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Stability and Computation of Topological Invariants of Solids in $\mathbb{R}^n$

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Abstract. In this work one proves that under quite general assumptions one can deduce the topology of a bounded open set in $\mathbb{R}^n$ from an approximation of it. For this, one introduces the weak feature size (wfs) that extends for nonsmooth objects the notion of local feature size. Our results apply to open sets with positive wfs. This class includes subanalytic open sets which cover many cases encountered in practical applications. The proofs are based upon the study of distance functions to closed sets and their critical points. The notion of critical point is the same as the one used in riemannian geometry [22], [9], [20] and nonsmooth analysis [10]. As an application, one gives a way to compute the homology groups of open sets from noisy samples of points on their boundary.

1. Introduction and Related Works

The contribution of this work is theoretical. However, it addresses a question arising in practice in the process of reverse engineering. Reverse engineering, in our context, is the process of building a geometric model for a physical object, given a set of points sampled on the object boundary. Does this geometric model, for example a polyhedron, capture the right topology of the initial object? Intuitively, this seems possible if the size of the features, such as the thickness of the thin parts, the diameter of holes, etc., are large with respect to the sampling accuracy and density.

In recent years, authors have worked out sampling conditions and associated reconstruction algorithms that allow the reconstructed geometric model to reflect correctly the topology of the sampled object. Amenta et al. [2] introduce the *local feature size*, or lfs, defined in each point as the distance to the medial axis of the object. Several algorithms are proved to provide a result homeomorphic to the sampled object [1], [2], [4], given a smooth object sampled with a local density at least around 20 times smaller than lfs.

Most studies assume exact sampling. In practice, measured points are assumed to lie within a given tolerance from the object boundary. The case of noisy sampling has been considered as well. In [14] it is proven that, as far as this noise is small with respect to lfs, the topology can still be captured.

However, the problem of a density criterion relying on the *local feature size* is that lfs vanishes on the boundary of nonsmooth objects. Theorems involving lfs do not help on nonsmooth objetcs, such as solids with sharp edges. Fortunately, algorithms proved correct in the case of smooth objects, behave relatively well in practice on solids with sharp edges.

In [6] and [7], the authors examine, given a Hausdorff distance approximation of an object O, the possibility of computing an approximation of the medial axis of O. For this purpose, a new measure of the "feature size" of an object is introduced, the *weak feature size*, or wfs (see Section 3). Wfs allows to consider nonsmooth objects, such as polyhedra or, more generally, piecewise analytic or semianalytic sets, for which wfs > 0. Wfs is defined as the minimum distance between the boundary and the set of singular points of the distance function (distance to the boundary). In other words, wfs is the minimum singular value of the function distance to the boundary (see Sections 2.2 and 3 below).

In the present work the noisy sampling is modelized by a possibly finite set within a given Hausdorff distance of the boundary of the original object. Using the tools developed in [6] and [7], one shows, roughly speaking, that if two objects O and O', such that $\mathrm{wfs}(O) > 2\varepsilon$ and $\mathrm{wfs}(O') > 2\varepsilon$, have their complement O^c and O'^c within a Hausdorff distance less than ε, $d_H(O^c, O'^c) < \varepsilon$, then O and O' have the same homotopy type (see Theorem 3.3). A consequence is that if one is given an ε Hausdorff approximation of the complement O^c of an open set O with $\mathrm{wfs}(O) > 4\varepsilon$, then the homotopy type of O is uniquely determined by the approximation. Indeed, we show that it is possible to express the homology of the set O through the notion of persistent homology [27], [15] from the given approximation (see Theorem 4.2).

If this approximation is a finite sample of the boundary, the algorithms for the computation of persistent Betti numbers (see [27] and [15]) may be used on a filtration in the Voronoi complex, named the λ-*medial axis*, defined in [23], [6], and [7], see Section 4.2.

We show that the persistent Betti numbers computation on the λ-medial axis filtration is guaranteed to provide the Betti numbers of the originally sampled set (Theorem 4.4). A similar computation for persistent homology on the α-complex filtration allows us to capture the homology of a thickening of the boundary of the initial object (Theorem 4.5). It has been independently observed by Cohen-Steiner and Edelsbrunner [12], in the context of work on topological persistence and Morse functions, that sampling conditions based on a variant of the wfs the *homological feature size*, and the use of topological persistence techniques allow us to capture the homology of a thickening of the boundary of the initial object.

2. Preliminaries

We introduce mathematical tools and results which are useful in the remainder of the paper. The proof of the results of this section are available in [23].

2.1. *The "Gradient" Vector Field of Distance Function*

We use the following definitions and notations. Throughout the paper, $\mathcal{O}$ and $\mathcal{M}$ always denote respectively a bounded open subset of $\mathbb{R}^n$ and its Medial Axis defined below. For any set X, let $\overline{X}$, X°, ∂X and X^c denote respectively the closure, the interior, the boundary and the complement of X. $\mathbb{B}_{x,r}$ and $\mathbb{B}^\circ_{x,r}$ respectively denote the closed and open ball of center x and radius r in $\mathbb{R}^n$. We denote by $\mathbb{S}_{x,r}$ the corresponding sphere, that is $\mathbb{S}_{x,r} = \mathbb{B}_{x,r} \backslash \mathbb{B}^\circ_{x,r}$. For any point $x \in \mathcal{O}$, we denote by $\Gamma(x)$ the set of the closest boundary points, that is,

$$\begin{aligned}\Gamma(x) &= \{y \in \mathcal{O}^{c},\ d(x, y) = d(x, \mathcal{O}^{c})\}\\ &= \{y \in \partial\mathcal{O},\ d(x, y) = d(x, \partial\mathcal{O})\}.\end{aligned}$$

Because $\partial\mathcal{O}$ is compact, $\Gamma(x)$ is a nonempty compact set. For a set E, let $|E|$ denote the cardinal of E.

Definition 2.1 (Medial Axis). The Medial Axis $\mathcal{M}$ of the open set $\mathcal{O}$ is the set of points x of $\mathcal{O}$ who have at least two closest boundary points:

$$\mathcal{M} = \{x \in \mathcal{O},\ |\Gamma(x)| \geq 2\}.$$

One denotes by $\mathcal{R}$ the distance function to the boundary of $\mathcal{O}$ defined by

$$\mathcal{R}(x) = d(x, \mathcal{O}^{c}) \qquad \text{for any} \quad x \in \mathcal{O}.$$

One can check, using the triangular inequality twice, that $\mathcal{R}$ is 1-Lipschitz. Given a point $x \in \mathcal{O}$, there always exists a unique closed ball with minimal radius enclosing $\Gamma(x)$ [23]. One defines a real-valued positive function $\mathcal{F}$: $\mathcal{F}(x)$ is the radius of this smallest closed ball enclosing $\Gamma(x)$ and one denotes its center by $\Theta(x)$ (see Fig. 1). In other words,

$$\mathcal{F}(x) = \inf\{r\colon \exists y \in \mathbb{R}^n, \mathbb{B}_{y,r} \supset \Gamma(x)\}.$$

One proves in [23] that $\mathcal{F}$ is *upper semicontinuous*, that is,

$$\forall \varepsilon \in \mathbb{R}, \qquad \{x \in \mathcal{O}, \mathcal{F}(x) < \varepsilon\} \text{ is open.}$$

