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Geometric Graphs with No Three Disjoint Edges

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Abstract. A *geometric graph* is a graph drawn in the plane so that the vertices are represented by points in general position and edges are represented by straight line segments. We show that a geometric graph on n vertices with no three pairwise disjoint edges has at most $2.5n$ edges. This result is tight up to an additive constant.

1. Introduction

An (*abstract*) *graph* G is a pair $(V(G), E(G))$ where $V(G)$ is the set of *vertices* and $E(G)$ is the set of *edges* $\{u, v\}$ each a two element subset of $V(G)$.

A *geometric graph* is a graph G drawn in the plane by straight line segments. It is defined as a pair $(V(G), E(G))$, where $V(G)$ is a finite set of points in general position in the plane, i.e. no three points are collinear, and $E(G)$ is a set of line segments with endpoints in $V(G)$. $V(G)$ and $E(G)$ are the *vertex set* and the *edge set* of G, respectively. Let H and G be two geometric graphs. We say that H is a (*geometric*) *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

A *topological graph* is defined similarly. It is a graph drawn in the plane in such a way that edges are Jordan curves. Two of these curves share at most one point and no curve passes through a vertex. Obviously, geometric graphs are a subclass of topological graphs. We say that two edges *cross* each other if they have an interior point in common. Two edges are *disjoint* if they have no point in common.

We investigate properties of subclasses of geometric or topological graphs with some geometrical constrains. One of the simplest questions is how to characterize graphs with no crossing edges. These graphs are known as *planar* graphs and have been studied for more than a hundred years.

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Kupitz, Erdős and Perles initiated, and many others continued, the investigation of the following general problem. Given a class $\mathcal{H}$ of so-called forbidden geometric subgraphs, determine or estimate the maximum number $t(\mathcal{H}, n)$ of edges that a geometric graph with n vertices can have without containing a subgraph belonging to $\mathcal{H}$.

There are many nice results for various forbidden classes—k pairwise crossing edges, k pairwise "parallel" edges, k pairwise disjoint edges, self-crossing paths, even cycles and many others (see [1], [2], [7], [8], [11], and [15]–[17]). For a survey of results on geometric graphs see [10].

We focus on geometric graphs with no $k + 1$ pairwise disjoint edges. For $k \geq 1$, let $\mathcal{D}_k$ denote the class of all geometric graphs consisting of k pairwise disjoint edges. Denote by $d_k(n) = t(\mathcal{D}_{k+1}, n)$ the maximum number of edges of a geometric graph on n vertices with no $k + 1$ pairwise disjoint edges.

Let us look at the history of this problem. One of the first investigations on geometric graphs, besides planar graphs, was motivated by repeated distances in the plane. Erdős asked how many times can the maximum distance among n points in the plane be repeated. Connect each pair of points with the maximum distance by an edge. It is clear that the resulting graph cannot have two disjoint edges. The convex hull of endpoints of two disjoint edges forms either a triangle or a quadrilateral. In both cases there is a distance longer than the length of the edge. That is a contradiction. The former question turns to the following: How many edges can a geometric graph with no two disjoint edges have? Erdős [4] proved the following theorem:

Theorem 1. $d_1(n) = n$.

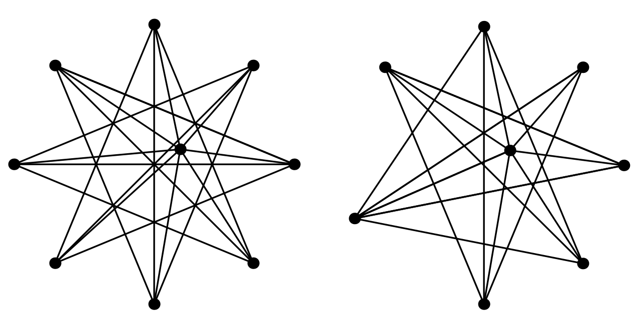
Alon and Erdős [3] (1989) proved $d_2(n) \leq 6n$. One year later O'Donnell and Perles (1990) improved it to $d_2(n) \leq 3.6n + c$. Later Goddard et al. [6] (1993) showed $d_2(n) \leq 3n$. At the end Mészáros [9] improved that to $d_2 \leq 3n - 1$. Combining some of the ideas of the proof of Goddard et al. [6] with a discharging method we show the following upper bound:

Theorem 2. $d_2(n) \leq \lfloor 2.5n \rfloor$.

