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Line Transversals to Translates of Unit Discs

Jesús Jerónimo Castro

Centro de Investigación en Matemáticas,
Guanajuato, Gto., C.P. 36000, México
jeronimo@cimat.mx

Abstract. Let $\mathcal{F}$ be a family of convex figures in the plane. We say that $\mathcal{F}$ has property T if there exists a line intersecting every member of $\mathcal{F}$. Also, the family $\mathcal{F}$ has property $T(k)$ if every *k-membered* subfamily of $\mathcal{F}$ has property T. Let B be the unit disc centered at the origin. In this paper we prove that if a finite family $\mathcal{F} = \{x_i + B\colon i \in I\}$ of translates of B has property $T(4)$ then the family $\mathcal{F}' = \{x_i + \lambda B\colon i \in I\}$, where $\lambda = (1+\sqrt{5})/2$, has property T. We also give some results concerning families of translates of the unit disc which has either property $T(3)$ or property $T(5)$.

1. Introduction

Let $\mathcal{F}$ be a family of compact convex sets in the plane. We say that $\mathcal{F}$ has property T if there exists a line intersecting every member of $\mathcal{F}$. Also, if $\mathcal{F}$ consist of at least k members, we say that $\mathcal{F}$ has property $T(k)$ if every *k-membered* subfamily of $\mathcal{F}$ has property T.

Over the years considerable effort has been devoted to finding conditions on the family $\mathcal{F}$ such that $T(k)$ implies T. This problem was posed by Vincensini [9] and the first result in that direction was the following result due to Santaló [6]:

Santaló's Theorem. *Let $\mathcal{F}$ be a family of parallelotopes in $\mathbb{E}^n$ with edges parallel to the coordinate axes. If $\mathcal{F}$ has property $T(2^{n-1}(2n-1))$ then $\mathcal{F}$ has property T.*

In 1964 Grünbaum [3] posed the following problem:

> Let C be a centrally symmetric figure and let $\mathcal{F} = \{x_i + C\}$ be a finite family of translates of C such that $\mathcal{F}$ has property $T(m)$. What is the smallest positive number $\lambda = \lambda(C, m)$ such that, for every such $\mathcal{F}$, the family $\mathcal{F}' = \{x_i + \lambda C\}$ has property T?

An upper bound for Grünbaum's problem was given by Eckhoff [2]:

Eckhoff's Theorem. *If a finite family $\mathcal{F}$ of translates of C in the plane has property $T(3)$ then there exists a parallel strip of C-width 1 intersecting all members of $\mathcal{F}$.*

Where by the *C-width* of a set X, in direction u, we mean the ratio between the width of X and the width of C, both taken in the direction u.

Heppes [5] proved the following result for the case of circles:

Heppes's Theorem. *If a finite family $\mathcal{F}$ of pairwise disjoint translates of a disc of diameter 1 in the plane has property $T(3)$, then there exists a parallel strip of width < 0.65 intersecting all members of $\mathcal{F}$.*

The aim of this paper is to give some results in this direction for families of translates of unit circles.

2. Some Results on Transversals

We begin with some observations:

That three translates of the euclidean unit circle (B), $x_i + B$ ($i = 1, 2, 3$), have a common line transversal is equivalent to saying that at least one altitude of the triangle $\triangle x_1 x_2 x_3$ has length at most 2 (see Fig. 1).

We can state the problem, for the case of unit circles, in the following equivalent way: Let X be a finite set of points in the plane such that every three of them are contained in some parallel strip of width 2. What is the smallest positive number α such that any such set X, with the above property, is contained in a parallel strip of width α?

Independently, Eckhoff [1] and Dolnikov (1972) conjectured that this minimum number α must be equal to $1+\sqrt{5}$. This is equivalent to saying that

$$\lambda(B, 3) = \frac{1+\sqrt{5}}{2}.$$

We will show that if we consider property $T(4)$ instead of $T(3)$ the number $(1+\sqrt{5})/2$ is the best. That this number cannot be improved is shown by a classical example (which we call the *pentagonal example*) consisting of unit circles centered at the set of vertices of a regular pentagon whose sides have length equal to $2(\sin 72^\circ)^{-1}$ (Fig. 2).