Of course, when $x \notin \overline{\mathcal{M}}$, we have $\Gamma(x) = \{\Theta(x)\}$ and $\mathcal{F}(x) = 0$. Moreover, $\mathcal{R}$ is differentiable at such a point x and its gradient $\nabla(x)$ is collinear to $(x\Theta(x))$ (see Theorem 4.8 of [17]). One extends the gradient of $\mathcal{R}$ to all points in $\mathcal{O}$ by the following formula:

$$\nabla(x) = \frac{x - \Theta(x)}{\mathcal{R}(x)}.$$

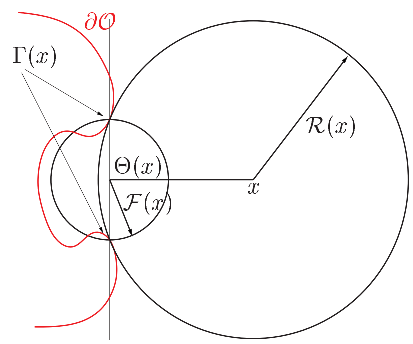

Fig. 1. A two-dimensional example with two closest points.

One has the following relation (see [23]):

$$\nabla(x)^2 = 1 - \frac{\mathcal{F}(x)^2}{\mathcal{R}(x)^2}, \tag{1}$$

which entails trivially

$$\mathcal{F}(x) = \mathcal{R}(x)\sqrt{1 - \nabla(x)^2}. \tag{2}$$

The map $x \mapsto \|\nabla(x)\|$ is lower semicontinuous (see [23]). The *singular points* of ∇ are the points x for which $\nabla(x) = 0$. When $\mathcal{O}^c$ is finite, that is for Voronoi diagrams, singular points are the intersections of the Delaunay cells with their dual Voronoi cell when they do intersect. Notice that in this setting a variant of the vector field ∇ (and which leads to the same critical points) has been used in [13] to study the flow complex of a finite set of points. In the general case, one has the following characterization of singular points (also observed in [14]).

Lemma 2.2. *A point x is a singular point of ∇ if and only if it lies in the convex hull of $\Gamma(x)$: $x \in \mathcal{CH}(\Gamma(x))$.*

Notice that this lemma shows that our notion of singular point is the same as the one considered in the setting of nonsmooth analysis (see [10], [11] and Section 3.2).

The vector field ∇ is not continuous. However, it is shown in [23] that Euler schemes using this vector field converge uniformly, when the integration step decreases, toward a continuous flow $\mathfrak{C}$:

$$\mathfrak{C}\colon \mathbb{R}^+ \times \mathcal{O} \to \mathcal{O}.$$

This flow is used in [23] to realize a homotopy equivalence between $\mathcal{O}$ and $\mathcal{M}$ (see Section 2.2). It also satisfies the following equalities proven in [23]:

$$\mathfrak{C}(t, x) = x + \int_0^t \nabla(\mathfrak{C}(\tau, x))\, d\tau \tag{3}$$

$$\mathcal{R}(\mathfrak{C}(t, x)) = \mathcal{R}(x) + \int_0^t \nabla(\mathfrak{C}(\tau, x))^2\, d\tau. \tag{4}$$

Moreover, $\mathcal{R}$ and $\mathcal{F}$ are increasing along the trajctories of $\mathfrak{C}$: for any $x \in \mathcal{O}$, the functions $t \to \mathcal{R}(\mathfrak{C}(t, x))$ and $t \to \mathcal{F}(\mathfrak{C}(t, x))$ are defined and increasing over $\mathbb{R}^+$.

2.2. *Homotopy Equivalence*

Two maps $f_0\colon X \to Y$ and $f_1\colon X \to Y$ are said to be *homotopic* if there is a continuous map H, $H\colon [0,1] \times X \to Y$, such that $\forall x \in X$, $H(0, x) = f_0(x)$ and $H(1, x) = f_1(x)$. Homotopy allows us to introduce the notion of homotopy equivalence which is defined below (see pages 171–172 of [19] or page 108 of [25] for more details).

Definition 2.3. Two spaces X and Y are said to have the same *homotopy type* if there are continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f$ is homotopic to the identity map of X and $f \circ g$ is homotopic to the identity map of Y.

Homotopy type is clearly a topological invariant: if two spaces X and Y are homeomorphic then they have the same homotopy type. In general, the converse is not true. The homotopy equivalence between topological sets enforces a one-to-one correspondence between connected components, cycles, holes, tunnels, cavities or higher-dimensional topological features of the two sets, as well as the way these features are related. More precisely, if X and Y have the same homotopy type, then their homotopy and homology groups are isomorphic.

In the case where $Y \subset X$ and g is the canonical inclusion, $\forall y \in Y$, $g(y) = y$, the homotopy equivalence may be proven using the following characterization:

Proposition 2.4. *If $Y \subset X$ and there exists a continuous map H, $H\colon [0,1] \times X \to X$ such that*:

- $\forall x \in X$, $H(0, x) = x$,
- $\forall x \in X$, $H(1, x) \in Y$,
- $\forall y \in Y, \forall t \in [0,1]$, $H(t, y) \in Y$,

then X and Y have same homotopy type. *If one replaces the third property by the stronger one*, $\forall y \in Y, \forall t \in [0,1]$, $H(t, y) = y$, *then H defines a* deformation retraction *of X toward Y*.

The definition of a deformation retraction given above is taken from p. 66 of [24].

2.3. *Hausdorff Distance*

The Hausdorff distance between sets is widely used in the paper. Basic definitions and properties of this distance are quickly recalled. For more general results and detailed proofs, the reader is referred to Section 9.11 of [3] for example.

Definition 2.5. Let A and B be two compact subsets of $\mathbb{R}^n$. The Hausdorff distance between A and B is defined by

$$d_H(A, B) = \max\left(\sup_{x\in A} d(x, B),\ \sup_{y\in B} d(y, A)\right).$$

The Hausdorff distance defines a distance on the set $\mathcal{K}(\mathbb{R}^n)$ of compact subsets of $\mathbb{R}^n$ which becomes a complete metric space. Moreover, if K is some fixed compact set, the metric space $(\mathcal{K}(\mathbb{R}^n)_K, d_{\mathrm{H}})$ of compact subsets contained in K is compact.

3. Weak Feature Size, Homotopy and Singular Points of Distance Functions

The aim of this section is to introduce the notion of weak feature size of a bounded open set $\mathcal{O}$ in $\mathbb{R}^n$ and to relate it with properties of the function $\mathcal{R}$ defined by the distance to the complement of $\mathcal{O}$.

As in the previous section, $\mathcal{O}$ is a bounded open subset of $\mathbb{R}^n$, $\mathcal{R}$: $\mathcal{O} \to \mathbb{R}^+$ is the distance function defined by $\mathcal{R}(x) = d(x, \mathcal{O}^{\mathrm{c}})$ and ∇ is the "gradient" of $\mathcal{R}$ as defined in Section 2.1.

Definition 3.1. The *weak feature size* of an open, bounded subset $\mathcal{O}$ of $\mathbb{R}^n$, denoted wfs($\mathcal{O}$), is the distance between the complement $\mathcal{O}^c$ and the set of singular points $\{x \in \mathcal{O}, \nabla(x) = 0\}$ of the vector field ∇.

