-edge ℓ_5 (which could be bad and which could also coincide with ℓ_4). Observe that when rotating from ℓ_4 to ℓ_5 only points from M and C_2 can be passed by the line and that one point spanning ℓ_5 belongs to either M or C_2. Thus $|C_2 \cap \overline{\ell_5^+}| \geq k - |S \cap \ell_4^+| - m + 1 \geq 3k - n + 3 - h$.

We finally claim that for each point in $C_2 \cap \overline{\ell_5^+}$ we can find a good k-edge, which together with the h good edges from H settles Case 2a. To prove the claim let $u \in C_2 \cap \overline{\ell_5^+}$ and consider the half-cone with apex at u, edges parallel to ℓ_3 and ℓ_5 and containing C_1. Because ℓ_5 and ℓ_3 are k-edges, we can guarantee that if we rotate a line around u we find at least two k-edges, uv and wu, and that at least one of them is good: Start from the line parallel to ℓ_5 which has less than k points to its right and more than k points to its left. Rotating counterclockwise around u until the line is parallel to ℓ_3 reverses the situation. Thus, during the rotation we first get an edge uv with k points to its right and then an edge uw with k points to its left. If r is to the left of uv then uv is good. Otherwise r has to be to the right of uw and thus wu is good. Finally observe that all good k-edges associated to points in $C_2 \cap \overline{\ell_5^+}$ in this last step are different from the good k-edges incident to points in H which were found in the k-rotation from ℓ_1 to ℓ_2 because the former ones have r to its left while the later ones have r to its right.

Case 2b: $c_1 = k - h$. In this case the arguments of Case 2a give $3k - n + 2$ good k-edges and one more is needed. Observe that, in this case, the points defining ℓ_2 are to the left of ℓ_1. Therefore, the point t in the tail of the k-edge defining ℓ_2 has a good k-edge incident to it: Points to the left of ℓ_1 and defining k-edges are found during the k-rotation, and the first time they are found, they have to define a good k-edge (recall that ℓ_2 was the first bad edge). As t does not belong to H, the good k-edge incident to t (having r to its right) was not counted previously. □

Theorem 11. *Let S be a set of n points in the plane in general position and let $E_k(S)$ be the number of $(\leq k)$-edges in S. For $0 \leq k < \lfloor (n-2)/2 \rfloor$ we have*

$$E_k(S) \geq 3\binom{k+2}{2} + \sum_{j=\lfloor n/3 \rfloor}^{k} (3j - n + 3).$$

Proof. The proof goes by induction on n. Observe that Lemma 8 guarantees that it is sufficient to prove the result for sets with triangular convex hull. Let p, q, r be the vertices of the convex hull of S and let $S_1 = S \backslash \{p, q, r\}$. For $k \leq \lfloor n/3 \rfloor - 1$ the result is already given by Theorem 9. If $k \geq \lfloor n/3 \rfloor + 1$ then

$$E_{k-2}(S_1) \geq 3\binom{k}{2} + \sum_{j=\lfloor (n-3)/3 \rfloor}^{k-2} (3j - (n-3) + 3) = 3\binom{k}{2} + \sum_{i=\lfloor n/3 \rfloor}^{k-1} (3i - n + 3).$$

Furthermore, as in the proof of Theorem 9 we know that there are exactly $3 + 6k$ $(\leq k)$-edges of S adjacent to p, q and r, so using Theorem 10 we conclude that

$$E_k(S) \geq E_{k-2}(S_1) + 3 + 6k + 3(k-1) - (n-3) + 3 \geq 3\binom{k+2}{2} + \sum_{j=\lfloor n/3 \rfloor}^{k} (3j - n + 3).$$

For $k = \lfloor n/3 \rfloor$, $E_{k-2}(S_1) \geq 3\binom{k}{2}$ and then

$$\begin{aligned} E_k(S) &\geq 3\binom{k}{2} + 3 + 6k + 3(k-1) - (n-3) + 3 \\ &= 3\binom{k+2}{2} + 3\left\lfloor \frac{n}{3} \right\rfloor - n + 3. \end{aligned}$$ □