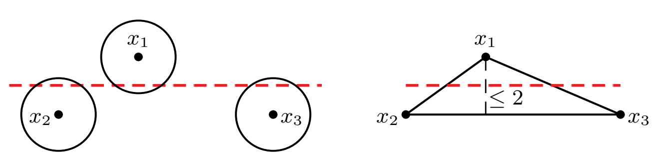

Fig. 1

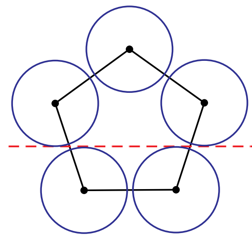

Fig. 2

In what follows, we consider only finite families of translates of a given convex figure. In order to prove Theorem 1 we will need the following lemma:

Lemma 1. *Let $\mathcal{F}$ be a finite family of unit discs in the plane which has property $T(3)$, and let $\triangle x_1x_2x_3$ be a triangle of centers of the discs which has maximum area. If $\triangle x_1x_2x_3$ has two altitudes of length at most 2, then there exists a parallel strip of width $1+\sqrt{5}$ which contains the set of centers.*

Let a, b, c be points in $\mathbb{E}^2$. We denote by $|ab|$ the length of the segment $[a, b]$ and by $d(a, bc)$ the distance from the point a to the line bc. Also, if C is a closed convex curve in the plane, we denote by $|C|$ the area of the convex hull of C. The following lemma is due to Straus [7]:

Straus's Lemma. *If C is a closed convex curve in $\mathbb{E}^2$ and one of the sides of an inscribed triangle T of maximum area lies on C, then the ratio of the areas satisfies $\sqrt{5}\cdot|T| \geq |C|$, with equality if and only if C is an affine regular pentagon.*

In the case where C is the boundary of a convex pentagon, it is not difficult to prove that there is a triangle, T, of maximum area whose vertices are vertices of the pentagon. Thus, one of the sides of T lies on C and so Straus's lemma applies.

Proof of Lemma 1. Obviously, there exists such a triangle $\triangle x_1x_2x_3$ since the set of centers, X, is finite. Suppose that $d(x_2, x_1x_3) \leq 2$ and $d(x_3, x_1x_2) \leq 2$. Let $\triangle y_1y_2y_3$ be the homothetic copy of $\triangle x_1x_2x_3$ with center of homothety at the centroid of $\triangle x_1x_2x_3$ and a constant of homothety equal to -2. We have $X \subset \triangle y_1y_2y_3$, otherwise we can find another triangle with vertices at points of X with greater area. Now, let $p\colon \mathbb{E}^2 \longrightarrow \mathbb{E}^2$ be an affine transformation with $a_i = p(x_i)$, for $i = 1, 2, 3$, such that $|a_1a_2| = |a_1a_3|$, $|a_1a_2|/|a_2a_3| = (1+\sqrt{5})/2$, and $|a_2a_3| = 2(\sin 72^\circ)^{-1}$. Also, let $b_i = p(y_i)$, for $i = 1, 2, 3$, and $X' = p(X)$. Clearly, $\triangle a_1a_2a_3$ is a triangle of X' of maximum area.

Let mn be a line parallel to a_1a_2 and let oq be parallel to a_1a_3, such that $d(n, b_1b_2) = d(q, b_1b_3) = 1+\sqrt{5}$ (see Fig. 3). Then X' must be contained either in the quadrilateral mnb_1b_2 or in the quadrilateral qob_3b_1. Otherwise, there exist points $d, e \in X'$ such that

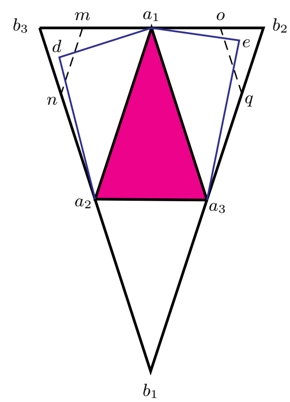

Fig. 3

$d(e, b_1b_3) > 1 + \sqrt{5}$ and $d(d, b_1b_2) > 1 + \sqrt{5}$. In this case we have

$$\frac{|a_1a_2a_3|}{|a_1da_2a_3e|} < \frac{1}{\sqrt{5}},$$

contradicting Straus's lemma. Then, without loss of generality, we can assume that X' is contained in the quadrilateral mnb_1b_2. Since the strip with boundary lines mn and b_1b_2 contains X', it follows that the strip with boundary lines $p^{-1}(mn)$ and $p^{-1}(b_1b_2)$ contains X and this strip has width at most $1 + \sqrt{5}$. □

Now, as we have said before, by replacing $T(3)$ by $T(4)$ we obtain the following result which was conjectured by Eckhoff [1]:

Theorem 1. *Let $\mathcal{F}$ be a finite family of unit discs in the plane. If $\mathcal{F}$ has property $T(4)$ then there exists a parallel strip of width $1 + \sqrt{5}$ which contains the set of centers.*

Proof of Theorem 1. Let $\triangle x_1x_2x_3$, where $\{x_1, x_2, x_3\} \subset X$, be a triangle of maximum area. By Lemma 1 we may assume that $\triangle x_1x_2x_3$ has exactly one altitude ≤ 2, so we may assume $(1 + \sqrt{5})/2 < d(x_2, x_1x_3) \leq 2$.