In the remainder of this paper, we focus on open sets with positive wfs. Weak Feature Size is closely related to stability properties of the topology of open sets. It is shown in Section 3.2 that open sets with positive wfs cover a large class of open sets including most of the ones encountered in practical applications.

3.1. *Weak Feature Size and Stablility of the Homotopy Type*

Let $d > 0$ be a positive real number. We denote by $\mathcal{O}_d$ the set of points of $\mathcal{O}$ at distance greater than d from the boundary,

$$\mathcal{O}_d = \{x \in \mathcal{O}, \mathcal{R}(x) > d\},$$

and by $\overline{\mathcal{O}_d}$ the closure of $\mathcal{O}_d$.

In Theorem 1 of [7] one proves that if $d <$ wfs($\mathcal{O}$) one can push $\mathcal{O}$ into $\mathcal{O}_d$ along the trajectories of the vector field ∇ to obtain the following result.

Theorem 3.2. *If $d <$ wfs($\mathcal{O}$), $\overline{\mathcal{O}_d}$ is a deformation retract of $\mathcal{O}$ and $\mathcal{O}_d$ has the homotopy type of $\mathcal{O}$.*

In other words, for any $d <$ wfs($\mathcal{O}$) one can shrink the open set $\mathcal{O}$ until $\mathcal{O}_d$ without erasing any topological feature. This result plays an important role in the rest of the paper. Moreover, since the trajectoties of ∇ are used to push $\mathcal{O}$ into $\mathcal{O}_d$, the retraction deformation H: $[0, 1] \times \mathcal{O} \to \mathcal{O}$ of previous theorem is such that for any $x \in \mathcal{O}$, $\mathcal{R}(H(x, t))$ is an increasing function of t (see [7] for a detailed proof).

Remark. One can directly prove a stronger result (not used in the following) using an isotopy lemma for distance functions (Proposition 3.4): under the hypothesis of the previous theorem, $\mathcal{O}$ and $\mathcal{O}_d$ are in fact homeomorphic.

One deduces from this result that two nearby bounded open sets with positive wfs have the same homotopy type.

Theorem 3.3. *Let $\mathcal{O}$ and $\mathcal{O}'$ be two bounded open sets in $\mathbb{R}^n$ and let $\varepsilon > 0$ be such that* $\mathrm{wfs}(\mathcal{O}) > 2\varepsilon$ *and* $\mathrm{wfs}(\mathcal{O}') > 2\varepsilon$. *If $d_{\mathrm{H}}(\mathcal{O}^{\mathrm{c}}, \mathcal{O}'^{\mathrm{c}}) < \varepsilon$ then $\mathcal{O}$ and $\mathcal{O}'$ have the same homotopy type.*

Proof. From $\mathrm{wfs}(\mathcal{O}) > 2\varepsilon$, there exists $\alpha > 0$ with $2\varepsilon + \alpha < \mathrm{wfs}(\mathcal{O})$. Let $f\colon \mathcal{O} \to \overline{\mathcal{O}_{2\varepsilon+\alpha}} \subset \mathcal{O}'$ be the deformation retraction of Theorem 3.2 that shrinks $\mathcal{O}$ into $\overline{\mathcal{O}_{2\varepsilon+\alpha}}$. Similarly, let $g\colon \mathcal{O}' \to \overline{\mathcal{O}'_{2\varepsilon+\alpha'}} \subset \mathcal{O}$ be the deformation retraction that shrinks $\mathcal{O}'$ into $\overline{\mathcal{O}'_{2\varepsilon+\alpha'}}$.

The maps $f\colon \mathcal{O} \to \mathcal{O}'$ and $g\colon \mathcal{O}' \to \mathcal{O}$ define the homotopy equivalence, according to Definition 2.3. One has to check that, for example, $g \circ f$ is homotopic to the identity in $\mathcal{O}$. Notice that

$$\overline{\mathcal{O}_{2\varepsilon+\alpha}} \subset \mathcal{O}'_\varepsilon \subset \mathcal{O}. \tag{5}$$

The natural homotopy consists in first applying the homotopy corresponding to f, that is pushing $x \in \mathcal{O}$ to $f(x) \in \overline{\mathcal{O}_{2\varepsilon+\alpha}}$. Then, from the inclusion (5), $f(x) \in \mathcal{O}'_\varepsilon$. Then one applies the homotopy corresponding to g, that is pushing $f(x) \in \mathcal{O}'$ to $g(f(x)) \in \overline{\mathcal{O}'_{2\varepsilon+\alpha'}}$.

Along the trajectory from $f(x)$ to $g(f(x))$, the distance to $\mathcal{O}'^{\mathrm{c}}$ increases, which means that the trajectory remains in $\mathcal{O}'_\varepsilon$ and, again using inclusion (5), it still remains in $\mathcal{O}$. As a consequence, the combination of the two previous homotopies induces an homotopy between the identity map of $\mathcal{O}$ and $g \circ f\colon \mathcal{O} \to \mathcal{O}$. □

Theorem 3.3 gives us hope to compute the homotopy type of a set given a Hausdorff distance approximation of its boundary. Let S be a closed set. From Theorem 3.3, if two open sets O and $\mathcal{O}'$ with a wfs greater than 4ε are such that $d_{\mathrm{H}}(S, \mathcal{O}^{\mathrm{c}}) < \varepsilon$ and $d_H(S, \mathcal{O}'^{\mathrm{c}}) < \varepsilon$ then they have same homotopy type.

Therefore, if S is known to be an ε Hausdorff approximation of some $\mathcal{O}^{\mathrm{c}}$ with $\mathrm{wfs}(\mathcal{O}) > 4\varepsilon$, one has "in theory" enough information to determine the homotopy type of $\mathcal{O}$. For example, we know that there exists at least $\mathcal{O}$ itself that satisfies $\mathrm{wfs}(\mathcal{O}) > 4\varepsilon$, and $d_{\mathrm{H}}(S, \mathcal{O}^{\mathrm{c}}) < \varepsilon$. If, starting from S, one is able to construct any set $\mathcal{O}'$ with $\mathrm{wfs}(\mathcal{O}') > 4\varepsilon$ and $d_{\mathrm{H}}(S, \mathcal{O}'^{\mathrm{c}}) < \varepsilon$, the homotopy type of $\mathcal{O}'$ gives the homotopy type of $\mathcal{O}$.

Remark. In the previous theorem, the bound 2ε is tight for any $n \geq 2$. Indeed, in Fig. 2, the "U" shape and the "O" shape do not have the same homotopy type: "U" is simply connected while "O" is not. The sides of the square of the dotted grid have

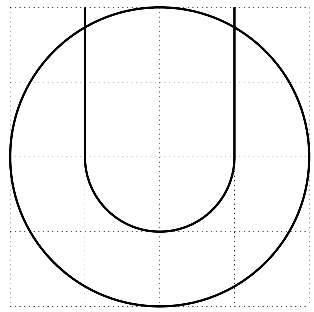

Fig. 2. The "U" shape and the "O" shape have not the same homotopy type.

length ε. One can check that the Hausdorff distance between the "U" shape and the "O" shape is ε. The two vertical bars of the "U" are actually not exactly vertical: the "U" is imperceptibly open, and therefore $\text{wfs}(\text{U}) = \infty$. The "O" is a circle of radius 2ε and one obviously has $\text{wfs}(\text{O}) = 2\varepsilon$.

