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Alcoved Polytopes, I*

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Abstract. The aim of this paper is to study alcoved polytopes, which are polytopes arising from affine Coxeter arrangements. This class of convex polytopes includes many classical polytopes, for example, the hypersimplices. We compare two constructions of triangulations of hypersimplices due to Stanley and Sturmfels and explain them in terms of alcoved polytopes. We study triangulations of alcoved polytopes, the adjacency graphs of these triangulations, and give a combinatorial formula for volumes of these polytopes. In particular, we study a class of matroid polytopes, which we call the multi-hypersimplices.

1. Introduction

The affine Coxeter arrangement of an irreducible crystallographic root system $\Phi \subset V \simeq \mathbb{R}^r$ is obtained by taking all integer affine translations $H_{\alpha,k} = \{x \in V \mid (\alpha, x) = k\}$, $\alpha \in \Phi$, $k \in \mathbb{Z}$, of the hyperplanes perpendicular to the roots. The regions of the affine Coxeter arrangements are simplices called alcoves. They are in a one-to-one correspondence with elements of the associated affine Weyl group. We define an *alcoved polytope* $\mathcal{P}$ as a convex polytope that is the union of several alcoves. In other words, an alcoved polytope is the intersection of some half-spaces bounded by the hyperplanes $H_{\alpha,k}$:

$$\mathcal{P} = \{x \in V \mid b_\alpha \le (\alpha, x) \le c_\alpha,\ \alpha \in \Phi\},$$

where b_α and c_α are some integer parameters. These polytopes come naturally equipped with coherent triangulations into alcoves. Alcoved polytopes include many interesting classes of polytopes: hypersimplices, order polytopes, some special matroid polytopes, Fomin–Zelevinsky's generalized associahedra, and many others. This is the first of two papers about alcoved polytopes. In this paper we concentrate on alcoved polytopes of the

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Lie type A case and on related combinatorial objects. In [LP1] we will treat the general case of an arbitrary root system.

Hypersimplices are integer polytopes which appear in algebraic and geometric contexts. For example, they are moment polytopes for torus actions on Grassmannians. They are also weight polytopes of the fundamental representations of the general linear group GL_n. The (k, n)th hypersimplex can be defined as the slice of the hypercube $[0, 1]^{n-1}$ located between the two hyperplanes $\sum x_i = k - 1$ and $\sum x_i = k$. It is well know that the normalized volume of this hypersimplex equals the Eulerian number $A_{k,n-1}$, i.e., the number of permutations of size $n - 1$ with $k - 1$ descents. Stanley [Sta1] explained this fact by constructing a triangulation of the hypersimplex into $A_{k,n-1}$ unit simplices. Another construction of a triangulation of the hypersimplex was given by Sturmfels [Stu]. It naturally appears in the context of Gröbner bases. These two constructions of triangulations are quite different.

In Section 2 we compare these triangulations and show that they are actually identical to each other and that they can be naturally described in terms of alcoved polytopes.[1] In Section 3 we extend the descriptions of this triangulation to general alcoved polytopes, and give a formula for the volume of an alcoved polytope. In Sections 4–6 we study in detail three examples of alcoved polytopes: the matroid polytopes, the second hypersimplex, and the multi-hypersimplices.

In the second part [LP1] of this paper, we will extend the hypersimplices to all Lie types and calculate their volumes. We will prove a general theorem on volumes of alcoved polytopes. We will give uniform generalizations of the descent and major index statistics, appropriate for our geometric approach.

Some alcoved polytopes have also been studied from a more algebraic perspective earlier; see [KKMS] and [BGT].

2. Four Triangulations of the Hypersimplex

Let us fix integers $0 < k < n$. Let $[n] := \{1, \dots, n\}$ and $\binom{[n]}{k}$ denote the collection of k-element subsets of $[n]$. To each k-subset $I \in \binom{[n]}{k}$ we associate the 01-vector $\varepsilon_I = (\varepsilon_1, \dots, \varepsilon_n)$ such that $\varepsilon_i = 1$, for $i \in I$; and $\varepsilon_i = 0$, for $i \notin I$.

The *hypersimplex* $\Delta_{k,n} \subset \mathbb{R}^n$ is the convex polytope defined as the convex hull of the points ε_I, for $I \in \binom{[n]}{k}$. All these $\binom{n}{k}$ points are actually vertices of the hypersimplex because they are obtained from each other by permutations of the coordinates. This $(n-1)$-dimensional polytope can also be defined as

$$\Delta_{k,n} = \{(x_1, \dots, x_n) \mid 0 \le x_1, \dots, x_n \le 1;\ x_1 + \cdots + x_n = k\}.$$

The hypersimplex is linearly equivalent to the polytope $\tilde{\Delta}_{k,n} \subset \mathbb{R}^{n-1}$ given by

$$\tilde{\Delta}_{k,n} = \{(x_1, \dots, x_{n-1}) \mid 0 \le x_1, \dots, x_{n-1} \le 1;\ k - 1 \le x_1 + \cdots + x_{n-1} \le k\}.$$

Indeed, the projection $p : (x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1})$ sends $\Delta_{k,n}$ to $\tilde{\Delta}_{k,n}$. The

[1] Haase [Ha] reported to us that he also discovered this equivalence (unpublished). Ziegler [Z] reported that the alcove triangulation of the hypersimplex appeared as an example in his 1997/98 class on triangulations.

hypersimplex $\tilde{\Delta}_{k,n}$ can be thought of as the region (slice) of the unit hypercube $[0,1]^{n-1}$ contained between the two hyperplanes $\sum x_i = k-1$ and $\sum x_i = k$.

Recall that a *descent* in a permutation $w \in S_n$ is an index $i \in \{1, \ldots, n-1\}$ such that $w(i) > w(i+1)$. Let $\operatorname{des}(w)$ denote the number of descents in w. The *Eulerian number* $A_{k,n}$ is the number of permutations in S_n with $\operatorname{des}(w) = k-1$ descents.

We normalize the volume form in $\mathbb{R}^{n-1}$ so that the volume of a unit simplex is 1 and, thus, the volume of a unit hypercube is $(n-1)!$. It is a classical result, implicit in the work of Laplace [L, p. 257ff], that the normalized volume of the hypersimplex $\Delta_{k,n}$ equals the Eulerian number $A_{k,n-1}$. One would like to present a triangulation of $\Delta_{k,n}$ into $A_{k,n-1}$ unit simplices. Such a triangulation into unit simplices is called a *unimodular triangulation*.

In this section we define four triangulations of the hypersimplex $\Delta_{k,n}$. One triangulation is due to Stanley [Sta1], one is due to Sturmfels [Stu], one arises from the affine Coxeter arrangement of type A, and the final one, which is new, we call the circuit triangulation. The main result of this section, Theorem 2.7, says that these four triangulations coincide. In addition, we describe the dual graphs of these triangulations.

2.1. *Stanley's Triangulation*

The hypercube $[0,1]^{n-1} \subset \mathbb{R}^{n-1}$ can be triangulated into $(n-1)$-dimensional unit simplices ∇_w labeled by permutations $w \in S_{n-1}$ given by

$$\nabla_w = \{(y_1, \ldots, y_{n-1}) \in [0,1]^{n-1} \mid 0 < y_{w(1)} < y_{w(2)} < \cdots < y_{w(n-1)} < 1\}.$$

Stanley [Sta1] defined a transformation of the hypercube $\psi : [0,1]^{n-1} \to [0,1]^{n-1}$ by $\psi(x_1, \ldots, x_{n-1}) = (y_1, \ldots, y_{n-1})$, where

$$y_i = (x_1 + x_2 + \cdots + x_i) - \lfloor x_1 + x_2 + \cdots + x_i \rfloor.$$

The notation $\lfloor x \rfloor$ denotes the integer part of x. The map ψ is piecewise-linear, bijective on the hypercube (except for a subset of measure zero), and volume preserving.

Since the inverse map ψ^{-1} is linear and injective when restricted to the open simplices ∇_w, it transforms the triangulation of the hypercube given by ∇_w's into another triangulation.

Theorem 2.1 [Sta1]. *The collection of simplices $\psi^{-1}(\nabla_w)$, $w \in S_{n-1}$, gives a triangulation of the hypercube $[0,1]^{n-1}$ compatible with the subdivision of the hypercube into hypersimplices. The collection of the simplices $\psi^{-1}(\nabla_w)$, where w^{-1} varies over permutations in S_{n-1} with $k-1$ descents, gives a triangulation of the kth hypersimplex $\tilde{\Delta}_{k,n}$. Thus the normalized volume of $\tilde{\Delta}_{k,n}$ equals the Eulerian number $A_{k,n-1}$.*

Proof. Let $\psi(x_1, \ldots, x_{n-1}) = (y_1, \ldots, y_{n-1}) \in \nabla_w$. For $i = 1, \ldots, n-2$, we have

$$\lfloor x_1 + \cdots + x_{i+1} \rfloor = \begin{cases} \lfloor x_1 + \cdots + x_i \rfloor & \text{if} \quad y_i < y_{i+1}, \\ \lfloor x_1 + \cdots + x_i \rfloor + 1 & \text{if} \quad y_i > y_{i+1}. \end{cases}$$

Thus $\lfloor x_1 + \cdots + x_{n-1} \rfloor = \mathrm{des}(w^{-1})$. In other words, if $\mathrm{des}(w^{-1}) = k - 1$, then $k - 1 \le x_1 + \cdots + x_{n-1} \le k$, i.e., $(x_1, \ldots, x_{n-1}) \in \tilde{\Delta}_{k,n}$. $\square$

2.2. *Sturmfels' Triangulation*

Let S be a multiset of elements from $[n]$. We define $\mathrm{sort}(S)$ to be the unique non-decreasing sequence obtained by ordering the elements of S. Let I and J be two k-element subsets of $[n]$, and let $\mathrm{sort}(I \cup J) = (a_1, a_2, \ldots, a_{2k})$. Then we set $U(I, J) = \{a_1, a_3, \ldots, a_{2k-1}\}$ and $V(I, J) = \{a_2, a_4, \ldots, a_{2k}\}$. For example, for $I = \{1, 2, 3, 5\}$, $J = \{2, 4, 5, 6\}$, we have $\mathrm{sort}(I \cup J) = (1, 2, 2, 3, 4, 5, 5, 6)$, $U(I, J) = \{1, 2, 4, 5\}$, and $V(I, J) = \{2, 3, 5, 6\}$.

We say that an ordered pair (I, J) is *sorted* if $I = U(I, J)$ and $J = V(I, J)$. We call an ordered collection $\mathcal{I} = (I_1, \ldots, I_r)$ of k-subsets of $[n]$ *sorted* if (I_i, I_j) is sorted for every $1 \le i < j \le r$. Equivalently, if $I_l = \{I_{l1} < \cdots < I_{lk}\}$, for $l = 1, \ldots, r$, then $\mathcal{I}$ is sorted if and only if $I_{11} \le I_{21} \le \cdots \le I_{r1} \le I_{12} \le I_{22} \le \cdots \le I_{rk}$. For such a collection $\mathcal{I}$, let $\nabla_{\mathcal{I}}$ denote the $(r-1)$-dimensional simplex with the vertices $\varepsilon_{I_1}, \ldots, \varepsilon_{I_r}$.

Theorem 2.2 [Stu]. *The collection of simplices $\nabla_{\mathcal{I}}$, where $\mathcal{I}$ varies over all sorted collections of k-element subsets in $[n]$, is a simplicial complex that forms a triangulation of the hypersimplex $\Delta_{k,n}$.*

It follows that the maximal by inclusion sorted collections, which correspond to the maximal simplices in the triangulation, all have the same size $r = n$.