As in the proof of Lemma 1 we have that $X \subset \triangle y_1y_2y_3$, where $\triangle y_1y_2y_3$ is the homothetic copy of $\triangle x_1x_2x_3$ with center of homothety the centroid of $\triangle x_1x_2x_3$ and constant of homothety equal to -2. Clearly, there must exist an $x \in X$ with $d(x, y_1y_3) > 1 + \sqrt{5}$, or else we are done, so let $e \in X$ be a point that maximizes this distance (see Fig. 4).

Since every four points of X are contained in a strip of width 2 and the triangle $\triangle x_1x_2x_3$ has exactly one altitude ≤ 2, we have either $d(x_2, x_3e) \leq 2$ or $d(x_2, x_1e) \leq 2$. Without loss of generality we may assume that $d(x_2, x_3e) \leq 2$.

Claim 1. *There exists a parallel strip which contains X, whose boundary lines are parallel to x_3e and whose width is less than $1 + \sqrt{5}$.*

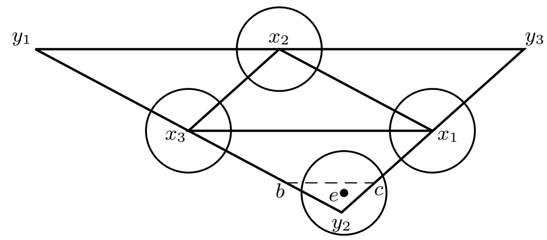

Fig. 4

Proof of Claim 1. Let $p\colon \mathbb{E}^2 \longrightarrow \mathbb{E}^2$ be an affine transformation with $x_i' = p(x_i)$, for $i = 1, 2, 3$, such that $|x_1'x_2'| = |x_1'x_3'|$, $|x_1'x_2'|/|x_2'x_3'| = (1 + \sqrt{5})/2$, and $|x_2'x_3'| = 2(\sin 72^\circ)^{-1}$. Also, let $y_i' = p(y_i)$, for $i = 1, 2, 3$, $e' = p(e)$, and $X' = p(X)$. We consider points $f \in [x_3', y_2']$, $g \in [x_1', y_2']$, $p \in [x_2', y_3']$, and $q \in [x_1', y_3']$, such that $d(f, y_1'y_3') = d(g, y_1'y_3') = d(p, y_1'y_2') = d(q, y_1'y_2') = 1 + \sqrt{5}$ (see Fig. 5). Let $m \in [f, y_2']$ and $n \in [g, y_2']$ be points such that mn is parallel to $y_1'y_3'$ and $e' \in [m, n]$. We deduce X' must be contained in the quadrilateral $mny_3'y_1'$. Now, we consider points $r \in [x_2', p]$, $s \in [x_1', q]$, such that rs is parallel to $y_1'y_2'$ and $|qs| = |gn|$. Since $\triangle x_1'x_2'x_3'$ is a triangle of maximum area we know by Straus's lemma that any point of X' must be below the line rs. This implies that X' must be contained in the pentagon $y_1'mnsr$. Let S_1 be the parallel strip whose boundary lines are l_1 and l_2, with $m \in l_1$, $r \in l_2$, and l_1 and l_2 parallel to $x_3'e'$. We then have $X' \subset S_1$ which implies that $X \subset p^{-1}(S_1)$. Now, we will prove that $p^{-1}(S_1)$ has width less than $1 + \sqrt{5}$.