The best known lower bound is due to Perles:

Theorem 3 (Perles). $d_2(n) \geq \lceil 2.5n \rceil - 3$.

Examples of such a graph for $n = 9$ and for $n = 8$ are given in the following figure (the construction for odd n can be easily generalized and the graph for even n is obtained from the previous one by the contraction of two neighboring vertices on the convex hull):



For $d_3(n)$ Goddard et al. [6] showed $3.5n \leq d_3(n) \leq 10n$. Later Tóth and Valtr [14] improved that to $4n - 9 \leq d_3(n) \leq 8.5n$.

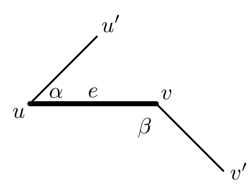
The first general upper bound $d_k(n) = O(n(\log n)^{k-3})$ was given again by Goddard et al. [6]. In 1993 Pach and Törőcsik [12] introduced the order relations on disjoint edges and as an application of Dilworth's theorem they showed that $d_k(n) \leq k^4 n$. That was the first upper bound linear in n. Tóth and Valtr [14] added a concept of zigzag and improved the bound to $d_k(n) \leq k^3(n + 1)$. Later Tóth [13] further improved the bound to $d_k(n) \leq 256k^2 n$. The original constant in Tóth's proof was a bit bigger. This one is due to Felsner [5].

It is believed that $d_k(n) \sim ckn$ in reality. Another interesting problem is to determine if it is true for those geometric graphs all of whose edges can be intersected by a line. If so, this would give an upper bound of $d_k(n) \leq c(k \log k)n$ for the general case. Just bisect the vertex set of the graph and count edges in both parts recursively.

2. Preliminaries

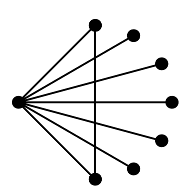
We show the proof of Theorem 1 because it is a beautiful illustration of a discharging method and it is very simple. Later we use a similar but more complicated approach to prove Theorem 2—the upper bound for $d_2(n)$.

Proof of Theorem 1 (Perles). For each vertex mark one edge incident to it. For vertices of degree one, there is no choice. At the other vertices, mark the right edge at the largest angle. If there remains an unmarked edge $e = uv$, we have the situation as in the following figure:



There must be edges uu' and vv' because we marked the right edge at the largest angle at every vertex. Angles α and β are less than the largest angles at the vertices u and v, so they are less than π. However, this implies that edges uu' and vv' are disjoint and so there cannot be an unmarked edge in the graph. Thus, the number of edges is not more than the number of vertices.

On the other hand there exist graphs achieving this bound:



□

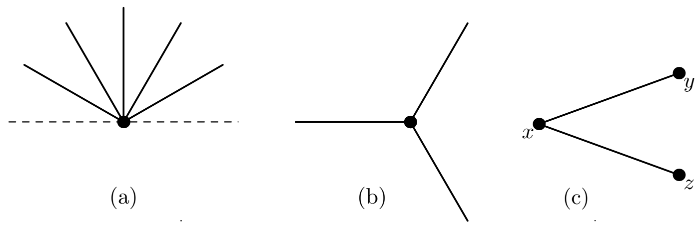


Fig. 1. (a) An example of a pointed vertex, (b) an example of a cyclic vertex and (c) there are edges xy and xz where edge xy is to the left of edge xz.

Definition 1. A vertex v is pointed if all edges incident to it lie in a halfplane whose boundary contains vertex v (see Fig. 1(a)).