Corollary 2.3. *The normalized volume of the hypersimplex $\Delta_{k,n}$ is equal to the number of maximal sorted collections of k-subsets in $[n]$.*

This triangulation naturally appears in the context of Gröbner bases. Let $k[x_I]$ be the polynomial ring in the $\binom{n}{k}$ variables x_I labeled by k-subsets $I \in \binom{[n]}{k}$. Define the map φ: $k[x_I] \to k[t_1, t_2, \ldots, t_n]$ by $x_I \mapsto t_{i_1} t_{i_2} \cdots t_{i_k}$, for $I = \{i_1, \ldots, i_k\}$. The kernel of this map is an ideal in $k[x_I]$ that we denote by $\mathcal{J}_{k,n}$. Recall that a sufficiently generic height function on the vertices ε_I of the hypersimplex $\Delta_{k,n}$ induces a term order on monomials in $k[x_I]$ and defines a Gröbner basis for the ideal $\mathcal{J}_{k,n}$. On the other hand, such a height function gives a coherent triangulation of $\Delta_{k,n}$. This gives a correspondences between Gröbner bases and coherent triangulations. The initial ideal associated with a Gröbner basis is square-free if and only if the corresponding triangulation is unimodular. For more details on Gröbner bases, see the Appendix.

Theorem 2.4 [Stu]. *The marked set of quadratic binomials*

$$\mathcal{G}_{k,n} = \left\{ \underline{x_I x_J} - x_{U(I,J)} x_{V(I,J)} \mid I, J \in \binom{[n]}{k} \right\}$$

is a Gröbner basis for $\mathcal{J}_{k,n}$ under some term order on $k[x_I]$ such that the underlined term is the initial monomial. The simplices of the corresponding triangulation are $\nabla_{\mathcal{I}}$, *where $\mathcal{I}$ varies over sorted collections of k-subsets of* $[n]$*. Moreover, this triangulation is unimodular.*

In Section 4.1 we state and prove a more general statement.

2.3. *Alcove Triangulation*

The *affine Coxeter arrangement* of type A_{n-1} is the arrangement of hyperplanes in $\mathbb{R}^{n-1}$ given by

$$H_{ij}^{l} = \{(z_1, \ldots, z_{n-1}) \in \mathbb{R}^{n-1} \mid z_i - z_j = l\}, \qquad \text{for} \quad 0 \le i < j \le n-1, \quad l \in \mathbb{Z},$$

where we assume that $z_0 = 0$. It follows from the general theory of affine Weyl groups, see [Hu], that the hyperplanes H_{ij}^{l} subdivide $\mathbb{R}^{n-1}$ into unit simplices, called *alcoves*.

We say that a polytope $\mathcal{P}$ in $\mathbb{R}^{n-1}$ is *alcoved* if $\mathcal{P}$ is an intersection of some half-spaces bounded by the hyperplanes H_{ij}^{l}. In other words, an alcoved polytope is a polytope given by inequalities of the form $b_{ij} \le z_j - z_i \le c_{ij}$, for some collection of integer parameters b_{ij} and c_{ij}. We denote this alcoved polytope by $\mathcal{P}(b_{ij}, c_{ij})$. If the parameters satisfy $b_{ij} = c_{ij} - 1$, for all i, j, then the corresponding polytope consists of a single alcove (or is empty). Each alcoved polytope comes naturally equipped with the triangulation into alcoves. Conversely, if $\mathcal{P}$ is a convex polytope which is a union of alcoves, then $\mathcal{P}$ is an alcoved polytope.

Assume that $z_i = x_1 + \cdots + x_i$, for $i = 1, \ldots, n-1$. The hypersimplex $\tilde{\Delta}_{k,n}$ is given by the following inequalities in the z-coordinates:

$$0 \le z_1 - z_0, \ldots, z_{n-1} - z_{n-2} \le 1, \qquad k - 1 \le z_{n-1} - z_0 \le k. \tag{1}$$

Thus the hypersimplex is an alcoved polytope. We call its triangulation into alcoves the *alcove triangulation*.

2.4. *Circuit Triangulation*

Let $G_{k,n}$ be the directed graph on the vertices ε_I, $I \in \binom{[n]}{k}$, of the hypersimplex $\Delta_{k,n}$ defined as follows. We regard the indices i of a vector $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ as elements of $\mathbb{Z}/n\mathbb{Z}$. Thus we assume that $\varepsilon_{n+1} = \varepsilon_1$. We connect a vertex $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ with a vertex ε' by an edge $\varepsilon \xrightarrow{i} \varepsilon'$ labeled by $i \in [n]$ whenever $(\varepsilon_i, \varepsilon_{i+1}) = (1, 0)$ and the vector ε' is obtained from ε by switching ε_i and ε_{i+1}. In other words, each edge in the graph $G_{k,n}$ is given by cyclically shifting a "1" in vector ε one step to the right to the next adjacent place. It is possible to perform such a shift if and only if the next place is not occupied by another "1".

A circuit in the graph $G_{k,n}$ of minimal possible length is given by a sequence of shifts of "1"s so that the first "1" in ε moves to the position of the second "1", the second "1" moves to the position of the third "1", and so on, finally, the last "1" cyclically moves to the position of the first "1". The length of such a circuit is n. We call such circuits in

$G_{k,n}$ *minimal*. Here is an example of a minimal circuit in G_{26}:

$$\begin{array}{ccccc} (1,0,1,0,0,0) & \xrightarrow{3} & (1,0,0,1,0,0) & \xrightarrow{1} & (0,1,0,1,0,0) \\ \uparrow_6 & & & & \downarrow_2 \\ (0,0,1,0,0,1) & \xleftarrow{5} & (0,0,1,0,1,0) & \xleftarrow{4} & (0,0,1,1,0,0) \end{array}$$

The sequence of labels of edges in a minimal circuit forms a permutation $w = w_1 \cdots w_n \in S_n$. For example, the permutation corresponding to the above minimal circuit is $w = 312456$.

If we do not specify the initial vertex in a minimal circuit, then the permutation w is defined modulo cyclic shifts $w_1 \cdots w_n \sim w_n w_1 \cdots w_{n-1}$. By convention, we pick the representative w of the class of permutations modulo cyclic shifts such that $w_n = n$. This corresponds to picking the initial point in a minimal circuit with lexicographically maximal 01-vector ε. Indeed, if $\varepsilon \xrightarrow{i} \varepsilon'$ is an edge in $G_{k,n}$, then $\varepsilon > \varepsilon'$ in the lexicographic order, for $i = 1, \ldots, n-1$, and $\varepsilon < \varepsilon'$, for $i = n$.

Lemma 2.5. *A minimal circuit in the graph $G_{k,n}$ is uniquely determined by the permutation w modulo cyclic shifts. A permutation $w \in S_n$ such that $w_n = n$ corresponds to a minimal circuit in the graph $G_{k,n}$ if and only if the inverse permutation w^{-1} has exactly $k-1$ descents.*

Proof. The inverse permutation w^{-1} has a descent for each pair $(i < j)$ such that $b = w_i = w_j + 1$. Thus a "1" was moved from the bth position to the $(b+1)$th position in ε before a "1" was moved from the $(b-1)$th position to the bth position. (Here ε denotes the initial vertex of the corresponding circuit.) This happens if and only if $\varepsilon_b = 1$. Since ε has k "1"s, this happens exactly k times. However, the occurrence corresponding to $w^{-1}(n) = n > w^{-1}(1)$ is not counted as a descent, so w^{-1} has exactly $k-1$ descents. Conversely, if w^{-1} has $k-1$ descents, then we obtain a vector ε with k "1"s, so that w corresponds to a minimal circuit containing ε. $\square$

For a permutation, $w = w_1 \cdots w_n \in S_n$, let (w) denote the long cycle in S_n given by $(w) = (w_1, \ldots, w_n)$ in cycle notation. Two permutations $u, w \in S_n$ are equivalent modulo cyclic shifts if and only if $(u) = (w)$. The reader should not confuse *circuits* in the graph $G_{k,n}$ with *cycles* in the symmetric group S_n.

Let $C_{k,n}$ denote the set of long cycles $(w) = (w_1, \ldots, w_{n-1}, n) \in S_n$ such that w^{-1} has exactly $k-1$ descents. For $(w) \in C_{k,n}$, let $c_{(w)}$ be the corresponding minimal circuit in the graph $G_{k,n}$, whose edges are labeled by $w_1, \ldots, w_n$. Lemma 2.5 shows that that the map $(w) \mapsto c_{(w)}$ is one-to-one correspondence between the set of long cycles $C_{k,n}$ and the set of minimal circuits in $G_{k,n}$.

Each minimal circuit $c_{(w)}$ in $G_{k,n}$ determines the simplex $\Delta_{(w)}$ inside the hypersimplex $\Delta_{k,n}$ with the vertex set $c_{(w)}$.

Theorem 2.6. *The collection of simplices $\Delta_{(w)}$ corresponding to all minimal circuits in $G_{k,n}$ forms a triangulation of the hypersimplex $\Delta_{k,n}$.*

We call this triangulation of the hypersimplex the *circuit triangulation*.

Theorem 2.7. *The following four triangulations of the hypersimplex are identical*: *Stanley's triangulation*, *Sturmfels' triangulation*, *the alcove triangulation*, *and the circuit triangulation*.

We prove Theorems 2.6 and 2.7 together. Let $\Gamma_{k,n}$ be the collection of (maximal) simplices of the triangulation of Theorem 2.7.

Proof. The fact that Stanley's triangulation coincides with the alcove triangulation follows directly from the definitions. We leave this as an exercise for the reader.

We show that the simplices $\Delta_{(w)}$ are exactly those in Sturmfels' triangulation. An ordered pair of subsets $I = \{i_1 < \cdots < i_k\}$ and $J = \{j_1 < \cdots < j_k\}$ is sorted if and only if the interleaving condition $i_1 \leq j_1 \leq i_2 \leq j_2 \leq \cdots \leq j_k$ is satisfied. When two vertices ε_I and ε_J belong to the same minimal circuit, a "1" from ε_I is moved towards the right in ε_J but never past the original position of another "1" in ε_I. Thus the interleaving $i_a \leq j_a \leq i_{a+1}$ condition is satisfied, and similarly we obtain the other interleaving inequalities. Conversely, the interleaving condition implies that each sorted collection belongs to a minimal circuit in $G_{k,n}$.

We now show that the circuit triangulation coincides with Stanley's triangulation. Recall that the latter triangulation occurs in the space $\mathbb{R}^{n-1}$. To be more precise, in order to obtain Stanley's triangulation we need to apply the projection $p : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1})$ to the circuit triangulation. We identify a permutation $w = w_1 \cdots w_{n-1} \in S_{n-1}$ with $k-1$ descents with the permutation $w_1 \cdots w_{n-1} n \in S_n$.