3.2. *Critical Values of Distance Functions*

The vector field ∇ and the wfs are closely related to the notion of critical points of distance functions. Critical points for distance functions to a point have been introduced in Riemannian geometry by Grove and Shiohama [22]. Distance functions to closed sets have been intensively studied (see [21] and [18], for example) and the aim of this section is to give properties of such functions that apply to our setting. As a consequence, we show that any bounded open subset with piecewise analytic boundary has a positive wfs.

Recall that a point $x \in \mathcal{O}$ is a singular point of ∇ (i.e. $\nabla(x) = 0$) if and only if it lies in the convex hull of $\Gamma(x)$ (Lemma 2.2). This characterization coincides with the definition of singular points of the generalized Clarke gradient of the function $\mathcal{R}$ (see [10], [11] and [18]). Singular points of ∇ are thus critical points of the function $\mathcal{R}$. The critical values of $\mathcal{R}$ are defined as the values taken by $\mathcal{R}$ at critical points. Notice that this notion of critical points is also the same as the one used in Riemannian geometry for distance functions to a point (see [22] and [9]). The wfs is thus the distance between $\mathcal{O}^c$ and the set $\mathbf{F}$ of singular points of $\mathcal{R}$ or, equivalently, the infimum of the critical values of $\mathbb{R}$.

In some way the properties of the distance function to a compact set are quite similar to those of the smooth functions. In particular, they satisfy an Isotopy Lemma [21], that we reproduce below (Proposition 3.4). Notice that Theorem 3.2 may be proven as a corollary of Proposition 3.4.

Proposition 3.4. *If* $0 < \rho_1 < \rho_2$ *are such that* $\overline{\mathcal{O}}_{\rho_1} \backslash \mathcal{O}_{\rho_2}$ *does not contain any critical point of* $\mathcal{R}$, *then all the levels* $\mathcal{R}^{-1}(\rho)$, $\rho \in [\rho_1, \rho_2]$, *are homeomorphic topological manifolds and*

$$\overline{\mathcal{O}}_{\rho_1} \backslash \mathcal{O}_{\rho_2} = \{x \in \mathcal{O} \colon \rho_1 \le \mathcal{R}(x) \le \rho_2\}$$

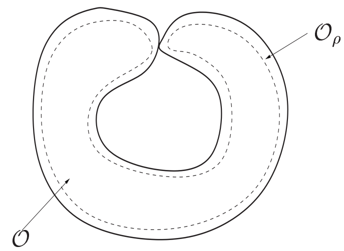

Fig. 3. An open set with positive wfs and nonmanifold boundary.

is homeomorphic to $\mathcal{R}^{-1}(\rho 1) \times [\rho_1, \rho_2]$. *As a consequence*, $\mathcal{O}\rho_1$ *and* $\mathcal{O}\rho_2$ *are homeomorphic and thus homotopy equivalent*.

As a consequence, if $\mathcal{O}$ has a positive wfs, then for all values $\rho \in]0, \text{wfs}[$, the level sets $\mathcal{O}_\rho$ are homeomorphic topological manifolds, even if the boundary of $\mathcal{O}$ is not a manifold (see Fig. 3).

If $n = 3$, the function $\mathcal{R}$ satisfies a Sard theorem [18]:

Proposition 3.5. *If* $n = 3$, *then the set of critical values of* $\mathcal{R}$,

$$\text{Crit}(\mathcal{R}) = \mathcal{R}(\mathbf{F}) = \mathcal{R}(\{x \in \mathcal{O}\colon\ \nabla(x) = 0\})$$

is a compact set with zero Lebesgue measure in $\mathbb{R}$.

Note that such a result is false without the assumption $n = 3$ (see [18]). Nevertheless, if the open set $\mathcal{O}$ is subanalytic one has a stronger result. The definition of subanalytic sets is rather technical and not presented here. In some way, this definition may be considered as a rigorous definition of piecewise analytic sets. Subanalytic sets include most of the sets encountered in practical applications (BRep solids, solids bounded by NURBS surfaces, solids with piecewise linear boundary,...). The reader may refer to [8] for more details and precise definitions.

Proposition 3.6. *Let* $\mathcal{O} \subset \mathbb{R}^n$ *be a subanalytic bounded open set*. *The set of critical values of* $\mathcal{R}$,

$$\text{Crit}(\mathcal{R}) = \mathcal{R}(\mathbf{F}) = \mathcal{R}(\{x \in \mathcal{O}\colon\ \nabla(x) = 0\}),$$

is finite. *In particular*, $\text{wfs}(\mathcal{O}) > 0$.

This result has been proven by Fu [18, p. 1045] for semialgebraic sets. The proof adapts easily to piecewise analytic sets and may be found in [7].

4. Homotopy and Homology of Sets with Positive wfs

We now study the behavior of the homotopy and singular homology groups of open sets $\mathcal{O}$ with positive wfs under small perturbations. Some of the ideas of this section are closely related to the notion of topological persistence (see [15]). To be conceptual, all the homology groups considered in what follows are with coefficients in $\mathbf{Z}/2$. Proofs of this section are clearly independent of the choice of the coefficients domain, so results of this section remain true if one replaces $\mathbf{Z}/2$ by another coefficients domain (e.g. $\mathbf{Z}$). If x is a point in a topological space X, one denotes by $\pi_1(X, x)$ the fundamental group of X with x as the base point.

4.1. *Stability of Homology and Homotopy*

If $\mathcal{O}$ and $\tilde{\mathcal{O}}$ are two bounded open sets in $\mathbb{R}^n$ and $\varepsilon > 0$ such that $d_{\mathrm{H}}(\mathcal{O}^{\mathrm{c}}, \tilde{\mathcal{O}}^{\mathrm{c}}) < \varepsilon$, then one has the inclusions

$$\mathcal{O}_{4\varepsilon} \subset \tilde{\mathcal{O}}_{3\varepsilon} \subset \mathcal{O}_{2\varepsilon} \subset \tilde{\mathcal{O}}_{\varepsilon} \subset \mathcal{O}. \tag{6}$$

These inclusions are used in the proofs of Proposition 4.1 and Theorem 4.2. The next proposition is the key argument to proving Theorem 4.2.

Proposition 4.1. *Let $\mathcal{O}$ and $\tilde{\mathcal{O}}$ be two bounded open sets in $\mathbb{R}^n$ and let $\varepsilon > 0$ be such that* $\mathrm{wfs}(\mathcal{O}) > 2\varepsilon$ *and $d_{\mathrm{H}}(\mathcal{O}^{\mathrm{c}}, \tilde{\mathcal{O}}^{\mathrm{c}}) < \varepsilon$.*

1. *Let $k \in \{0, \ldots, n\}$ and let c_1 and c_2 be two k-chains in $\tilde{\mathcal{O}}_{3\varepsilon}$. Then c_1 and c_2 are homologous in $\mathcal{O}$ if and only if c_1 and c_2 are homologous in $\tilde{\mathcal{O}}_{\varepsilon}$.*
2. *Let γ_1 and γ_2 be two continuous loops in $\tilde{\mathcal{O}}_{3\varepsilon}$. The loops γ_1 and γ_2 are homotopic in $\mathcal{O}$ if and only if γ_1 and γ_2 are homotopic in $\tilde{\mathcal{O}}_{\varepsilon}$.*

Notice that no assumption is done on the wfs of $\tilde{\mathcal{O}}$.