As a main consequence of Theorem 11, we can obtain a new lower bound for the rectilinear crossing number of the complete graph:

Theorem 12. *For each positive integer n,*

$$\overline{\mathrm{cr}}(K_n) \geq \left(\frac{41}{108} + \varepsilon\right)\binom{n}{4} + O(n^3) \geq 0.379688\binom{n}{4} + O(n^3).$$

Proof. As shown in [3] and [22], the number of $(\leq k)$-edges and the crossing number of K_n are strongly related. More precisely, if we denote by $\overline{\mathrm{cr}}(S)$ the number of crossings when the complete graph is drawn with the set of vertices S, then

$$\overline{\mathrm{cr}}(S) = \sum_{k<(n-2)/2} (n-2k-3)\, E_k(S) + O(n^3). \tag{1}$$

Writing $3\binom{k+2}{2} + \sum_{j=\lfloor n/3 \rfloor}^{k} (3j - n + 3) = \hat{E}_k$, we get

$$\overline{\mathrm{cr}}(K_n) \geq \sum_{k<(n-2)/2} (n-2k-3)\, \hat{E}_k = \frac{41}{108}\binom{n}{4} + O(n^3).$$

The lower bound can be slightly improved by exploiting a bound for $(\leq k)$-edges which is better than $\hat{E}_k$ when k is close to $n/2$. In order to do so, we can use the claim by Ábrego et al. [2] that the bound in [12] is bigger than $\hat{E}_k$ when $k \geq 0.4864n$, which then yields the bound

$$\overline{\mathrm{cr}}(K_n) \geq 0.379688\binom{n}{4} + O(n^3). \qquad \square$$

Observation 3. Using Theorem 11 and Lemma 6 it can be shown that the configurations of 19 and 21 points in [5] are optimal for the number of crossings: their $(\leq k)$-edge vectors are, respectively, $(3, 9, 18, 30, 45, 63, 86, 115, 171)$ and $(3, 9, 18, 30, 45, 63, 84, 111, 144, 210)$. Because they match the bound in Theorem 11 for $k < (n-3)/2$, we have that $\overline{\mathrm{cr}}(K_{19}) = 1318$ and $\overline{\mathrm{cr}}(K_{21}) = 2055$.

Observation 4. Let us recall that, for n odd, j-edges with $j = (n-3)/2$ are halving edges. Let $h_n = \max_{|S|=n} e_{\lfloor (n-2)/2 \rfloor}$ be the maximum number of halving lines that a set of $\sim n$ points can have. In Table 1 we present a summary of the values of h_n for $13 \leq n \leq 27$: the value $h_{14} = 22$ and the upper bound for h_{16} were shown in [13], while the

Table 1. Values and bounds of h_n for $13 \leq n \leq 27$.

	n														
	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
h_n	31	22	39	27 28	47	33 36	56	38 43	66	44 51	75 76	51 60	85 87	57 69	96 99

lower bound for h_{16} appeared in [16]. The rest of the lower bounds come from the examples in [5] and the upper bounds can be derived by applying Theorem 11 with $k = \lfloor (n-2)/2 \rfloor - 1$, namely $h_n \leq \binom{n}{2} - E_{\lfloor (n-2)/2 \rfloor - 1}$.

4. Concluding Remarks

In this paper we have presented a new lower bound for the number of $(\leq k)$-edges of a set of n points in the plane in general position. As a corollary of this, a new lower bound for the rectilinear crossing number of K_n is obtained. The basis of the technique is a property about the structure of sets minimizing the number of $(\leq k)$-edges or the rectilinear crossing number: such sets always have a triangular convex hull.

There are still a host of open problems and conjectures about these and related questions, among which we emphasize the following:

- Prove that the new lower bound is optimal for some range of k. Based on computational experiments, we conjecture that the bound is optimal for $k \leq \lfloor 5n/12 \rfloor - 1$.
- Prove that all sets maximizing the number of halving lines have a convex hull with at most h vertices. We conjecture that $h = 4$ is sufficient.
- Prove that sets minimizing the crossing number maximize the number of halving lines.

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