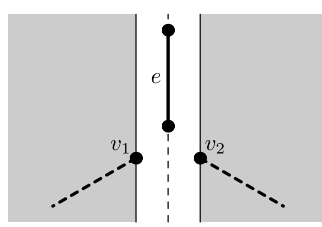
Definition 2. A vertex v which is not pointed is cyclic. This means that in every open halfplane determined by a line passing through vertex v there is an edge incident to v (see Fig. 1(b)).

Definition 3. We say that an edge xy is to the left of an edge xz if the ray $\overrightarrow{xz}$ can be obtained from the ray $\overrightarrow{xy}$ by a clockwise turn of less than π. Similarly we define when an edge is to the right of another edge (see Fig. 1(c)).

3. Cyclic Vertices

Lemma 1. *A geometric graph with two cyclic vertices and an edge, whose supporting line strictly separates these two cyclic vertices, contains three pairwise disjoint edges.*

Proof.



In the above picture there are two cyclic vertices and an edge which separates them strictly. We can find an edge in each grey halfplane, because the vertices v_1 and v_2 are cyclic. This yields three pairwise disjoint edges. □

Lemma 2. *A geometric graph with three cyclic vertices contains three pairwise disjoint edges.*

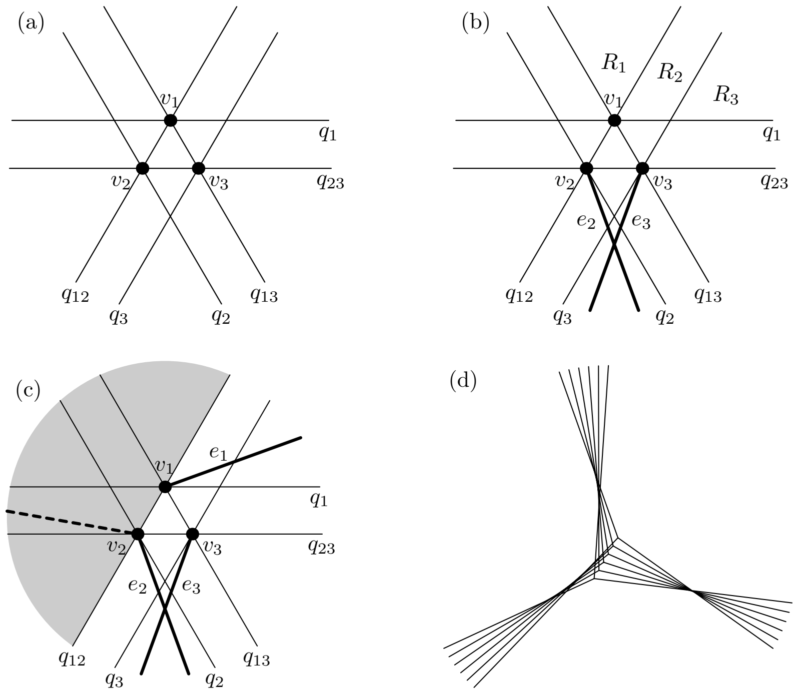


Fig. 2. (a) Position of three cyclic vertices. (b) Edge e_1 can without loss of generality lie only in regions R_1, R_2 and R_3. (c) If edge e_1 lies in regions R_2 and R_3, there must be an edge in the open halfplane determined by q_{12}, which does not cross edges e_1 and e_3. These edges are pairwise disjoint. (d) For $k \geq 4$, there are geometric graphs with k cyclic vertices and without four pairwise disjoint edges. This figure is for $k = 6$.

Proof. Denote the cyclic vertices by v_1, v_2 and v_3. Let q_i be the line passing through v_i parallel to the line passing through the other two cyclic vertices (for $i = 1, 2, 3$). See Fig. 2(a).

There must be an edge e_i incident to v_i lying in the open halfplane determined by the line q_i, not containing the other cyclic vertices. At least two edges from e_i, $i = 1, 2, 3$, must cross. Otherwise we have three disjoint edges. Without loss of generality edges e_2 and e_3 cross. Where can the third edge lie? See Fig. 2(b).