Proof. We only give the proof of part 1, the proof of part 2 being similar. Among the inclusions (6), one uses here

$$\tilde{\mathcal{O}}_{3\varepsilon} \subset \mathcal{O}_{2\varepsilon} \subset \tilde{\mathcal{O}}_{\varepsilon} \subset \mathcal{O}.$$

If c_1 and c_2 are homologous in $\tilde{\mathcal{O}}_{\varepsilon}$, then there exists a $(k+1)$-cycle $C \subset \tilde{\mathcal{O}}_{\varepsilon}$ such that $\partial C = c_1 + c_2$ where ∂ denotes the boundary operator. From $\tilde{\mathcal{O}}_{\varepsilon} \subset \mathcal{O}$ it follows that $C \subset \mathcal{O}$ and c_1 and c_2 are homologous in $\mathcal{O}$.

Suppose now that c_1 and c_2 are two k-cycles in $\tilde{\mathcal{O}}_{3\varepsilon}$ that are homologous in $\mathcal{O}$. This means there exists a $(k+1)$-cycle $C \subset \mathcal{O}$ such that $\partial C = c_1 + c_2$. The cycles c_1 and c_2 are compact sets in $\mathcal{O}_{2\varepsilon}$, so there exists $\alpha > 0$ such that c_1 and c_2 are included in $\overline{\mathcal{O}_{2\varepsilon+\alpha}}$ and $2\varepsilon + \alpha < \mathrm{wfs}(\mathcal{O})$. There exists (Theorem 3.2) a continuous map $\varphi\colon \mathcal{O} \to \overline{\mathcal{O}_{2\varepsilon+\alpha}}$ which is a deformation retraction. φ restricted to $\mathcal{O}_{2\varepsilon+\alpha}$ is the identity map, so one has

$$\partial\varphi_{\#}(C) = \varphi_{\#}(\partial C) = \varphi_{\#}(c_1) - \varphi_{\#}(c_2) = c_1 + c_2,$$

where $\varphi_\#$ is the homomorphism induced by φ between the modules of k-chains. So c_1 and c_2 are homologous in $\overline{\mathcal{O}_{2\varepsilon+\alpha}}$.

To conclude the proof it suffices to notice that $\overline{\mathcal{O}_{2\varepsilon+\alpha}} \subset \tilde{\mathcal{O}}_\varepsilon$. □

The following theorem shows that even if one does not know $\mathcal{O}$ but only an approximation $\tilde{\mathcal{O}}$ of it, one can still "compute" its homology groups.

Theorem 4.2. *Let $\mathcal{O}$ and $\tilde{\mathcal{O}}$ be two bounded open sets in $\mathbb{R}^n$, let $\varepsilon > 0$ be such that* $\mathrm{wfs}(\mathcal{O}) > 4\varepsilon$ *and* $d_{\mathrm{H}}(\mathcal{O}^c, \tilde{\mathcal{O}}^c) < \varepsilon$ *and let* $k \in \{0, \dots, n\}$ *be an integer. Denote by* $i\colon \tilde{\mathcal{O}}_{3\varepsilon} \to \tilde{\mathcal{O}}_\varepsilon$ *the canonical inclusion map and by* $i_*\colon \mathrm{H_k}(\tilde{\mathcal{O}}_{3\varepsilon}, \mathbf{Z}/2) \to \mathrm{H_k}(\tilde{\mathcal{O}}_\varepsilon, \mathbf{Z}/2)$ *the induced map between homology groups. One has*

$$\mathrm{H_k}(\mathcal{O}, \mathbf{Z}/2) \simeq \mathrm{im}(\mathrm{i}_*\colon \mathrm{H_k}(\tilde{\mathcal{O}}_{3\varepsilon}, \mathbf{Z}/2) \to \mathrm{H_k}(\tilde{\mathcal{O}}_\varepsilon, \mathbf{Z}/2)).$$

Denoting the map induced by i between fundamental groups also by i_*, one has

$$\pi_1(\mathcal{O}, x) \simeq \mathrm{im}(i_*\colon \pi_1(\tilde{\mathcal{O}}_{3\varepsilon}, x) \to \pi_1(\tilde{\mathcal{O}}_\varepsilon, x)).$$

Using the terminology of [15], the previous result means that the homology groups of $\mathcal{O}$ can be deduced from the homology groups of $\tilde{\mathcal{O}}_{3\varepsilon}$ by "removing" the cycles of persistence less than 2ε in the filtration defined by the open sets $\tilde{\mathcal{O}}_d$, $d > 0$. In other words, the homology groups of $\mathcal{O}$ are determined by the way $\tilde{\mathcal{O}}_{3\varepsilon}$ is included in $\tilde{\mathcal{O}}_\varepsilon$.

Proof. First, one has

$$\mathrm{im}(i_*\colon \mathrm{H_k}(\tilde{\mathcal{O}}_{3\varepsilon}, \mathbf{Z}/2) \to \mathrm{H_k}(\tilde{\mathcal{O}}_\varepsilon, \mathbf{Z}/2)) \simeq \mathrm{H_k}(\tilde{\mathcal{O}}_{3\varepsilon}, \mathbf{Z}/2)/\mathrm{Ker}(\mathrm{i}_*),$$

where $\mathrm{Ker}(i_*)$ denotes the kernel of the homomorphism i_*.

Let $j\colon \tilde{\mathcal{O}}_{3\varepsilon} \to \mathcal{O}$ be the canonical inclusion map and let j_* be the induced homomorphism between corresponding homology groups. Consider the following sequence of inclusion maps:

$$\mathcal{O}_{4\varepsilon} \to \tilde{\mathcal{O}}_{3\varepsilon} \to \mathcal{O}.$$

Because $4\varepsilon < \mathrm{wfs}(\mathcal{O})$, it follows from Theorem 3.2 that $\mathcal{O}_{4\varepsilon}$ is a deformation retract of $\mathcal{O}$. As a consequence the composition of the two previous maps, which is the inclusion map $\mathcal{O}_{4\varepsilon} \to \mathcal{O}$, induces an isomorphism between corresponding homology groups. Thus the composition of the two homomorphisms

$$\mathrm{H_k}(\mathcal{O}_{4\varepsilon}, \mathbf{Z}/2) \to \mathrm{H_k}(\tilde{\mathcal{O}}_{3\varepsilon}, \mathbf{Z}/2) \to \mathrm{H_k}(\mathcal{O}, \mathbf{Z}/2)$$

is an isomorphism. It follows that j_* is surjective and

$$\mathrm{H_k}(\mathcal{O}, \mathbf{Z}/2) \simeq \mathrm{H_k}(\tilde{\mathcal{O}}_{3\varepsilon}, \mathbf{Z}/2)/\mathrm{Ker}(\mathrm{j}_*).$$

To conclude the proof, it suffices to remark that Proposition 4.1 implies that $\mathrm{Ker}(i_*) = \mathrm{Ker}(j_*)$.