We claim that the projected simplex $p(\Delta_{(w)})$ is exactly the simplex $\psi^{-1}(\nabla_w)$ in Stanley's triangulation. Indeed, the map $\psi^{-1} : (y_1, \ldots, y_{n-1}) \mapsto (x_1, \ldots, x_{n-1})$ restricted to the simplex $\nabla_w = \{0 < y_{w(1)} < \cdots < y_{w(n-1)} < 1\}$, is given by $x_1 = y_1$ and

$$x_{i+1} = \begin{cases} y_{i+1} - y_i & \text{if} \quad w^{-1}(i+1) > w^{-1}(i), \\ y_{i+1} - y_i + 1 & \text{if} \quad w^{-1}(i+1) < w^{-1}(i) \end{cases}$$

for $i = 1, \ldots, n-2$. The vertices of the simplex ∇_w are the points $v_0, \ldots, v_{n-1} \in \mathbb{R}^{n-1}$ such that $v_r = (y_1, \ldots, y_{n-1})$ is given by $y_{w(1)} = \cdots = y_{w(r)} = 0$ and $y_{w(r+1)} = \cdots = y_{w(n-1)} = 1$. The map ψ^{-1} sends the vertex $v_0 = (0, \ldots, 0)$ to the point $(x_1, \ldots, x_{n-1})$ such that $x_1 = 0$ and $x_{i+1} = 1$ if $w^{-1}(i+1) < w^{-1}(i)$ and $x_{i+1} = 0$ if $w^{-1}(i+1) > w^{-1}(i)$, for $i = 1, \ldots, n-2$. The vertex v_r is obtained from v_{r-1} by changing $y_{w(n-r)}$ from 0 to 1. Thus $\psi^{-1}(v_r)$ differs from $\psi^{-1}(v_{r-1})$ exactly in the coordinates $x_{w(n-r)}$ and $x_{w(n-r)+1}$. Here $x_n = k - (x_1 + \cdots + x_{n-1})$. In fact, going from $\psi^{-1}(v_{r-1})$ to $\psi^{-1}(v_r)$ we move a "1" from $x_{w(n-r)+1}$ to $x_{w(n-r)}$. Finally, moving from v_{n-1} to v_0 we are changing x_1 from 1 to 0. Thus as we go from $\psi^{-1}(v_r)$ to $\psi^{-1}(v_{r+1})$ we are traveling along the edges of the graph $G_{k,n}$ in the reverse direction. So the vertices $\psi^{-1}(v_r)$ of the simplex $\psi^{-1}(\nabla_w)$ are exactly the vertices of $p(\Delta_{(w)})$. This completes the proof of the theorem. □

Remark 2.8. An explicit bijection θ between maximal sorted collections of k-subsets of $[n]$ and permutations $w \in S_n$ with $k-1$ descents satisfying $w_n = n$ can be constructed as follows. Let $\mathcal{I} = (I_1, \ldots, I_n)$ be such a collection. Every number in $[n]$ must occur in $\bigcup_i I_i$. Set $(a_1, \ldots, a_{kn}) = \mathrm{sort}(\bigcup_i I_i)$. Let α_k be such that $a_{\alpha_k} = k$ and $a_{\alpha_k + 1} = k + 1$.

Then $\theta(\mathcal{I}) = w_1 w_2 \cdots w_{n-1} n$, where $w_i \equiv \alpha_i \pmod{n-1}$ with representatives taken from $[n-1]$. This bijection is compatible with the correspondences in Theorem 2.7. For example, $\alpha_i \bmod n-1$ tells us when a "1" is moved from ε_i to ε_{i+1} where by convention a "1" is moved from ε_n to ε_1 in the last edge of a circuit.

2.5. *Adjacency of Maximal Simplices in the Hypersimplex*

We say that two simplices in a triangulation are *adjacent* if they share a common facet. We describe the adjacent simplices in the triangulation $\Gamma_{k,n}$, using first the construction of the circuit triangulation.

Theorem 2.9. *Two simplices $\Delta_{(u)}$ and $\Delta_{(w)}$ of $\Gamma_{k,n}$ are adjacent if and only if there exists $i = 1, \ldots, n$ such that $u_i - u_{i+1} \neq \pm 1 \pmod{n}$ and the cycle (w) is obtained from (u) by switching u_i with u_{i+1}, i.e., $(w) = (u_i, u_{i+1})(u)(u_i, u_{i+1})$. Here again we assume that $u_{n+1} = u_1$.*

Proof. The two simplices $\Delta_{(u)}$ and $\Delta_{(w)}$ are adjacent if and only if exactly one pair of their vertices differ. This means that the corresponding minimal circuits $c_{(u)}$ and $c_{(w)}$ differ in exactly one place. Let $\varepsilon' \xrightarrow{u_i} \varepsilon \xrightarrow{u_{i+1}} \varepsilon''$ be three vertices in order along the minimal cycle $c_{(u)}$. Then we can obtain another cycle $c_{(w)}$ from $c_{(u)}$ by changing only ε if and only if $u_i - u_{i+1} \neq \pm 1 \pmod{n}$ so that $\varepsilon' \xrightarrow{u_{i+1}} \varepsilon^* \xrightarrow{u_i} \varepsilon''$ are valid edges. When $u_i - u_{i+1} = \pm 1 \pmod{n}$ we are either moving the same "1" twice or moving two adjacent "1"s one after another. In both cases, the order of the shifts cannot be reversed, and so ε cannot be replaced by another vertex. $\square$

Alternatively, let $\mathcal{I} = (I_1, \ldots, I_n)$ be a sorted subset corresponding to the maximal simplex $\nabla_{\mathcal{I}}$ of $\Gamma_{k,n}$. Let $t \in [n]$ and $I_t = \{i_1, i_2, \ldots, i_k\}$. Then we can replace I_t in $\mathcal{I}$ by another $I_t' \in \binom{[n]}{k}$ to obtain an adjacent maximal simplex $\nabla_{\mathcal{I}'}$ if and only if the following holds. We must have $I_t' = \{i_1, \ldots, i_a', \ldots, i_b', \ldots, i_k\}$ for some $a \neq b \in [n]$ and $i_a' \neq i_b'$, satisfying $i_a - i_a' = i_b' - i_b = \pm 1 \pmod{n}$ and also both k-subsets $\{i_1, \ldots, i_a', \ldots, i_b, \ldots, i_k\}$ and $\{i_1, \ldots, i_a, \ldots, i_b', \ldots i_k\}$ must lie in $\mathcal{I}$. For example, we may replace $\{1, 3, 5, 8\}$ by $\{1, 2, 6, 8\}$ if and only if both $\{1, 2, 5, 8\}$ and $\{1, 3, 6, 8\}$ lie in $\mathcal{I}$.

We can give $\Gamma_{k,n}$ the structure of a graph by letting the simplices be the vertices and letting an edge join two simplices if the two simplices are adjacent. Figures 1 and 2 give examples of these graphs which we also denote as $\Gamma_{k,n}$.

We investigate degrees of vertices in these graphs for $k = 2$ in Section 5.2.

3. Triangulations and Volumes of Alcoved Polytopes

In this section we generalize (Theorem 3.1) the triangulations of Section 2 to all alcoved polytopes (see Section 2.3). As a consequence of these descriptions of the alcove triangulation, we obtain a curious formula (Theorem 3.2) expressing the volume of an alcoved

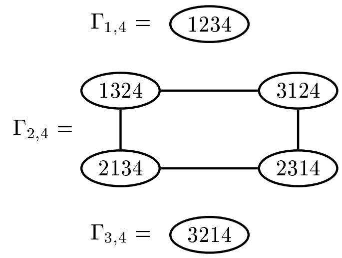

Fig. 1. The graphs of the triangulations of $\Delta_{1,4}$, $\Delta_{2,4}$, and $\Delta_{3,4}$.

polytope as a sum of the number of lattice points in certain other alcoved polytopes. This formula has a root system theoretic explanation which we will give in [LP1]. Finally, we explain a construction of the dual graph of the alcove triangulation, which we call the *alcove lattice*.

3.1. *Triangulations of Alcoved Polytopes*

Let $\mathcal{P} = \mathcal{P}(b_{ij}, c_{ij}) \subset \mathbb{R}^n$ be an alcoved polytope which we realize in the x-coordinates. In other words, $\mathcal{P}$ is an $(n-1)$-dimensional polytope lying in a hyperplane $x_1 + x_2 + \cdots + x_n = k$ for some $k \in \mathbb{Z}$; and given by the inequalities $b_{ij} \leq x_{i+1} + \cdots + x_j \leq c_{ij}$ for each pair (i, j) satisfying $0 \leq i < j \leq n-1$. By translating $\mathcal{P}$ by $(m, \ldots, m)$ for some $m \in \mathbb{Z}$ to obtain an affinely equivalent polytope, we can assume that all the coordinates of the points of $\mathcal{P}$ are non-negative. Let $Z_\mathcal{P} = \mathcal{P} \cap \mathbb{Z}^n \subset \mathbb{N}^n$ denote the set of integer points lying inside $\mathcal{P}$.

Let $G_\mathcal{P}$ be the directed graph defined as follows in analogy with $G_{k,n}$ in Section 2.4. The graph $G_\mathcal{P}$ has vertices labeled by points $a \in Z_\mathcal{P}$. Two vertices $a, b \in Z_\mathcal{P}$ are connected by an edge $a \to b$ labeled i if there exists an index $i \in [1, n]$ such that $a + e_{i+1} - e_i = b$, where e_i, e_{i+1} are the coordinate vectors and $e_{n+1} := e_1$. Let $C_\mathcal{P}$ denote the set of minimal circuits of $G_\mathcal{P}$, which have length n.

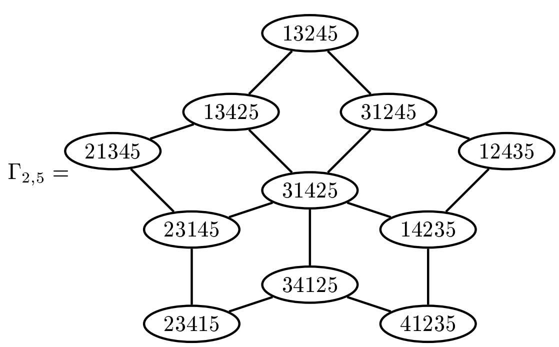

Fig. 2. Graph of the triangulation of the hypersimplex $\Delta_{2,5}$.

For an integer vector $a = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$ with non-negative coordinates lying on $x_1 + x_2 + \cdots + x_n = k$, we let I_a denote the multiset of size k of $\{1, 2, \ldots, n\}$ with a_1 "1"s, a_2 "2"s and so on. If I, J are multisets of size k with elements from $\{1, 2, \ldots, n\}$ then we can define $U(I, J)$ and $V(I, J)$ by sorting $I \cup J$ as in Section 2.2. Similarly, we define the notions of sorted and sort-closed for collections of multisets. In the following theorem note that $Z_{\mathcal{P}}$ and $G_{\mathcal{P}}$ are defined without needing $\mathcal{P}$ to be alcoved.

The proof of the following theorem is exactly analogous to the arguments of Theorem 2.7.

Theorem 3.1. *Let $\mathcal{P} \subset \mathbb{R}^n$ be an $(n-1)$-dimensional polytope lying in $x_1 + x_2 + \cdots + x_n = k$ so that all points of $\mathcal{P}$ have non-negative coordinates. Then the following are equivalent*:

(1) *$\mathcal{P}$ is an alcoved polytope.*
(2) *The set $\mathcal{I} = \{I_a \mid a \in Z_{\mathcal{P}}\}$ is sort-closed. A triangulation of $\mathcal{P}$ consists of the maximal simplices with vertices $\{a_1, a_2, \ldots, a_n\}$ for each sorted collection $(I_{a_1}, I_{a_2}, \ldots, I_{a_n})$ where $I_{a_k} \in \mathcal{I}$.*
(3) *The set $C_{\mathcal{P}}$ of minimal circuits of $G_{\mathcal{P}}$ gives rise to a triangulation of $\mathcal{P}$: if $C = (c^{(1)}, c^{(2)}, \ldots, c^{(n)}) \in C_{\mathcal{P}}$ then a maximal simplex is given by* $\operatorname{conv}(c^{(1)}, c^{(2)}, \ldots, c^{(n)})$.

When these conditions hold, all three above triangulations agree.

An alcoved polytope also gives rise to a Gröbner basis $\mathcal{G}_{\mathcal{P}}$ of the associated toric ideal $\mathcal{J}_{\mathcal{P}}$. The reader will be able to write it down following Section 4.1.