Let t be the point of segment $[p, q]$ such that $|pt| = |pr| = |fm|$ (see Fig. 6). Thus the segments $[t, m]$, $[p, f]$, and $[x_2', x_3']$ are parallel. Since $|x_1'g| > \frac{1}{2}|x_1'y_2'|$ it follows that $\angle x_1'x_3'e' > \frac{1}{2}\angle x_1'x_3'y_2' = \angle prt$. This in turn implies that l_2 intersects the interior of

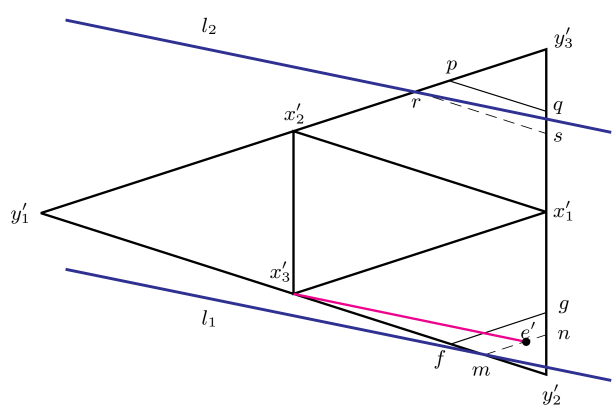

Fig. 5

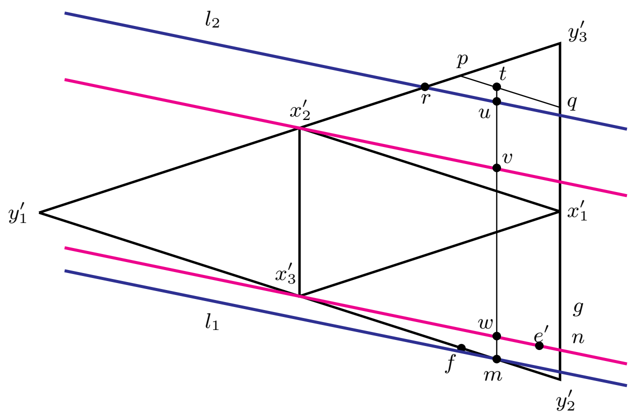

Fig. 6

the segment $[t, m]$ in a point u. Moreover, let $x_2'v$ be parallel to $x_3'e'$, with $v \in [t, m]$, and let $w = [x_3', e'] \cap [t, m]$.

Let S_2 be the strip whose boundary lines are $x_2'v$ and $x_3'w$. Since $|vw| = |x_2'x_3'| = |x_1'y_2'|$, $|tm| = |qy_2'|$, and $|qy_2'|/|x_1'y_2'| = (1+\sqrt{5})/2$, we have

$$\frac{|um|}{|vw|} < \frac{|tm|}{|vw|} = \frac{|qy_2'|}{|x_1'y_2'|} = \frac{1+\sqrt{5}}{2};$$

this implies that the ratio between the width of S_1 and the width of S_2 is less than $(1+\sqrt{5})/2$. As affine transformations preserve the ratio between lengths of parallel segments, then we get that the ratio between the width of the strip $p^{-1}(S_1)$ and the width of the strip $p^{-1}(S_2)$ is less than $(1+\sqrt{5})/2$. Since the width of $p^{-1}(S_2)$ is at most 2 (since $d(x_2, x_3e) \leq 2$), the width of $p^{-1}(S_1)$ is less than $1+\sqrt{5}$. □

Thus, we conclude that there exists a parallel strip of width $1+\sqrt{5}$ which contains X. □

Remark 1. Notice that in the proof of Theorem 1 we only require that the translates of B by the vectors x_1, x_2, x_3, and e possess a common transversal, i.e., it is not necessary that the whole family $\mathcal{F}$ possesses the property $T(4)$. Furthermore, since we know that the number $(1+\sqrt{5})/2$ is necessary for the pentagonal example, we have proved that $\lambda(B, 4) = (1+\sqrt{5})/2$.

With stronger conditions it is possible to obtain a better constant, $2\sqrt{2}$, although this constant could probably be reduced. We obtain this result by a nice application of the following theorem due to Hadwiger and Debrunner [4]:

Hadwiger–Debrunner's Theorem. *Given any family of parallelograms with parallel edges, such that any three can be intersected by an ascending line, there exists an ascending line intersecting all the parallelograms.*

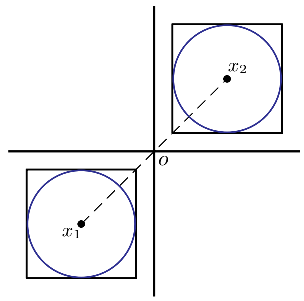

Fig. 7

Theorem 2. *Let $\mathcal{F}$ be a family of unit discs in the plane which has property $T(5)$. Then there exists a parallel strip of width $2\sqrt{2}$ which contains the set of centers.*

Proof of Theorem 2. Let x_1, x_2 be points in the set of centers, X, which are further apart and let o be the midpoint of the segment $[x_1, x_2]$. If $d(x_1, x_2) \le 2\sqrt{2}$, then for every point $x \in X$ we have that the distance from x to the line perpendicular to x_1x_2 through o is $\le \sqrt{2}$. So the conclusion of the theorem follows. Else, we may assume that $d(x_1, x_2) > 2\sqrt{2}$. Consider the coordinate axes through o in such a way that x_1x_2 is an angle-bisector to the angle formed by the axes (see Fig. 7). With each disc in $\mathcal{F}$ we associate a circumscribed square which has sides parallel to the coordinate axes; in this way we obtain a finite family, F', of translates of a square with side 2.