This proof immediately adapts to the case of fundamental groups using the second part of Proposition 4.1. □

Remark. Notice that in Proposition 4.1, one only needs the assumption $\mathrm{wfs}(\mathcal{O}) > 2\varepsilon$ while in Theorem 4.2 one needs $\mathrm{wfs}(\mathcal{O}) > 4\varepsilon$ to be satisfied. Notice also that previous proposition and theorem generalize immediately to higher homotopy groups.

4.2. *Using λ-Medial Axis*

In [7] a subset of the medial axis, called the λ-medial axis and denoted $\mathcal{M}_\lambda$ is introduced. Using the definitions and notations of Section 2.1, for an open set $\mathcal{O}$, its medial axis $\mathcal{M}(\mathcal{O})$ can be defined as

$$\mathcal{M}(\mathcal{O}) = \{x \in \mathcal{O}\,;\;\; \mathcal{F}(x) > 0\}.$$

For $\lambda > 0$, the λ-medial axis $\mathcal{M}_\lambda(\mathcal{O})$ is defined by

$$\mathcal{M}_\lambda(\mathcal{O}) = \{x \in \mathcal{O}\,;\;\; \mathcal{F}(x) \geq \lambda\}.$$

Notice that, because $\mathcal{F}$ is upper semicontinuous, $\mathcal{M}_\lambda(\mathcal{O})$ is a closed set. For a finite set S, the medial axis $\mathcal{M}(S^c)$ is the union of the cells of the Voronoi diagram of S of dimension strictly less than the dimension n of the ambient space. Moreover, the function $\mathcal{F}$ being constant on each Voronoi cell, $\mathcal{M}_\lambda(S^c)$ is a union of some Voronoi cells of the Voronoi diagram of S (see [7]). Since $\mathcal{M}_\lambda(S^c)$ is closed, it is thus a subcomplex of the Voronoï diagram of S. Given the Voronoi diagram, it is straightforward to compute $\mathcal{M}_\lambda(S^c)$, by selecting the cells on which $\mathcal{F}$ is greater than or equal to λ.

The set $\mathbf{F} = \{x \in \mathcal{O}\,;\;\; \nabla(x) = 0\}$ of critical points of the distance function is compact because $\mathcal{O}$ is bounded and $x \mapsto \|\nabla(x)\|$ is lower semicontinuous. Therefore, the set $\mathcal{R}(\mathbf{F})$ of *critical values* of the distance function is compact. We have the following lemma:

Lemma 4.3. *Let $\mathcal{O}$ be a bounded open set. If $\varepsilon > 0$ is not a critical value of the distance function $\mathcal{R}$, then $\mathcal{O}_\varepsilon$ and $\mathcal{M}_\varepsilon$ have the same homotopy type.*

Proof. Because $\forall x \in \mathcal{O},\ \mathcal{F}(x) \leq \mathcal{R}(x)$, one has of course $\mathcal{M}_\varepsilon \subset \mathcal{O}_\varepsilon$. Because the set $\mathcal{R}(\mathbf{F})$ of critical values is compact, there is $\alpha > 0$ such that there are no critical values in $[\varepsilon, \varepsilon + \alpha]$. Therefore, it is possible to shrink $\mathcal{O}_\varepsilon$ on $\overline{\mathcal{O}_{\varepsilon+\alpha}}$, using for example Proposition 3.4.

Let us take $\beta > 0$ such that

$$\sqrt{1 - \beta^2}(\varepsilon + \alpha) > \varepsilon. \tag{7}$$

If D is a bound on the diameter of $\mathcal{O}$, $t \mapsto \mathcal{R}(\mathfrak{C}(t, x))$ is bounded by D. Then, from (4), there must be some $t \in [0, D/\beta^2]$ with $\|\nabla(\mathfrak{C}(t, x))\| < \beta$.

We consider the following deformation $x \mapsto f(x)$. First one pushes $x \in \mathcal{O}_\varepsilon$ toward $y \in \overline{\mathcal{O}_{\varepsilon+\alpha}}$ by the deformation retraction on $\overline{\mathcal{O}_{\varepsilon+\alpha}}$, one thus has

$$\mathcal{R}(y) \geq \varepsilon + \alpha. \tag{8}$$

The second part of the deformation consists in applying the flow $\mathfrak{C}$, for $t \in [0, D/\beta^2]$: $f(x) = \mathfrak{C}(D/\beta^2, y)$.

For at least some t, one has $\|\nabla(\mathfrak{C}(t, y))\| < \beta$. Let us denote this point by $z = \mathfrak{C}(t, y)$. One has $\|\nabla(z)\| < \beta$ and, from (8), $\mathcal{R}(z) \geq \varepsilon + \alpha$ which entails, by (2) and (7), $\mathcal{F}(z) \geq \varepsilon$. This means $z \in \mathcal{M}_\varepsilon$. However, because $t \mapsto \mathcal{F}(\mathfrak{C}(t, y))$ is increasing (see [23]), this entails that $f(x) = \mathfrak{C}(D/\beta^2, y)$ belongs to $\mathcal{M}_\varepsilon$. The map f meets the condition of the characterization of Proposition 2.4 for homotopy equivalence. □

Theorem 4.2 allows us to capture the homology of a set $\mathcal{O}$ from an approximation $\tilde{\mathcal{O}}$ by looking at the image, by the homomorphism i_* induced by the canonical inclusion i, of $\mathrm{H_k}(\tilde{\mathcal{O}}_{3\varepsilon})$ toward $\mathrm{H_k}(\tilde{\mathcal{O}}_\varepsilon)$. In fact, one has a similar result using respectively $\mathcal{M}_{3\varepsilon}(\tilde{\mathcal{O}})$ and $\mathcal{M}_\varepsilon(\tilde{\mathcal{O}})$, which is more convenient for a practical computation of the homology of $\mathcal{O}$ from finite samples.

Theorem 4.4. *Let $\mathcal{O}$ and $\tilde{\mathcal{O}}$ be two bounded open sets in $\mathbb{R}^n$, let $\varepsilon > 0$ be such that ε and 3ε are not critical values of the distance function to $\tilde{\mathcal{O}}^{\mathrm{c}}$ and such that* $\mathrm{wfs}(\mathcal{O}) > 4\varepsilon$ *and $d_{\mathrm{H}}(\mathcal{O}^{\mathrm{c}}, \tilde{\mathcal{O}}^{\mathrm{c}}) < \varepsilon$. Let $k \in \{0, \ldots, n\}$ be an integer. Denote by $i\colon \mathcal{M}_{3\varepsilon}(\tilde{\mathcal{O}}) \to \mathcal{M}_\varepsilon(\tilde{\mathcal{O}})$ the canonical inclusion map and by i_* the induced map between homology groups. One has*

$$\mathrm{H_k}(\mathcal{O}, \mathbf{Z}/2) = \mathrm{im}(\mathrm{i}_*\colon \mathrm{H_k}(\mathcal{M}_{3\varepsilon}(\tilde{\mathcal{O}}), \mathbf{Z}/2) \to \mathrm{H_k}(\mathcal{M}_\varepsilon(\tilde{\mathcal{O}}), \mathbf{Z}/2)).$$