We identify $w \in S_{n-1}$ with $w_1 w_2 \cdots w_{n-1} n \in S_n$ as usual. Recall that $\Delta_{(w)}$ denotes the simplex $(\psi \circ p)^{-1}(\nabla_w)$ where we view p as a map from $\{x \in \mathbb{R}^n \mid x_1 + \cdots + x_n = k\}$ to $\mathbb{R}^{n-1}$. For an alcoved polytope $\mathcal{P}$, define the polytopes $\mathcal{P}_{(w)}$ by

$$\mathcal{P}_{(w)} = \{x \in \mathbb{R}^n \mid (\Delta_{(w)} + x) \subset \mathcal{P}\}.$$

Denote by $I(\mathcal{P})$ the number $|\mathbb{Z}^n \cap \mathcal{P}| = \#Z_{\mathcal{P}}$ of lattice points in $\mathcal{P}$.

Theorem 3.2. *Each of the polytopes $\mathcal{P}_{(w)}$ is an alcoved polytope. The normalized volume of $\mathcal{P}$ is given by*

$$\operatorname{Vol}(\mathcal{P}) = \sum_w I(\mathcal{P}_{(w)}),$$

where the sum is over all permutations $w \in S_{n-1}$.

Proof. The alcoved triangulation is obtained by copying and translating the triangulations of Theorem 2.7 by an integer vector so that the corresponding simplices cover the polytope $\mathcal{P}$. For example, if $(c^{(1)}, c^{(2)}, \ldots, c^{(n)}) \in C_{\mathcal{P}}$ then there exists $c \in \mathbb{Z}^n$ so that $c^{(i)} - c$ is a 0-1 vector for each i. Thus $(c^{(1)} - c, c^{(2)} - c, \ldots, c^{(n)} - c)$ is a minimal circuit in some $G_{k,n}$ corresponding to some simplex $\Delta_{(w)}$; see Section 2.4. The simplex $\Delta_{(w)} + c = \operatorname{conv}(c^{(1)}, c^{(2)}, \ldots, c^{(n)})$ corresponds to the circuit $(c^{(1)}, c^{(2)}, \ldots, c^{(n)})$. This proves the second statement of the theorem.

Let $\mathcal{P}$ be given by the inequalities $b_{ij} \le x_{i+1} + \cdots + x_j \le c_{ij}$ within the hyperplane $x_1 + x_2 + \cdots + x_n = l$. We check that $\mathcal{P}_{(w)}$ is an alcoved polytope. In fact this follows from the fact that $\Delta_{(w)}$ is itself an alcoved polytope and is given by some inequalities $d_{ij} \le x_{i+1} + \cdots + x_j \le f_{ij}$ and a hyperplane $x_1 + x_2 + \cdots + x_n = k$, where we pick d_{ij} and f_{ij} so that all equalities are achieved by some point in $\Delta_{(w)}$. Then $\mathcal{P}_{(w)}$ is the intersection of the inequalities $b_{ij} - d_{ij} \le x_{i+1} + \cdots + x_j \le c_{ij} - f_{ij}$ with the hyperplane $x_1 + x_2 + \cdots + x_n = l - k$, which by definition is an alcoved polytope. $\square$

For the hypersimplex, all the polytopes $\mathcal{P}_{(w)}$ are either empty or a single point.

To conclude this section we give one further interpretation of the volumes of alcoved polytopes in terms of maps of the circle with marked points. Let S^1 be the unit circle and let $S^1_{(n)}$ denote a circle with n distinct marked points $p_0, p_1, \ldots, p_{n-1}$ arranged in clockwise order. Let $\mathcal{P}$ be an alcoved polytope with parameters b_{ij} and c_{ij} as in Section 3.1. Let $M_{\mathcal{P}}$ denote the set of homotopy classes of continuous maps $f\colon S^1_{(n)} \to S^1$ satisfying:

- The map f is always locally bijective and locally orientation preserving. Informally, this means that f traces out S^1 in the clockwise direction and never stops.
- The images of marked points are distinct.
- For each $0 \le i < j \le n-1$, the number d of pre-images of $f(p_i)$ under f in the open interval (p_i, p_j) satisfies $b_{ij} \le d < c_{ij}$.

Two maps f and g belong to the same homotopy class if and only if they can be deformed into one another by a homotopy, in the usual sense, while always satisfying the conditions above. The following proposition follows from the preceding discussion.

Proposition 3.3. *Let $\mathcal{P}$ be an alcoved polytope. Then the simplices in the triangulation $\Gamma_{\mathcal{P}}$ are in bijection with the elements of $M_{\mathcal{P}}$.*

3.2. *Alcove Lattice and Alcoved Polytopes*

Define the *alcove lattice* Λ_n as the infinite graph whose vertices correspond to alcoves (i.e., regions of type A_{n-1} affine Coxeter arrangement) and edges correspond to pairs of adjacent alcoves. For example, Λ_3 is the infinite hexagonal lattice. For an alcoved polytope $\mathcal{P}$, define its graph $\Gamma_{\mathcal{P}}$ as the finite subgraph of Λ_n formed by alcoves in $\mathcal{P}$. For the graphs of Section 2.4, we have $\Gamma_{k,n} = \Gamma_{\Delta_{k,n}}$.

According to Section 14 of [LP2], we have the following combinatorial construction of the lattice Λ_n. Let $[\lambda_0, \ldots, \lambda_{n-1}]$ denote an element of $\mathbb{Z}^n/(1, \ldots, 1)\mathbb{Z}$. In other words, we assume that $[\lambda_0, \ldots, \lambda_{n-1}] = [\lambda'_0, \ldots, \lambda'_{n-1}]$ whenever the λ'_i are obtained from the λ_i by adding the same integer. The vertices of Λ_n can be identified with the following subset of $\mathbb{Z}^n/(1, \ldots, 1)\mathbb{Z}$, see [LP2]:

$$\Lambda_n = \{[\lambda_0, \ldots, \lambda_{n-1}] \mid \text{theintegers} \lambda_0, \ldots, \lambda_{n-1} \text{havedifferentresiduesmodulo} n\}.$$

Two vertices $[\lambda_0, \ldots, \lambda_{n-1}]$ and $[\mu_0, \ldots, \mu_{n-1}]$ of Λ_n are connected by an edge whenever there exists a pair (i, j), $0 \le i \ne j \le (n-1)$, such that $\lambda_i + 1 \equiv \lambda_j \bmod n$ and $(\mu_0, \ldots, \mu_{n-1}) = (\lambda_0, \ldots, \lambda_{n-1}) + e_i - e_j$, where e_i, e_j are the coordinate vectors in

$\mathbb{Z}^n$. In this construction, $\lambda_0, \ldots, \lambda_{n-1}$ are the z-coordinates of the central point of the associated alcove scaled by the factor n, see [LP2]. This construction immediately implies the following description of the graph of the alcoved polytope $\mathcal{P}(b_{ij}, c_{ij})$, defined as in Section 2.3.

Proposition 3.4. *For an alcoved polytope $\mathcal{P} = \mathcal{P}(b_{ij}, c_{ij})$, its graph $\Gamma_{\mathcal{P}}$ is the induced subgraph of Λ_n given by the subset of vertices*

$$\left\{[\lambda_0, \ldots, \lambda_{n-1}] \in \Lambda_n \mid n \cdot b_{i,j} \le \lambda_i - \lambda_j \le n \cdot c_{i,j}, \text{for} i, j \in [0, n-1]\right\}.$$

The vertices of the graph $\Gamma_{\mathcal{P}}$ are in bijection with the elements of $C_{\mathcal{P}}$ defined in the previous section. Let $C = (c^{(1)}, c^{(2)}, \ldots, c^{(n)}) \in C_{\mathcal{P}}$ be a minimal circuit in the graph $\Gamma_{\mathcal{P}}$. The integer points $\{c^{(1)}, c^{(2)}, \ldots, c^{(n)}\}$ are the vertices of an alcove A_C, in the x-coordinates. The vertex $[\lambda_1, \lambda_2, \ldots, \lambda_n]$ of Λ_n associated to A_C is given by $\lambda_i = \alpha_1 + \alpha_2 + \cdots + \alpha_i$ where $\alpha := (\alpha_1, \ldots, \alpha_n)$ is given by $\alpha = \sum_{i=1}^n c^{(i)}$. The point $(1/n)[\lambda_1, \lambda_2, \ldots, \lambda_n] \in \mathbb{R}^n/(1, \ldots, 1)\mathbb{R}$ is the central point of the alcove A_C in the z-coordinates.

Example 3.5. The kth hypersimplex is given by inequalities (1). Thus the vertex set of the graph $\Gamma_{\Delta_{k,n}}$ is the subset of Λ_n given by the inequalities $0 \le \lambda_1 - \lambda_0, \lambda_2 - \lambda_1, \ldots, \lambda_{n-1} - \lambda_{n-2} \le n$, and $(k-1) \cdot n \le \lambda_{n-1} - \lambda_0 \le k \cdot n$.

One can give an abstract characterization of subgraphs of Λ_n corresponding to alcoved polytopes. We say that an induced subgraph H of some graph G is *convex* if, for any pair of vertices u, v in H, and any path P in G from u to v of minimal possible length, all vertices of P are in H. In [LP1] we will prove, in the more general context of an arbitrary Weyl group, that an induced subgraph Γ in Λ_n is the graph of some alcoved polytope if and only if Γ is convex.

4. Matroid Polytopes

Let $\mathcal{M}$ be a collection of k-subsets of $[n]$. The polytope $\mathcal{P}_{\mathcal{M}}$ is the convex hull in $\mathbb{R}^n$ of the points $\{\varepsilon_I \mid I \in \mathcal{M}\}$ and is a subpolytope of the hypersimplex $\Delta_{k,n}$. In this section we classify the polytopes $\mathcal{P}_{\mathcal{M}}$ that are alcoved and in these cases give a combinatorial interpretation for their volumes.

Our main interest lies in the case when $\mathcal{M}$ is a matroid on the set $[n]$. In this case the polytopes $\mathcal{P}_{\mathcal{M}}$ are known as *matroid polytopes*—we give explicit examples of matroid polytopes which are alcoved. Matroid polytopes have recently been studied intensively, motivated by applications to tropical geometry [FS], [Sp]: the *amoeba* of a linear subspace $V \subset \mathbb{C}^n$ is asymptotically described by its *Bergman fan*, and this fan is closely related to the normal fan of the matroid polytope $\mathcal{P}_{\mathcal{M}}$ of the matroid of V; see also [AK]. Some of the results in this section have been obtained earlier by Blum [Bl] in the context of Koszul algebras.

4.1. *Sort-Closed Sets*

Definition 4.1. A collection $\mathcal{M}$ of k-subsets of $[n]$ is *sort-closed* if for every two elements I and J in $\mathcal{M}$, the subsets $U(I, J)$ and $V(I, J)$ are both in $\mathcal{M}$.

A sorted subset of $\mathcal{M}$ is a subset of the form $\{I_1, \ldots, I_r\} \subset \mathcal{M}$ such that $(I_1, \ldots, I_r)$ is a sorted collection of k-subsets of $[n]$.

Theorem 4.2. *The triangulation $\Gamma_{k,n}$ of the hypersimplex induces a triangulation of the polytope $\mathcal{P}_{\mathcal{M}}$ if and only if $\mathcal{M}$ is sort-closed. The normalized volume of $\mathcal{P}_{\mathcal{M}}$ is equal to the number of sorted subsets of $\mathcal{M}$ of size* $\dim(\mathcal{P}_{\mathcal{M}}) + 1$.

The proof is analogous to that of Theorem 2.4. We work in the polynomial ring $k[x_I \mid I \in \mathcal{M}]$. The ideal $\mathcal{J}_{\mathcal{M}}$ is the kernel of the ring homomorphism φ: $k[x_I \mid I \in \mathcal{M}] \to k[t_1, t_2, \ldots, t_n]$ given by $x_I \mapsto t_{i_1} t_{i_2} \cdots t_{i_k}$, for $I = \{i_1, \ldots, i_k\}$. The following result is essentially equivalent to Proposition 3.1 of [Bl].