If e_1 lies in region R_1, use Lemma 1. In the other cases there must be an edge in the open halfplane determined by the line q_{12}, not containing other cyclic vertices, because vertex v_2 is cyclic. See Fig. 2(c). This edge and edges e_1 and e_3 are pairwise disjoint. $\square$

We note that for $k \geq 4$ there is a geometric graph G with k cyclic vertices without four pairwise disjoint edges (see Fig. 2(d)).

4. The Upper Bound

In this section we prove Theorem 2: $d_2(n) \leq \lfloor 2.5n \rfloor$.

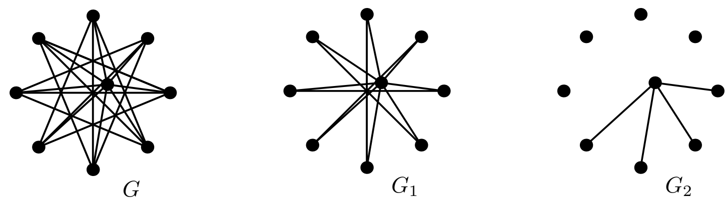


Fig. 3. (Example) Graphs G, G_1 and G_2. In the first round delete the rightmost edge at each pointed vertex of G and obtain graph G_1. In the second round delete the leftmost edge at each pointed vertex of G_1 and if there are two vertices c and d cyclic in G then for each vertex cyclic in G_1 delete the edge to the left of segment cd. We obtain graph G_2.

4.1. *Sketch of the Proof*

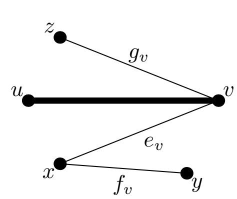
Let $G = (V, E)$ be a geometric graph with no three disjoint edges. Denote the number of cyclic vertices in G by γ. We know by Lemma 2 that $\gamma \le 2$. We construct two subgraphs $G_i = (V_i, E_i)$, $i = 1, 2$, as follows. For each pointed vertex in G delete the rightmost edge. Denote the resulting graph by G_1 (see Fig. 3). For each pointed vertex in G_1 delete the leftmost edge (if any). If there are two vertices c and d cyclic in G then for each vertex cyclic in G_1 delete the edge to the left of segment cd. The vertex cyclic in G_1 must be one of the vertices c and d because $\gamma \le 2$. Denote the resulting graph by G_2. Deleting in the second round is for each pointed vertex in G_1 (not in G)!

We show that graph G_2 contains no two disjoint edges (Lemma 4). We have deleted at most $2n - \gamma$ edges to get graph G_2.

Finally we use a discharging method to show that graph G_2 has at most $(n + 2\gamma)/2$ edges (Lemma 8). We then conclude that G has at most $(2n - \gamma) + (n + 2\gamma)/2 \le 2.5n$ edges. We need many auxiliary lemmas to prove that the discharging method works.

4.2. *The Proof*

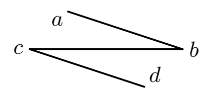
Lemma 3. *For each vertex v and edge $e = uv \in E_2$ containing v there exist vertices $x, y, z \in V$ and edges $e_v(e)$, $f_v(e)$, $g_v(e) \in E$ such that $e_v(e) = vx$ is to the left of edge uv, edge $f_v(e) = xy$ is to the right of edge $e_v(e)$ and edge $g_v(e) = vz$ is to the right of edge uv. If vertex v is pointed in G, then edges $e_v(e)$ and $g_v(e)$ are determined uniquely as the edges $e_v \in E_1 - E_2$ and $g_v \in E - E_1$ deleted at vertex v. If vertex x is pointed in G, then edge $f_v(e)$ is determined uniquely as the edge $f_v \in E - E_1$ deleted at vertex x.*



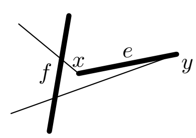
Proof. We start in vertex v. If v is pointed in G_1, then there must be an edge e_v to the left of edge uv, because we had deleted the leftmost edge in the second round. If v is cyclic in G_1, then e_v exists by the definition of a cyclic vertex. Similarly, there must be edges f_v and g_v to the right of edges e_v and uv because we had deleted the rightmost edges in the first round or by the definition of a cyclic vertex. It can happen that $y = z$ or that f_v crosses g_v. □