Proof. We use the notation $\mathrm{H_k}(\cdot)$ for $\mathrm{H_k}(\cdot, \mathbf{Z}/2)$. In the diagram below, right and right-up arrows are group homomorphisms induced by canonical inclusions. The vertical arrows are the group isomorphisms corresponding to the deformation defined in the proof of Lemma 4.3. We claim that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H_k}(\tilde{\mathcal{O}}_{3\varepsilon}) & \longrightarrow & \mathrm{H_k}(\tilde{\mathcal{O}}_\varepsilon) \\ \downarrow\uparrow & \nearrow & \downarrow\uparrow \\ \mathrm{H_k}(\mathcal{M}_{3\varepsilon}(\tilde{\mathcal{O}})) & \longrightarrow & \mathrm{H_k}(\mathcal{M}_\varepsilon(\tilde{\mathcal{O}})) \end{array}$$

Let us consider $c \in \mathrm{H_k}(\tilde{\mathcal{O}}_{3\varepsilon})$ and its image $c' \in \mathrm{H_k}(\mathcal{M}_{3\varepsilon}(\tilde{\mathcal{O}}))$ by the isomorphism induced by the deformation of the proof of Lemma 4.3. If $\gamma \in \tilde{\mathcal{O}}_{3\varepsilon}$ is a k-chain in the class $c \in \mathrm{H_k}(\tilde{\mathcal{O}}_{3\varepsilon})$ and $\gamma' \in \mathcal{M}_{3\varepsilon}(\tilde{\mathcal{O}})$ is a k-chain in the class $c' \in \mathrm{H_k}(\mathcal{M}_{3\varepsilon}(\tilde{\mathcal{O}}))$, then γ and γ' are homologous in $\tilde{\mathcal{O}}_{3\varepsilon}$ (that is $\gamma - \gamma'$ is a boundary in $\tilde{\mathcal{O}}_{3\varepsilon}$). Therefore c and c' have same image by the homomorphism induced by the inclusion in $\mathrm{H_k}(\tilde{\mathcal{O}}_\varepsilon)$. This proves that the upper-left part of the diagram commutes.

Recall that the canonical inclusion of $\mathcal{M}_\varepsilon(\tilde{\mathcal{O}})$ in $\tilde{\mathcal{O}}_\varepsilon$ induces an isomorphism. It results that if $\gamma \in \mathcal{M}_{3\varepsilon}(\tilde{\mathcal{O}})$ is a chain from the class $c \in \mathrm{H_k}(\mathcal{M}_{3\varepsilon}(\tilde{\mathcal{O}}))$, its images by the respective canonical inclusions in $\tilde{\mathcal{O}}_\varepsilon$ and $\mathcal{M}_\varepsilon(\tilde{\mathcal{O}})$ belong to respective isomorphic classes in $\mathrm{H_k}(\tilde{\mathcal{O}}_\varepsilon)$ and $\mathrm{H_k}(\mathcal{M}_\varepsilon(\tilde{\mathcal{O}}))$. This proves that the lower-right part of the diagram commutes.

It results that the image of $\mathrm{H_k}(\tilde{\mathcal{O}}_{3\varepsilon})$ in $\mathrm{H_k}(\tilde{\mathcal{O}}_\varepsilon)$ is isomorphic to the image of $\mathrm{H_k}(\mathcal{M}_{3\varepsilon}(\tilde{\mathcal{O}}))$ in $\mathrm{H_k}(\mathcal{M}_\varepsilon(\tilde{\mathcal{O}}))$. This, together with Theorem 4.2, concludes the proof. □

Notice that the commutative diagram is still valid with homotopy groups. We will see in Section 5 that Theorem 4.4 together with the algorithms of topological persistence (see

[15]) allow us to compute the homology of a bounded open set $\mathcal{O}$ with $\mathrm{wfs}(\mathcal{O}) > 0$ from a noisy sampling.

4.3. *Homology of Thickenings of Compact Sets with Positive wfs*

The *wfs of a compact subset* K of $\mathbb{R}^n$ is the wfs of its complement $\mathbb{R}^n \backslash K$. Note that $\mathbb{R}^n \backslash K$ is not bounded but one can define its wfs in the same way as for bounded open sets. Let $K \subset \mathbb{R}^n$ be a compact set such that $\mathrm{wfs}(K) > 0$. One denotes by $K^\varepsilon = \{x \in \mathbb{R}^n\colon d(x, K) < \varepsilon\}$ the ε-thickening of K. In this section one shows how our previous results adapt immediately to the topology of K^ε.

Theorem 4.5. *Let K and $\tilde{K}$ be two compact subsets of $\mathbb{R}^n$, let $\varepsilon > 0$ be such that $\mathrm{wfs}(K) > 4\varepsilon$ and $d_{\mathrm{H}}(K, \tilde{K}) < \varepsilon$ and let $x \in K$. Let $\alpha > 0$ be such that $\alpha + 4\varepsilon < \mathrm{wfs}(K)$ and denote by $i\colon \tilde{K}^{\alpha+\varepsilon} \to \tilde{K}^{\alpha+3\varepsilon}$ the canonical inclusion map and by i_* the induced map between homotopy or homology groups. For any $0 < \lambda < \mathrm{wfs}(K)$ one has*

$$\mathrm{H_k}(\mathrm{K}^\lambda, \mathbf{Z}/2) = \mathrm{im}(i_*\colon \mathrm{H_k}(\tilde{\mathrm{K}}^{\alpha+\varepsilon}, \mathbf{Z}/2) \to \mathrm{H_k}(\tilde{\mathrm{K}}^{\alpha+3\varepsilon}, \mathbf{Z}/2)),$$
$$\pi_1(K^\lambda, x) = \mathrm{im}(i_*\colon \pi_1(\tilde{K}^{\alpha+\varepsilon}, x) \to \pi_1(\tilde{K}^{\alpha+3\varepsilon}, x)).$$

The part of this result about homology has been independently proven in [12] using topological persistence theory.

Proof. First note that it follows from isotopy Proposition 3.4, that for any $0 < \lambda,\ \mu < \mathrm{wfs}(K)$, the two thickenings K^λ and K^μ are isotopic. It is thus sufficient to prove the theorem for $\lambda = \alpha$. Now the proof follows, in the same way as in previous proofs, from the following inclusions:

$$K^\alpha \subset \tilde{K}^{\alpha+\varepsilon} \subset K^{\alpha+2\varepsilon} \subset \tilde{K}^{\alpha+3\varepsilon} \subset K^{\alpha+4\varepsilon}$$

and the fact that K^α is a deformation retract of $K^{\alpha+2\varepsilon}$ which is itself a deformation retract of $K^{\alpha+4\varepsilon}$. □

Such a result combined with results on homological persistence [15] and alpha-shapes [16] leads to an algorithm to compute the homology groups of K from a noisy sample of points (see Section 5 below). A case of particular interest is when K is the boundary of an open set. It is important to notice that even if $\mathrm{wfs}(K) > 0$, the homology groups of K and the homology groups of its thickenings are not always the same: consider the following example (see also 2.4.8 of [26]). Let $K \subset \mathbb{R}^2$ be the union of the four sets $K_1 = \{(x, y)\colon x = 0, -2 \le y \le 1\}$, $K_2 = \{(x, y)\colon 0 \le x \le 1, y = -2\}$, $K_3 = \{(x, y)\colon x = 1, -2 \le y \le 0\}$ and $K_4 = \{(x, y)\colon 0 < x \le 1, y = \sin(2\pi/x)\}$ (see Fig. 4). It is an easy exercise to check that K is a simply connected compact set with positive wfs, while the thickenings of K are homeomorphic to annuli and that K is the boundary of a topological disk [26, 2.4.8].