Proposition 4.3. *Suppose that $\mathcal{M}$ is sort-closed. Then there is a term order, $\prec_{\mathcal{M}}$, on $k[x_I | I \in \mathcal{M}]$ so that the reduced Gröbner basis of $\mathcal{J}_{\mathcal{M}}$ is given by the non-zero marked binomials of the form*

$$\{\underline{x_I x_J} - x_{U(I,J)} x_{V(I,J)}\} \tag{2}$$

with the first monomial being the leading term.

Sketch of Proof. Since this proof is essentially the same as that of Sturmfels in Chapter 14 of [Stu], we only sketch the argument; see the Appendix for background.

We say a monomial $x_A x_B \cdots x_V = x_{a_1 \cdots a_k} x_{b_1 \cdots b_k} \cdots x_{v_1 \cdots v_k}$ is *sorted* if the ordered collection of sets $(A, B, \ldots, V)$ is sorted. If a monomial is not sorted, then there is a pair of adjacent variables $x_I x_J$ which is unsorted. Using the binomial $x_I x_J - x_{U(I,J)} x_{V(I,J)}$ we can sort this pair. We can sort a monomial modulo the ideal generated by the marked binomials of (2) in a finite number of steps. Using Theorem 3.12 of [Stu], we conclude that there is a term order $\prec_{\mathcal{M}}$ which selects the marked term for each binomial of (2). Finally, one checks that the sorted monomials are exactly the $\prec_{\mathcal{M}}$-standard monomials. $\square$

Proof of Theorem 4.2. The "if" direction follows from Proposition 4.3 and Theorem A.1. We may assume that $\prec_{\mathcal{M}}$ arises from a weight vector since only finitely many binomials are involved in (2). For the "only if" direction, suppose $\mathcal{P} = \mathcal{P}_{\mathcal{M}}$ is a convex polytope which is a union of simplices in $\Gamma_{k,n}$. Since the triangulation $\Gamma_{k,n}$ is coherent, this triangulation $\Gamma_{\mathcal{P}}$ is also coherent. We know already that all the faces of $\Gamma_{\mathcal{P}}$ are sorted collections of k-subsets of $[n]$ (if we identify a simplex with its set of vertices). By the correspondence of Theorem A.1 and Proposition A.2, $\Gamma_{\mathcal{P}}$ arises from some term-order $\prec_{\mathcal{M}}$ which gives rise to an initial ideal which is the Stanley–Reisner ideal of the triangulation $\Gamma_{\mathcal{P}}$. Let $(I, J) \in \mathcal{M} \times \mathcal{M}$ not be sorted, then $m_1 = x_I x_J \in \mathrm{in}_{\prec_{\mathcal{M}}}(\mathcal{J}_{\mathcal{M}})$. As in the proof of Proposition 4.3, this means there is another monomial m_2 so that $m_1 - m_2 \in$

$\mathcal{J}_{\mathcal{M}}$. After a finite iteration of this argument, we see that $x_I x_J = x_A x_B \mod \mathcal{J}_{\mathcal{M}}$ for some $\prec_{\mathcal{M}}$-standard monomial $x_A x_B$. However, this means that $x_A x_B$ must be a sorted monomial since it is an edge of an alcove. Thus $A = U(I, J)$ and $B = V(I, J)$ satisfy $A, B \in \mathcal{M}$ so that $\mathcal{M}$ is sort-closed. □

4.2. *Sort-Closed Matroids*

Let k and n be positive integers satisfying $k \leq n$. A (non-empty) collection $\mathcal{M}$ of k-subsets (called bases) of $[n]$ (the ground set) is a *matroid* if it satisfies the following axiom (Exchange Axiom):

> Let I and J be two bases of $\mathcal{M}$. Then for any $i \in I$ there exists $j \in J$ so that $(I - \{i\}) \cup \{j\}$ is a base of $\mathcal{M}$.

The matroid $\mathcal{M}$ is then said to be a rank k matroid on n elements. If I is a base of $\mathcal{M}$ we write $I \in \mathcal{M}$. To a matroid $\mathcal{M}$ (of rank k on n elements) we associate the *matroid polytope* $\mathcal{P}_{\mathcal{M}}$ as in Section 4.1. We say that $\mathcal{M}$ is *sort-closed* if it is sort-closed as a collection of k-subsets. Thus by Theorem 4.2, the triangulation $\Gamma_{k,n}$ of the hypersimplex induces a triangulation of the polytopes $\mathcal{P}_{\mathcal{M}}$ for sort-closed matroids $\mathcal{M}$. Sort-closed matroids were introduced by Blum [Bl], who called them base-sortable matroids.

We now describe two classes of matroids which are sort-closed. Let Π be a set partition of $[n]$ with parts $\{\pi_i\}_{i=1}^r$ of sizes $|\pi_i| = a_i$, and let $\underline{b} = (b_1, \ldots, b_r)$ and $\underline{c} = (c_1, \ldots, c_r)$ be two sequences of non-negative integers. We call the data $(\Pi, \underline{b}, \underline{c})$ a *weighted set partition*. Define $\mathcal{M}_{\Pi,\underline{b},\underline{c},k}$ to be the collection of k-subsets I of $[n]$ such that

$$b_j \leq |I \cap \pi_j| \leq c_j \tag{3}$$

for all j.

Lemma 4.4. *The collection of k-subsets $\mathcal{M}_{\Pi,\underline{b},\underline{c},k}$ defined above is a matroid.*

Proof. Let I and J be two such subsets and let $i \in I$, say $i \in \pi_s$ for some s. If $|I \cap \pi_k| = |J \cap \pi_k|$ for all k or if $|I \cap \pi_s| \leq |J \cap \pi_s|$, one can again find some $j \in J \cap \pi_s - (I - \{i\})$ to add to $I - \{i\}$ to form a base. Otherwise there is some t such that $|I \cap \pi_t| < |J \cap \pi_t| \leq c_t$ in which case one can find some $j \in (J \cap \pi_t)$ to add to $I - \{i\}$ without violating any of the inequalities in (3). This verifies the exchange axiom. □

This class of matroids is closed under duality. The dual of $\mathcal{M}_{\Pi,\underline{b},\underline{c},k}$ is $\mathcal{M}_{\Pi,\underline{b}',\underline{c}',n-k}$ where $b'_j = |\pi_j| - c_j$ and $c'_j = |\pi_j| - b_j$. We call the polytope $\Delta_{\Pi,\underline{b},\underline{c},k}$ associated to the matroid $\mathcal{M}_{\Pi,\underline{b},\underline{c},k}$ a *weighted multi-hypersimplex*. When $b_j = 0$ and $c_j = 1$ for all j we denote the matroid and polytope by $\mathcal{M}_{\Pi,k}$ and $\Delta_{\Pi,k}$, respectively, and we call the polytope $\Delta_{\Pi,k}$ a *multi-hypersimplex*. Up to affine equivalence the polytope $\Delta_{\Pi,k}$ depends only on the multiset $\{a_i\}_{i=1}^r$. The polytope $\Delta_{\Pi,k}$ is the intersection of the hyperplane $x_1 + \cdots + x_n = k$ with a product of simplices $\Delta_\Pi \simeq \Delta_{a_1} \times \cdots \times \Delta_{a_r}$ just as the hypersimplices are slices of cubes. In the z-coordinates, this polytope is determined by intersecting the hypersimplex $\tilde{\Delta}_{k,n}$ with the inequalities

$$0 \le z_{a_1} - z_0 \le 1, \quad 0 \le z_{a_1+a_2} - z_{a_1} \le 1, \quad \dots, \quad 0 \le z_n - z_{a_1+a_2+\cdots+a_{r-1}} \le 1,$$

where we assume $z_n = k$. A weighted multi-hypersimplex can be viewed as a slice of a product of unions of hypersimplices. In particular, when $b_j = c_j - 1$ for all j, the polytope $\Delta_{\Pi,\underline{b},\underline{c},k}$ is the slice $x_1 + \cdots + x_n = k$ of

$$\Delta_{\Pi,\underline{b},\underline{c}} \simeq \tilde{\Delta}_{c_1,a_1+1} \times \tilde{\Delta}_{c_2,a_2+1} \times \cdots \times \tilde{\Delta}_{c_r,a_r+1},$$

where the hypersimplex $\tilde{\Delta}_{c_i,a_i}$ (in the notation of Section 2) lives in the coordinates $(x_{a_1+\cdots+a_{i-1}+1}, \dots, x_{a_1+\cdots+a_i})$. Let $\Pi(a_1, \dots, a_r)$ denote the set partition

$$\{\pi_1 = \{1, \dots, a_1\}, \dots, \pi_r = \{a_1 + \cdots + a_{r-1} + 1, \dots, a_1 + \cdots + a_r = n\}\}.$$

Proposition 4.5. *The matroid $\mathcal{M}_{\Pi,\underline{b},\underline{c},k}$ with $\Pi = \Pi(a_1, \dots, a_r)$ and any $\underline{b}, \underline{c} \in \mathbb{N}^r$ is sort-closed.*

Proof. Let $I, J \in \mathcal{M}_{\Pi,\underline{b},\underline{c},k}$. Suppose to the contrary that one of $U(I, J)$ or $V(I, J)$ were not a base. Let $(q_1, q_2, \dots, q_{2k}) = \mathrm{sort}(I \cup J)$. If $|U(I, J) \cap \pi_s| > c_s$ or $|V(I, J) \cap \pi_s| > c_s$ then it must be the case that for some i the entries $q_i, q_{i+2}, \dots, q_{i+2c_s}$ belonged to the same part $\pi_s \in \Pi$. Then $q_{i+1}, q_{i+3}, \dots, q_{i+2c_s-1}$ belong to π_s as well since $q_{i+2k} \le q_{i+2k+1} \le q_{i+2k+2}$. This is impossible as I and J were legitimate bases to begin with and contain at most c_s elements from π_s each. A similar argument guarantees that $|U(I, J) \cap \pi_s| \ge b_s$ and $|V(I, J) \cap \pi_s| \ge b_s$ for all s. □

A matroid $\mathcal{M}$ is *cyclically transversal* if it is a transversal matroid specified by a set of (not necessarily disjoint) cyclic intervals $\{S_1, \dots, S_k\}$ of $[n]$. Recall that the bases of a transversal matroid are the k-element subsets $I = \{i_1, \dots, i_k\}$ of $[n]$ such that $i_s \in S_s$.

Proposition 4.6 [Bl, Theorem 5.2 (without proof)]. *Let $\mathcal{M}$ be a cyclically transversal matroid defined by the subsets $\{S_1, \dots, S_k\}$. Then $\mathcal{M}$ is sort-closed.*

Since the proof is omitted in [Bl], we give a simple direct proof here.

Proof. Let I be a k-element subset of $[n]$. By the Hall marriage theorem, I is a base of $\mathcal{M}$ if and only if

$$\left| I \cap \bigcup_{r \in R} S_r \right| \ge |R| \tag{4}$$

for every subset R of $[k]$. Now let I and J be bases of $\mathcal{M}$ and we now check (4) for $U(I, J)$ and $V(I, J)$. Since each S_i is a cyclic interval of $[n]$ it suffices to consider the case where $\bigcup_{r\in R} S_r$ is itself a cyclic interval $[a, b]$. By hypothesis, the multiset $I \cup J$ intersects $[a, b]$ in at least $2|R|$ elements. Thus each of $U(I, J)$ and $V(I, J)$ will intersect $[a, b]$ in at least $|R|$ elements. □

Let us describe $\mathcal{P}_{\mathcal{M}} \subset \mathbb{R}^n$ for a cyclically transversal matroid explicitly in terms of inequalities. It is given by the hyperplane $x_1 + x_2 + \cdots + x_n = k$, the inequalities $0 \leq x_i \leq 1$ together with the inequalities

$$\sum_{s \in S_R} x_s \geq |R|$$

with $S_R = \bigcup_{r \in R} S_r$ for every subset R of $[k]$.