As we can see, the squares centered at x_1 and x_2 have only ascending transversals. Furthermore, since $\mathcal{F}$ has property $T(5)$, and so $\mathcal{F}'$, we have that every three members of $\mathcal{F}'$ have an ascending transversal. It follows, by Hadwiger–Debrunner's theorem that there exists a line l which is a common transversal to every member of $\mathcal{F}'$. Therefore, any center of a square belonging to $\mathcal{F}'$ has a distance at most $\sqrt{2}$ from l. We conclude that there exists a parallel strip of width $2\sqrt{2}$ which contains the set of centers. □

Now, denote by Q a square of side 2 and consider a family $\mathcal{F}$ of translates of Q. If $\mathcal{F}$ has property $T(6)$ then there exists a line transversal to each member of this family, as was shown by Santaló. Assume that the family has property $T(3)$. Then we obtain that the constant 1 in Eckhoff's result is the best possible. The following result was previously proved by Eckhoff [2], however, for completeness we give a proof for it here:

Theorem 3. *Let $\mathcal{F}$ be a finite family of translates of Q which has property $T(3)$. Then there exists a parallel strip of Q-width* 1 *which intersects every member of $\mathcal{F}$. Moreover, this constant cannot be reduced*, *i.e.*, $\lambda(Q, 3) = 2$.

Proof of Theorem 3. By Eckhoff's theorem we know that there exists a parallel strip of Q-width at most 2 which contains the set of centers. Let $\mathcal{F} = \{P_1, P_2, P_3, P_4\}$ be a family consisting of four translates of the unit square, where the translates have centers x_1, x_2, x_3, and x_4, respectively. Choose the translates so that $x_1x_2x_3x_4$ is a square of side

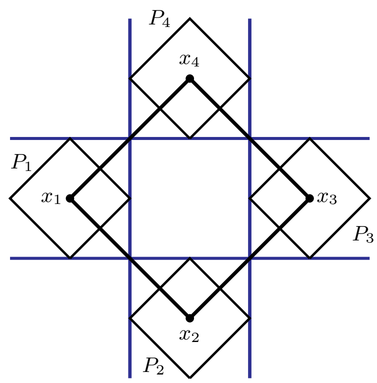

Fig. 8

2 whose sides are parallel to the sides of the unit square. As we can see in Fig. 8, $\mathcal{F}$ has property $T(3)$ but no property T, i.e., there exists no common transversal for the family $\mathcal{F}$. Since any parallel strip containing the set $\{x_1, x_2, x_3, x_4\}$ must contain the square $x_1x_2x_3x_4$, it must have Q-width at least 2, therefore, the number 1 given in the theorem cannot be reduced. □

Remark 2. Let K be a convex body in the plane. The following was observed by Eckhoff:

$$\lambda(K, 3) = 2 \qquad \text{if and only if } K \text{ is a parallelogram.}$$

3. Final Remarks

The Eckhoff–Dolnikov conjecture remains open for the case where the triangle of maximum area has only one altitude ≤ 2. It is possible that the techniques used in this paper could be useful for the proof of this conjecture.

Letting $\mathcal{F}$ be a finite family of translates of a figure of constant width, we find that Lemma 1, Theorem 1, and Theorem 2 are still valid. This can be seen from the following reduction noted by Tverberg [8]: Let K be a convex set and let K' be the centrally symmetric set obtained by setting $K' = \frac{1}{2}(K + (-K))$. Let $\mathcal{F} = \{x_i + K\colon i \in I\}$ be a family of translates of K and let $\mathcal{F}' = \{x_i + K'\colon i \in I\}$ be the associated family of translates of K'. Then it is easily seen that $\mathcal{F}$ and $\mathcal{F}'$ share the same properties with respect to disjointness, transversals, and GPs. We know that in the case when K is a figure of constant width, K' is a euclidean disc, hence our assertion follows.

Obviously, the number $2\sqrt{2}$ in Theorem 2 could be reduced using better arguments.

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