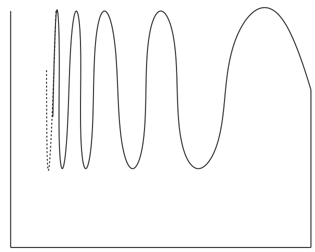

Fig. 4. A compact set with positive wfs whose homology groups differ from the ones of its thickenings.

5. Applications

Let us consider a bounded open set $\mathcal{O}$ such that $\mathcal{O}$ and $(\mathcal{O}^c)^\circ$ have positive wfs.

The results of Section 4 combined with algorithms of [15] for the computation of persistent homology allow us to compute the homology groups of $\mathcal{O}$ as well as the homology groups of a thickening of the boundary of $\mathcal{O}$, given a noisy set of points sampled on the boundary of $\mathcal{O}$. The main interest of the algorithm we provide is twofold. First, no assumption on the smoothness of the boundary of $\mathcal{O}$ is needed. Second, noisy samples are allowed, i.e. it is only required that the Hausdorff distance between $\partial\mathcal{O}$ and the sample is bounded by some constant depending on $\text{wfs}(\mathcal{O} \cup (\mathcal{O}^c)^\circ)$. We use the following notion of noisy sample that presents some similarities with the one introduced in [14]. Let $\mathcal{O}$ be a bounded open subset of $\mathbb{R}^n$ whose boundary is denoted by

$$S = \partial\mathcal{O} = \mathcal{O}^c \cap \overline{\mathcal{O}}.$$

Definition 5.1. A finite sample of points $\mathcal{E}$ such that the Hausdorff distance between S and $\mathcal{E}$ is less than ε is called an *ε-noisy sample* of S.

Homology through the Voronoi Filtration $\mathcal{M}_\lambda$. We now consider an ε-noisy sample $\mathcal{E}$ of S. We consider a ball of radius R, with R large enough for $\mathbb{B}^\circ_{0,R/2}$ to contain $\mathcal{E}$ and all the Voronoi vertices of the Voronoi diagram of $\mathcal{E}$. From the Voronoi diagram of $\mathcal{E}$, it is possible to compute the filtration of the $\mathcal{M}_\lambda(\tilde{\mathcal{O}})$ where $\tilde{\mathcal{O}} = \mathbb{B}^\circ_{0,R}\backslash\mathcal{E}$. If one assumes $\text{wfs}(\mathbb{B}^\circ_{0,R}\backslash S) > 4\varepsilon$, Theorem 4.4 can be applied to the open sets $\mathcal{O} = \mathbb{B}^\circ_{0,R}\backslash S$ and $\tilde{\mathcal{O}}$.

The Delaunay filtration dual to the Voronoi filtration $\mathcal{M}_\lambda$ is simplicial. It is then possible to use techniques described in [15] and [27] on the filtration corresponding to the $\mathcal{M}_\lambda(\tilde{\mathcal{O}})$ when λ decreases from 3ε to ε to compute the homology of $i_*(\text{H}_\text{k}(\mathcal{M}_{3\varepsilon}(\tilde{\mathcal{O}}), \mathbf{Z}/2))$ and therefore, by Theorem 4.4, of $\text{H}_\text{k}(\mathbb{B}^\circ_{0,\text{R}}\backslash \text{S}, \mathbf{Z}/2)$.

Notice that S^c has exactly one unbounded connected component, which can be identified in the filtration $\mathcal{M}_\lambda(\tilde{\mathcal{O}})$. Therefore, if one knows that $\mathcal{O}^c$ has only one connected component (which means that $\mathcal{O}$ has no voids), then the homology of $\mathbb{B}^\circ_{0,R}\backslash S$ gives the homology of $\mathcal{O}$.

Homology through α-Shape Filtration. Consider, in Theorem 4.5, the case where $\tilde{K}$ is a finite set. $\tilde{K}^r$ is then a union of balls, which is known to have the homotopy type of the *dual complex* or *α-complex* of $\tilde{K}$. One can use precisely the filtration mentioned in [15] on the α-complex and the associated algorithm for the computation of persistent homology. Then, according to Theorem 4.5, counting the cycles classes that "survive" between the "times" ε and 3ε gives the Betti numbers of K^λ.

Finiteness of Homotopy Types. Theorem 3.3 also leads to a homotopy finiteness theorem for bounded open sets with positive wfs.

Theorem 5.2 (Homotopy Finiteness). *Given an integer* $n \geq 1$ *and two positive reals* $\varepsilon > 0$ *and* $D > 0$, *there are at most finitely many homotopy types among bounded open sets* $\mathcal{O}$ *in* $\mathbb{R}^n$ *satisfying* $\mathrm{wfs}(\mathcal{O}) > \varepsilon$ *and diameter*$(\mathcal{O}) < D$. *As a consequence*, *the number of homotopy types among bounded open sets in* $\mathbb{R}^n$ *with positive wfs is countable*.

This theorem shows that the geometry of a bounded open set in $\mathbb{R}^n$ involves constraints on its topology. This is the same kind of result as the ones known for Riemannian manifolds (see [20], [21] and [9]).

Proof. Since one considers open sets with diameter less than D, one can suppose that they are all included in the cube $H = [-D, D]^n$. Let $h = \varepsilon/2\sqrt{n}$ and let

$$G_\varepsilon = \left\{(k_1 h, k_2 h, \ldots, k_n h) \colon \frac{-D}{h} \leq k_i \leq \frac{D}{h}\right\}$$

be the h-regular grid in H and let $\mathcal{O}$ be an open set included in H. Notice that $d_{\mathrm{H}}(H, G) = \varepsilon/4$. One associates to $\mathcal{O}$ a subset of G_ε defined by $G_{\mathcal{O}} = \{x \in G_\varepsilon \colon d(x, \mathcal{O}^c) < \varepsilon/4\}$. Since G_ε is finite, one thus defines a map between the open sets of H and the finite set of subsets of G_ε.

Now, if two open sets $\mathcal{O}$ and $\tilde{\mathcal{O}}$ included in H are such that $G_{\mathcal{O}} = G_{\tilde{\mathcal{O}}}$, then $d_{\mathrm{H}}(\mathcal{O}^{\mathrm{c}}, \tilde{\mathcal{O}}^{\mathrm{c}}) < \varepsilon/2$. Moreover, if $\mathrm{wfs}(\mathcal{O}) > \varepsilon$ and $\mathrm{wfs}(\tilde{\mathcal{O}}) > \varepsilon$, it follows from Theorem 3.3 that $\mathcal{O}$ and $\tilde{\mathcal{O}}$ have the same homotopy type. □

Note that given n, ε and D, the number of different homotopy types for open subset of $\mathbb{R}^n$ with wfs greater than ε and diameter less than D is bounded by $2^{(4D\sqrt{n}/\varepsilon+1)^n}$. Such a bound is far from being optimal.

To conclude note that the previous theorem together with Proposition 3.6 has the following consequence in real analytic geometry.

Corollary 5.3. *The number of homotopy types among bounded subanalytic open sets in* $\mathbb{R}^n$ *with positive wfs is countable*.

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