We end this section with the question: what other matroids are sort-closed?

5. The Second Hypersimplex

There is a description of the triangulation of the second hypersimplex, developed in [LST] and Chapter 9 of [Stu], in terms of graphs known as thrackles. In this section we apply this description to rank two matroids, and give a precise description of the dual graph of the triangulation.

Triangulations of the second hypersimplex $\Delta_{2,n}$ arise in the study of metrics on a finite set of points [Dr], and recently a thorough classification of the triangulations of $\Delta(2, 6)$ was performed in [SY]. This classification of triangulations is an important problem in phylogenetic combinatorics. Our study of the dual graphs of the triangulations is partly motivated by this connection: the graph $\Gamma_{2,n}$ of the triangulation is essentially what is known as the *tight span* of the corresponding metric, and generalizes the phylogenetic trees derived from the metric; see [Dr].

5.1. *Thrackles and Rank Two Matroids*

When the rank k is equal to two (which we assume throughout this section), every matroid $\mathcal{M}$ arises as $\mathcal{M}_{\Pi,2}$ for some set partition Π. Throughout this section we assume that Π has the form $\Pi = \Pi(a_1, \ldots, a_r)$. Following Chapter 9 of [Stu] and [LST], we associate a graph on $[n]$ to each maximal simplex of the matroid polytope $\mathcal{M}_{\Pi,2}$. The vertices are drawn on a circle so that they are labeled clockwise in increasing order. Throughout this section a "graph on $[n]$" refers to such a configuration of the vertices in the plane. Since the bases are two element subsets of $[n]$, we may identify them with the edges.

Lemma 5.1. *Let $A = (a_1, a_2)$ and let $B = (b_1, b_2)$ be two bases. Then the pair A, B is sorted if and only if the edges (a_1, a_2) and (b_1, b_2) intersect (not necessarily in their interior) when drawn on the circle.*

Proof. Sorted implies that $a_1 \leq b_1 \leq a_2 \leq b_2$ which immediately gives the lemma. □

Thus sorted subsets of $\binom{[n]}{2}$ correspond to graphs on $[n]$ drawn on a circle, so that every pair of edges cross. These graphs are known as *thrackles*. Note that two edges sharing a vertex are considered to cross.

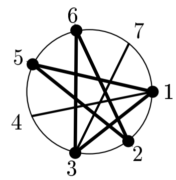

Fig. 3. A thrackle $G(C)$. The cycle C has been drawn in bold.

Proposition 5.2. *Let Π be a set partition and let $\Delta_{\Pi,2}$ have dimension d. The maximal simplices in the alcoved triangulation of $\Delta_{\Pi,2}$ are in one-to-one correspondence with thrackles on $[n]$ with $d+1$ edges such that all edges are bases.*

Proof. Follows immediately from Theorem 4.2 and Lemma 5.1. $\square$

Without the condition on the number of edges in the thrackles of Proposition 5.2, one would obtain graphs corresponding to all simplices (not just the maximal ones) of the triangulation.

When the dimension of $\Delta_{\Pi,2}$ is $n-1$, each thrackle G is determined by picking an odd-cycle C such that all the edges cross pairwise. The remaining edges of G join a vertex not on C to the unique "opposite" vertex lying on C (so that the edge crosses every edge of C); see Fig. 3. We call the resulting thrackle $G(C)$. Let C be a cycle, with pairwise crossing edges, of length $2k+1$ with vertices $V(C) = \{v_1, v_2, \ldots, v_{2k+1}\} \subset [n]$ labeled so that $v_1 < v_2 < \cdots < v_{2k+1}$. Then the edges of C are of the form (v_i, v_{k+i+1}), where the indices are taken modulo $2k+1$. Thus the condition that all the edges of C are bases is equivalent to $|V(C) \cap \pi_i| \le k$ for all i. In fact, this is enough to guarantee that $G(C)$ corresponds to a valid maximal simplex of $\Delta_{\Pi,2}$—that the remaining edges not on the cycle are bases is implied.

Suppose G arises from a sorted subset $(I_1, \ldots, I_r)$. Let $w = \theta(I_1, \ldots, I_r)$ where θ is the bijection of Remark 2.8. The vertices i not on the odd cycle of G are exactly the positions such that $w_i = w_{i-1} + 1$.

Proposition 5.3. *Let $a_1, \ldots, a_r$ be positive integers and let $n = a_1 + \cdots + a_r$. Then the $(n-1)$-dimensional volume of the second multi-hypersimplex $\Delta_{\Pi(a_1,\ldots,a_r),2}$ is given by*

$$\begin{aligned}\mathrm{Vol}(\mathcal{P}_{\Pi(a_1,\ldots,a_r),2}) &= \sum_{k=1}^{\infty}\left(\sum_{c_1,\ldots,c_r \le k;\ c_1+\cdots+c_r=2k+1} \binom{a_1}{c_1}\cdots\binom{a_r}{c_r}\right)\\ &= 2^{n-1} - \sum_{i=1}^{r}\sum_{b,d\ge 0}\binom{a_i}{2b+d+1}\binom{n-a_i}{d}.\end{aligned}$$

Proof. The first formula follows from enumerating odd subsets $S \subset \{1, 2, \ldots, n\}$ with size $2k+1$ satisfying $c_i := |S \cap \pi_i| \le k$ for all i. The second formula comes from counting the odd subsets $S' \subset \{1, 2, \ldots, n\}$, where $|S' \cap \pi_i| > |S'|/2$ for some $i \in \{1, \ldots, r\}$, and subtracting them from all odd subsets of $\{1, \ldots, n\}$. $\square$

One can also describe the simplices of the polytopes $\mathcal{P}_{\mathcal{M}}$ for higher rank matroids as hypergraphs G satisfying the following conditions:

(1) Every hyperedge $A \subset [n]$ of G is a base of $\mathcal{M}$.
(2) Let $A = \{a_1 < \cdots < a_k\}$, $B = \{b_1 < \cdots < b_k\}$ be a pair of hyperedges belonging to G. Let C_A be the cycle on $[n]$ drawn with usual edges $(a_1, a_2), \ldots, (a_k, a_1)$ and similarly for C_B. Then each edge of C_A must touch C_B and vice versa.

Remark 5.4. Let $\tilde{\Gamma}_{2,n}$ be the simplicial complex associated with the triangulation $\Gamma_{2,n}$, and let $f(\tilde{\Gamma}_{2,n}) = \sum_{i=1}^{d} f_i(\tilde{\Gamma}_{2,n})\, t^i$ be its f-polynomial, where f_i is the number of i-dimensional simplices in this complex. Using generating function techniques, one can deduce the following expression for these polynomials:

$$\sum_{n\geq 2} f(\tilde{\Gamma}_{2,n}, t)\, x^n = \left[\frac{tq^2(1+q)(t^2q^2+t^2q-tq+1)}{(1-tq)^2(1-2tq-tq^2)}\right]_{q\mapsto x/(1-x)}.$$

5.2. *Adjacency of Alcoves in the Second Hypersimplex*

Let $\Delta \in \Gamma_{k,n}$ be a (maximal) simplex. We say that Δ has *degree* d if it is adjacent to d other simplices. We call Δ an *internal simplex* if none of its facets lies on the boundary of $\Delta_{k,n}$. In this case Δ has maximal degree, namely n.

Proposition 5.5. *The two simplices of $\Gamma_{2,n}$ corresponding to two thrackles G and G' (via the correspondence of Proposition 5.2) are adjacent if and only if there are four distinct vertices labeled $a, a+1, b, b+1 \mod n$ such that G contains the edges (a, b), $(a-1, b)$, $(b+1, a)$ and G' is obtained from G by changing the edge (a, b) to $(a-1, b+1)$; see Fig. 4.*

Proof. The proposition follows immediately from Proposition 5.2 and Theorem 2.9 applied to the case $k = 2$ (more precisely, the comments after the proof of the theorem). □

In fact the proposition is true also for smaller dimensional faces of the simplices in $\Gamma_{2,n}$. Let C be an odd cycle such that all edges cross and let $|C|$ denote its length.

Theorem 5.6. *Let Δ be the simplex of $\Gamma_{2,n}$ corresponding to the thrackle $G(C)$. If $G(C)$ is a triangle then $\Delta = \Delta_{2,3}$ is the unique simplex in $\Gamma_{2,3}$ (and has degree zero). If*

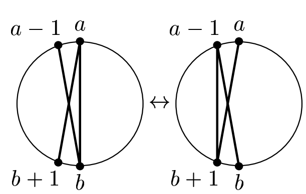

Fig. 4. The two simplices in $\Gamma_{2,n}$ corresponding to two thrackles G and G' are adjacent if G and G' are related by the above move.

$|C| = 3$ and $G(C)$ has two vertices of degree two then Δ has degree two. Otherwise, Δ has degree $|C|$.

Proof. The theorem follows from Proposition 5.5. Indeed, we can perform the move shown in Fig. 4 to an edge (a, b) if it joins two vertices a, b each of degree at least two, with the exception of the case where a and b both have degree exactly two and are joined to the same vertex c. When the edge (a, b) can be replaced, the change is necessarily unique. The case where a and b both have degree two and are joined to the same vertex c occurs only when C is a three-cycle. In all other cases, every edge of C can be replaced, and so Δ has degree $|C|$. $\square$

The following corollary is immediate from Theorem 5.6.

Corollary 5.7. *For $d > 1$, the second hypersimplex $\Gamma_{2,n}$ has $\binom{n}{2d+1}$ simplices with degree $2d + 1$. No simplex has even degree greater than or equal to four. In particular, if n is odd then $\Gamma_{2,n}$ contains a unique internal simplex. If n is even, then $\Gamma_{2,n}$ has no internal simplices.*

6. Multi-Hypersimplices and Multi-Eulerian Polynomials

In this section we investigate the volumes of the *multi-hypersimplices*, defined in Section 4.2. They are slices of the product $\Delta_{a_1} \times \Delta_{a_2} \times \cdots \times \Delta_{a_r}$ of simplices or more generally of hypersimplices. We define the multi-Eulerian numbers to be the volumes of these polytopes. They are generalizations of the usual Eulerian numbers. These numbers, like the usual Eulerian numbers, satisfy a number of interesting enumerative identities. In the first non-trivial case (Proposition 6.3), we determine the multi-Eulerian numbers explicitly in terms of Eulerian numbers.

6.1. *Descent-Restricted Permutations and Alcoved Polytopes*

We consider an alcoved polytope $\mathcal{P}$ (as in Section 2.3) which lies within a hypersimplex $\Delta_{k,n}$. In the x-coordinates, $\mathcal{P} \subset \mathbb{R}^n$ is defined by the hyperplane $x_1 + x_2 + \cdots + x_n = k$, the inequalities $0 \le x_i \le 1$ together with inequalities of the form

$$b_{ij} \le x_{i+1} + \cdots + x_j \le c_{ij}$$

for integer parameters b_{ij} and c_{ij} for each pair (i, j) satisfying $0 \le i < j \le n - 1$.

Let $W_{\mathcal{P}} \subset S_{n-1}$ be the set of permutations $w = w_1 w_2 \cdots w_{n-1} \in S_{n-1}$ satisfying the following conditions:

(1) w has $k - 1$ descents.
(2) The sequence $w_i \cdots w_j$ has at least b_{ij} descents. Furthermore, if $w_i \cdots w_j$ has exactly b_{ij} descents, then $w_i < w_j$.
(3) The sequence $w_i \cdots w_j$ has at most c_{ij} descents. Furthermore, if $w_i \cdots w_j$ has exactly c_{ij} descents, then we must have that $w_i > w_j$.

In the above conditions we assume that $w_0 = 0$.

Let $p\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ denote the projection as in Section 2. We may apply Stanley's piecewise-linear map ψ to $p(\mathcal{P})$. Using Theorem 2.7, we see that a unimodular triangulation of $p(\mathcal{P})$ is given by the set of simplices $\psi^{-1}(\nabla_{w^{-1}})$ as $w = w_1 w_2 \cdots w_{n-1}$ varies over permutations in $W_{\mathcal{P}}$. As a corollary we obtain the following result.

Proposition 6.1. *The volume of $\mathcal{P}$ is equal to $|W_{\mathcal{P}}|$.*

6.2. *Multi-Eulerian Polynomials*

Let Π be a fixed set partition with parts of sizes $\{a_i\}_{i=1}^r$ of total size $n = \sum_i a_i$ and let $\underline{b} = (b_1, \ldots, b_r)$, $\underline{c} = (c_1, \ldots, c_r) \in \mathbb{N}^r$. Define the *weighted multi-Eulerian number* $A_{\Pi,\underline{b},\underline{c},k} = \mathrm{Vol}(\Delta_{\Pi,\underline{b},\underline{c},k})$ as the normalized $(n-1)$-dimensional volume of the corresponding weighted multi-hypersimplex. We consider polytopes of smaller dimension to have volume 0 for what follows. Now define the *weighted multi-Eulerian polynomial* $A_{\Pi,\underline{b},\underline{c}}(t)$ as

$$A_{\Pi,\underline{b},\underline{c}}(t) = \sum_{k=1}^{r} A_{\Pi,\underline{b},\underline{c},k} t^k.$$

Note that when Π is the set partition with all $a_i = 1$ and $b_i = 0$ and $c_i = 1$ for all i, then $A_{\Pi,\underline{b},\underline{c}}(t)$ reduces to the usual Eulerian polynomial $A_{n-1}(t) = \sum_k A_{k,n-1} t^k$. If Π is a set partition of $[n]$, we denote by Π^* the set partition of $[n+1]$ with an additional part of size one containing $n+1$. If $\underline{d} = (d_1, \ldots, d_r) \in \mathbb{Z}^r$, then we denote $\underline{d}^* \in \mathbb{Z}^{r+1}$ to be the integer vector with an additional coordinate $d_{r+1} = 0$ and $\underline{d}' \in \mathbb{Z}^{r+1}$ similarly with $d_{r+1} = 1$. Then $\Delta_{\Pi^*,\underline{b}^*,\underline{c}',k}$ is affinely equivalent to (and has the same normalized volume as) the intersection of $k-1 \le x_1 + \cdots + x_n \le k$ with the product of unions of hypersimplices:

$$\left(\bigcup_{i=b_1+1}^{c_1} \Delta_{i,a_1+1}\right) \times \cdots \times \left(\bigcup_{i=b_r+1}^{c_r} \Delta_{i,a_r+1}\right).$$

Thus

$$A_{\Pi^*,\underline{b}^*,\underline{c}'}(1) = \binom{n}{a_1, a_2, \ldots, a_r} \left(\sum_{i=b_1+1}^{c_1} A_{i,a_1}\right) \cdots \left(\sum_{i=b_r+1}^{c_r} A_{i,a_r}\right).$$

In particular if $b_j = 0$ and $c_j = 1$ then this value is simply a multinomial coefficient. For this special case, we omit $\underline{b}$ and $\underline{c}$ in the notation and omit the prefix *weighted* from the names. We write out the combinatorial interpretation for $A_{\Pi,k}$ explicitly.

Proposition 6.2. *Let k be a positive integer and let Π be a set partition of $[n]$ with parts of sizes $\{a_i\}_{i=1}^r$ as before. The following quantities are equal*:

(1) *the multi-Eulerian number $A_{\Pi,k}$*,
(2) *the number of permutations of $w \in S_{n-1}$ with $k-1$ descents such that the substring $w_{a_1} w_{a_1+a_2} \cdots w_{a_1+a_2+\cdots+a_{r-1}}$ has $k-1$ descents*,
(3) *the number of sorted subsets $(I_1, \ldots, I_n)$ of $\mathcal{M}_\Pi$ of size n.*

Proof. As described in Section 4.2 we may consider multi-hypersimplices subpolytopes of the hypersimplex so the proposition follows from Theorems 2.7 and 4.2 and Proposition 6.1. □

Note that with $\Pi = \Pi(a_1, a_2, \ldots, a_r)$, then $A_{\Pi,k}$ is a function symmetric in the inputs $\{a_i\}$. Let $\Pi = \Pi(a, 1^{n-a})$. By Proposition 6.2(6.2), $A_\Pi(t)$ is the generating function by descents for permutations satisfying $w_1 < w_2 < \cdots < w_a$. Thus

$$A_{\Pi(a,1^{n-a})}(t) = \sum_{w \in S_{n-a}} \binom{a + w_1 - 2}{a - 1} t^{\mathrm{des}(w)+1}.$$

We give an explicit formula for $a = 2$.

Proposition 6.3. *For $n \geq 3$, $A_{\Pi(2,1^{n-2})}(t) = t(d/dt)A_{n-1}(t)$, where $A_{n-1}(t)$ is the usual Eulerian polynomial. In other words,*

$$\sum_{w \in S_{n-1}} w_1 t^{\mathrm{des}(w)+1} = \sum_{w \in S_{n-1}} (\mathrm{des}(w) + 1) t^{\mathrm{des}(w)+1}. \tag{5}$$

We prove this statement bijectively. Let $w \in S_n$. A *circular descent* of w is either a usual descent or the index n if $w_n > w_1$. Define the *circular descent number* $\mathrm{cdes}(w)$ as the number of circular descents. Let C_n denote the subgroup of S_n generated by the long cycle $c = (12 \cdots n)$, written in cycle notation. We have the following easy lemma.

Lemma 6.4. *The statistic* cdes *is constant on double cosets $C_n \backslash S_n / C_n$.*

Proof. Left multiplication by the long cycle c maps $w_1 w_2 \cdots w_n$ to $(w_1 + 1)(w_2 + 1) \cdots (w_n + 1)$ where "$n + 1$" is identified with "1". Right multiplication by c maps $w_1 w_2 \cdots w_n$ to $w_2 w_3 \cdots w_n w_1$. The lemma is immediate from the definition of circular descent number. □

Actually in the following we use this lemma for S_{n+1}.

Proof of Proposition 6.3. If $w = w_1 w_2 \cdots w_n \in S_n$ and $w' = w_1 w_2 \cdots w_n (n + 1) \in S_{n+1}$ then $\mathrm{cdes}(w') = \mathrm{des}(w) + 1$. By this and our earlier comments, the left-hand side of (5) is the generating function for permutations in S_{n+1} satisfying $w_1 < w_2$ and $w_{n+1} = n + 1$, according to their circular descent number. Alternatively, we may view this as the cdes-generating function of right cosets $\bar{w} \in S_{n+1}/C_{n+1}$ satisfying the property that the two numbers w_i, w_{i+1} cyclically located after $w_{i-1} = n + 1$ satisfy $w_i < w_{i+1}$ for any representative $w \in \bar{w}$. Here the indices are taken modulo $n + 1$ and by Lemma 6.4, $\mathrm{cdes}(\bar{w}) := \mathrm{cdes}(w)$ does not depend on the representative w of $\bar{w}$ and so is well defined.

The right-hand side of (5) is the generating function for permutations $u = u_1 u_2 \cdots u_n \in S_n$ satisfying $u_n = n$ where one of the circular descents has been marked, again according to circular descent number. If $u_i > u_{i+1}$ is the marked circular descent (where $i + 1$ is to be taken modulo n), then we can insert the number $n + 1$ between u_i and u_{i+1} to obtain a coset $\bar{v} \in S_{n+1}/C_{n+1}$ with the same number of circular descents. If $v \in \bar{v}$ and $v_i = n + 1$ then we automatically have $v_{i-1} > v_{i+1}$ (in fact this is a cdes-preserving bijection between cosets $\bar{v}$ satisfying this property and permutations of $u \in S_n$ satisfying $u_n = n$ with a marked circular descent). Let $c = (123 \cdots (n+1))$ be the generator of C_{n+1} and consider $v' = c^{n+1-v_{i-1}} v$ for any $v \in \bar{v}$ where i is determined by $v_i = n + 1$. Let $\bar{v}' = v'C$. By Lemma 6.4, $\mathrm{cdes}(\bar{v}') = \mathrm{cdes}(\bar{v})$. However, it is easy to see that $\bar{v}'$ is exactly one of the cosets which are enumerated by the left-hand side of (5). Thus we obtain a cdes-preserving bijection $\bar{v} \mapsto \bar{v}'$ between two classes of cosets in S_{n+1}/C_{n+1}, enumerated by the two sides of (5).

We illustrate the bijection with an example, where we will pick representatives of appropriate cosets at our convenience. Let $u = 53162748 \in S_8$ satisfying $u_8 = 8$ with marked circular descent index 4 corresponding to $u_4 = 6 > 2 = u_5$. This is an object enumerated by the right-hand side of (5), with $n = 8$. Inserting “9” between “6” and “2” we obtain $v = 531692748 \in S_9$. Multiplying on the left by c^3 (where $c = (123456789)$) adds three to every value, changing “6” to “9”, giving the permutation $v' = 864935172$. Multiplying on the right by c^4 we move the 9 to the last position to get $w = 351728649 \in \bar{v}' = v'C$, which satisfies $w_9 = 9$ and $w_1 < w_2$. This is exactly a permutation enumerated by the left-hand side of (5). Note that $\mathrm{cdes}(u) = 5 = \mathrm{cdes}(w)$ and that all the steps can be reversed to give a bijection. $\square$

It would be interesting if algebro-geometric proofs of some of our results concerning (weighted) multi-Eulerian numbers could be given; see Section 7.4.

7. Final Remarks

7.1. *The h-Vectors of Alcoved Polytopes*

Let $\tilde{\Gamma}_{\mathcal{P}}$ denote the simplicial complex associated to the collection of simplices in the alcoved triangulation of $\mathcal{P}$. The triangulation $\Gamma_{\mathcal{P}}$ of an alcoved polytope $\mathcal{P}$ is unimodular and this implies that the h-polynomial $h(\tilde{\Gamma}_{\mathcal{P}}, t)$ is a non-negative polynomial. We have given many interpretations for the volume $h(\tilde{\Gamma}_{\mathcal{P}}, 1)$ of an alcoved polytope. It would be interesting to obtain statistics on these interpretations which give $h(\tilde{\Gamma}_{\mathcal{P}}, t)$.

In fact, the Erhart polynomial of $\mathcal{P}$ equals the Hilbert polynomial of the quotient ring $k[x_I]/\mathcal{J}_{\mathcal{P}}$ associated to the polytope $\mathcal{P}$; see the Appendix. The h-polynomial is the numerator of the associated generating function.

7.2. *Relation with Order Polytopes*

Let P be a poset naturally labeled with the numbers $[n]$. The order polytope $\mathcal{O}_P \subset \mathbb{R}^n$ is defined by the inequalities $0 \le x_i \le 1$ together with $x_i \le x_j$ for every pair of elements $i, j \in P$ satisfying $i > j$. It is clear that $\mathcal{O}_P$ is affinely equivalent to an alcoved polytope via the transformation $y_i = x_i - x_{i-1}$. The triangulation Γ_P of $\mathcal{O}_P$ thus obtained is known as the *canonical triangulation* of $\mathcal{O}_P$.

Denote by $\mathcal{L}_P$ the set of linear extensions of P; see [Sta3]. The simplices of Γ_P can be labeled by $w \in \mathcal{L}_P$ and this is compatible with the labeling used throughout this paper. It is known [Sta2] that we have

$$h(\tilde{\Gamma}_P, t) = \sum_{w \in \mathcal{L}_P} t^{\mathrm{des}(w)}.$$

The descents counted by this h-vector, however, disagree with the way we have been counting descents in this paper. More precisely, we have been concerned with the number of descents $\mathrm{des}(w^{-1})$ of the inverse permutations labeling the simplices.

7.3. *Weight Polytopes for Type A_n and Alcoved Polytopes*

A *weight polytope* is the convex hull of the weights which occur in some highest weight representation of a Lie algebra. For example, the vertices of the hypersimplex $\Delta_{k,n}$ are exactly the weights which occur in the kth fundamental representation of $\mathrm{sl_n}$.

We identify the integral weights L of $\mathrm{sl_n}$ with the integer vectors $(a_1, \dots, a_n)$ satisfying $a_1 + \cdots + a_n = l$, for some fixed l. A weight polytope $\mathcal{P}_\lambda$ is specified completely by giving the highest weight $\lambda = (\lambda_1, \dots, \lambda_n)$. We assume all the coordinates λ_i are non-negative. A weight μ lies inside $\mathcal{P}_\lambda$ if λ dominates μ in the usual sense: $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all i.

Let $\mu, \nu \in L$ be two weights. Define $U(\mu, \nu), V(\mu, \nu) \in L$ by requiring that $U(I_\mu, I_\nu) = I_{U(\mu,\nu)}$ and $V(I_\mu, I_\nu) = I_{V(\mu,\nu)}$, in the notation of Section 3.1. Alternatively, $U(\mu, \nu)$ and $V(\mu, \nu)$ are determined by requiring that $\mu + \nu = U(\mu, \nu) + V(\mu, \nu)$ and $V(\mu, \nu) - U(\mu, \nu) = \sum_i b_i \alpha_i$ for some $b_i \in \{0, 1\}$, where $\alpha_i = e_{i+1} - e_i$ are the simple roots. We call $\mathcal{P}_\lambda$ *sort-closed* if the set of weights $\mathcal{P}_\lambda \cap L$ is sort-closed, as in Theorem 3.1.

Proposition 7.1. *A weight polytope $\mathcal{P}_\lambda$ is sort-closed if and only if λ is equal to $aw_i + bw_{i+1}$ for non-negative integers a, b and where $w_k = (1, \dots, 1, 0, \dots, 0)$ (k "1"s) are the fundamental weights.*

Proof. The "if" direction is easy as $\mathcal{P}_\lambda$ can be specified by $x_1 + \cdots + x_n = i \cdot a + (i+1) \cdot b$ and the inequalities $0 \leq x_j \leq a$ for each $j \in [1, n]$ and we can use Theorem 3.1. For the other direction, we may assume the highest weight λ is an n-tuple with highest value a and lowest value 0. Suppose λ is not of the form of the proposition, then there are two more values b, c not equal to a satisfying $a > b \geq c > 0$ so that λ is of the form $(a, \dots, a, b, \dots, b, c, \dots, c, \dots, 0)$.

Explicitly construct a pair of weights $\delta = (a, 0, b, c, \dots)$ and $\mu = (a-1, 1, b+1, c-1, \dots)$ where the tails of the two n-tuples are identical, and δ is just a permutation of the coordinates of λ. Both δ and μ are dominated by λ and hence lie in $\mathcal{P}_\lambda$. However, $U(\delta, \mu) = (a, 0, b+1, c-1, \dots)$ does not lie in $\mathcal{P}_\lambda$. $\square$

Sturmfels [Stu, Chapter 14] considered exactly this class of sort-closed weight polytopes. The following corollary follows immediately from Theorem 3.1 and Proposition 7.1.

Corollary 7.2. *A weight polytope in the x-coordinates with highest weight λ is alcoved if and only if λ is of the form $a\omega_i + b\omega_{i+1}$ for a and b non-negative integers. In particular, every weight polytope for A_2 is alcoved.*

7.4. *Geometric Motivation: Degrees of Torus Orbits*

Let $Gr_{k,n}$ denote the *Grassmannian manifold* of k-dimensional subspaces in the complex linear space $\mathbb{C}^n$. Elements of $Gr_{k,n}$ can be represented by $k \times n$ matrices of maximal rank k modulo left action of GL_k. The $\binom{n}{k}$ maximal minors p_I of such a matrix, where I runs over k-element subsets in $\{1, \ldots, n\}$, form projective coordinates on $Gr_{k,n}$, called the *Plücker coordinates*. The map $Gr_{k,n} \to (p_I)$ gives the *Plücker embedding* of the Grassmannian into the projective space $\mathbb{C}P^{\binom{n}{k}-1}$. Two points $A, B \in Gr_{k,n}$ are in the same *matroid stratum* if $p_I(A) = 0$ is equivalent to $p_I(B) = 0$, for all I. The matroid $\mathcal{M}_A$ of A has as set of bases $\{I \in \binom{[n]}{k} \mid p_I(A) \neq 0\}$.

The complex torus $T = (\mathbb{C}\backslash\{0\})^n$ acts on $\mathbb{C}^n$ by stretching the coordinates

$$(t_1, \ldots, t_n) : (x_1, \ldots, x_n) \mapsto (t_1x_1, \ldots, t_nx_n).$$

This action lifts to an action of the torus T on the Grassmannian $Gr_{k,n}$. This action was studied in [GGMS]. The authors showed that the geometry of the closure of a torus orbit $X_A = \overline{T \cdot A}$ depends (only) on the matroid stratum of A. The variety X_A is a toric variety and its associated polytope is exactly the polytope $\mathcal{P}_{\mathcal{M}_A}$ associated to the the matroid $\mathcal{M}_A$ from Section 4.2. Our study of the volume of the polytopes $\mathcal{P}_{\mathcal{M}}$ was motivated by the well-known fact (see [F]) that

$$\deg(X_A) = \mathrm{Vol}(\mathcal{P}_{\mathcal{M}_A}),$$

where $\deg(X_A)$ denotes the degree of X_A as a projective subvariety of $\mathbb{C}P^{\binom{n}{k}-1}$ and Vol denotes the normalized volume with respect to the lattice generated by the coordinate vectors e_i. Note that by definition only *representable* matroids $\mathcal{M}$ arise as $\mathcal{M}_A$ in this manner. Our Theorem 4.2 gives a combinatorial description of the degree of a torus orbit closure corresponding to a stratum of a sort-closed matroid. In fact, Proposition 4.3 (and Proposition A.2) shows that these torus orbit closures are projectively normal, a fact known for all torus orbit closures; see [W] and [Da].

It has been conjectured (see Conjecture 13.19 of [Stu]) that the ideal of a smooth projectively normal toric variety is always generated by quadratic binomials. The toric varieties associated to simple alcoved polytopes give more examples of this.

Appendix. Coherent Triangulations and Gröbner Bases

We give a brief introduction to the relationship between coherent triangulations of integer polytopes and Gröbner bases. See [Stu] for further details.

Let k be a field and let $k[x] = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables. A total order $\prec$ on $\mathbb{N}^n$ is a *term order* if it satisfies:

- The zero vector is the unique minimal element.
- For any $\underline{a}, \underline{b}, \underline{c} \in \mathbb{N}^n$, such that $\underline{a} \prec \underline{b}$ we have $\underline{a} + \underline{c} \prec \underline{b} + \underline{c}$.

One way to create a term order is by giving a weight vector $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{R}^n$. Then for sufficiently generic weight vectors a term order $\prec$ is given by $\underline{b} \prec \underline{c}$ if and only if $\omega \cdot \underline{b} < \omega \cdot \underline{c}$. In this situation we say that ω represents $\prec$.

Given a polynomial $f \in k[x]$ one defines the initial monomial $\mathrm{in}_\prec(f)$ as the monomial $\underline{x}^{\underline{a}}$ with the largest $\underline{a}$ under $\prec$. For an ideal I of $k[x]$ one defines the initial ideal $\mathrm{in}_\prec(I)$ as the ideal generated by the initial monomials of elements of I. The monomials which do not lie in $\mathrm{in}_\prec(I)$ are called the standard monomials. A finite subset $\mathcal{G} \subset I$ is a *Gröbner basis* for I with respect to $\prec$ if $\mathrm{in}_\prec(\mathcal{G})$ generates $\mathrm{in}_\prec(I)$. The Gröbner basis is called reduced if for two distinct elements $g, g' \in \mathcal{G}$, no term of g' is divisible by $\mathrm{in}_\prec(g)$.

Now let $\mathcal{A} = \{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n\}$ be a finite subset of $\mathbb{Z}^d$. We define a ring homomorphism $k[x] \to k[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$ by

$$x_k \longmapsto \underline{t}^{\underline{a}_k}.$$

The kernel $\mathcal{J}_\mathcal{A}$ of this map is an ideal known as a *toric ideal*.

We now describe the relationship between Gröbner bases of $\mathcal{J}_\mathcal{A}$ and coherent triangulations of the convex hull of $\mathcal{A}$. For any term order $\prec$, the initial complex $\Delta_\prec(\mathcal{J}_\mathcal{A})$ of $\mathcal{J}_\mathcal{A}$ is the simplicial complex defined as follows. A subset $F \subset \{1, 2, \dots, n\}$ is a face of $\Delta_\prec(\mathcal{J}_\mathcal{A})$ if there is no polynomial $f \in \mathcal{J}_\mathcal{A}$ such that the support of $\mathrm{in}_\prec(f)$ is F. Thus the *Stanley–Reisner ideal* of $\Delta_\prec(\mathcal{J}_\mathcal{A})$ is the radical of $\mathrm{in}_\prec(\mathcal{J}_\mathcal{A})$.

A triangulation of a set $\mathcal{A} \in \mathbb{Z}^d$ (more specifically its convex hull) is *coherent* if one can find a piecewise-linear convex function ν on $\mathbb{R}^d$ such that the domains of linearity are exactly the simplices of the triangulation. Alternatively, the triangulation is coherent if one can find a "height" vector ω such that the projection of the "lower" faces of the convex hull of $\{(\underline{a}_1, \omega_1), (\underline{a}_2, \omega_2), \dots, (\underline{a}_n, \omega_n)\}$ is exactly the triangulation. We denote such a triangulation of $\mathcal{A}$ by $\Delta_\omega(\mathcal{A})$. The function ν and the vector ω can be related by setting $\omega_n = \nu(\underline{a}_n)$.

The main result we need is the following [Stu, Chapter 8]:

Theorem A.1. *The coherent triangulations of $\mathcal{A}$ are the initial complexes of the toric ideal $\mathcal{J}_\mathcal{A}$. More precisely, if $\omega \in \mathbb{R}^n$ represents $\prec$ for $\mathcal{J}_\mathcal{A}$ then $\Delta_\prec(\mathcal{J}_\mathcal{A}) = \Delta_\omega(\mathcal{A})$.*

In the case when $\mathcal{J}_\mathcal{A}$ is a homogeneous toric ideal we can say more.

Proposition A.2. *Let $\mathcal{A}$ be such that $\mathcal{J}_\mathcal{A}$ is a homogeneous toric ideal. Then the initial ideal* $\mathrm{in}_\prec(\mathcal{J}_\mathcal{A})$ *is square-free if and only if the corresponding regular triangulation $\Delta_\prec$ of $\mathcal{A}$ is unimodular. In that case, let $Y_\mathcal{A}$ be the projective toric variety defined by the ideal $\mathcal{J}_\mathcal{A}$. Then $Y_\mathcal{A}$ is projectively normal and the Hilbert polynomial of $Y_\mathcal{A}$ equals the Erhart polynomial of the convex hull of $\mathcal{A}$.*

In this last case, the $\prec$-standard monomials correspond exactly to the simplices of the triangulation.

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