We use the following arguments to show that some edges are disjoint:

Observation 1 (Z Argument). *If a, b, c, d is a path in G and ab lies to the right of bc and cd lies also to the right of bc (the shape of the letter Z), then edges ab and cd are disjoint. Similarly if the edges lie to the left of edge bc.*



Observation 2 (Both Ends in One Halfplane Argument). *Let $e = xy$ and f be edges. If there exist rays beginning in the vertices x, y which intersect the supporting line of edge f from the same side, then edges e and f are disjoint.*

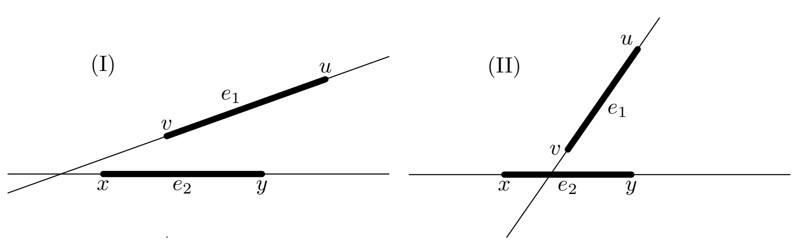


When we consider some of the edges e_v, f_v and g_v in the following proofs, we know by Lemma 3 that they exist. Sometimes we omit verification of the fact that three edges from the proof are pairwise disjoint. It is a direct application of previous arguments and we leave it to the reader.

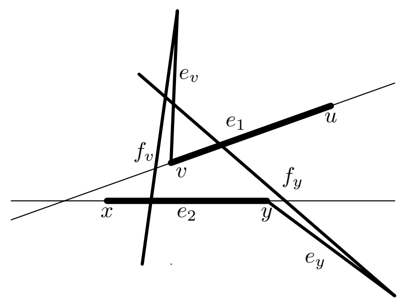
In the figures the edges of G_2 are drawn by thick lines. When we want to show three disjoint edges in some figure, they are drawn in grey.

Lemma 4 (About Two Disjoint Edges in G_2). *Graph G_2 contains no two disjoint edges.*

Proof. (By contradiction) Assume that there are two disjoint edges uv and xy in G_2. There are two possible positions of two disjoint edges. Either the line supporting one edge intersects the second edge or it does not. See the following figure:

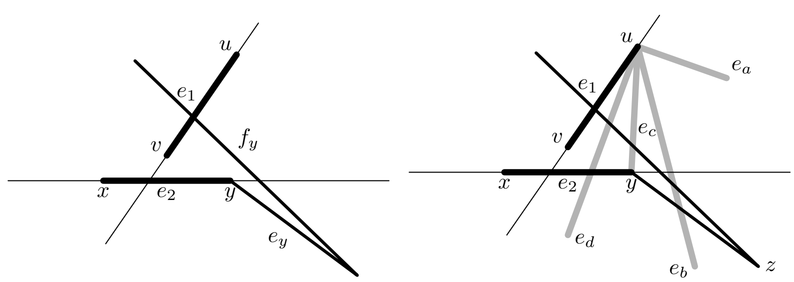


Case I. Edge f_v cannot be disjoint with edge e_2, otherwise we have three disjoint edges—e_1, e_2 and f_v. If edge f_v is incident to vertex y, there are two disjoint edges e_1 and f_v in one halfplane determined by edge e_2. Then consider edge g_x incident to vertex x lying in the opposite halfplane and we get three disjoint edges. Similarly for edge f_y. See the following figure:



Denote the supporting lines of edges f_v and f_y by p and q. Assume without loss of generality that the lines p and q cross above edge e_1 (or at infinity). Then edges f_v, e_y and e_1 are pairwise disjoint.

Case II. Edge f_y cannot be disjoint with edge e_1 again because otherwise we have three disjoint edges e_1, e_2 and f_y. The case when f_y leads to an end vertex of edge e_1 can be handled almost in the same way as the case of a proper crossing. See the following figure on the left:



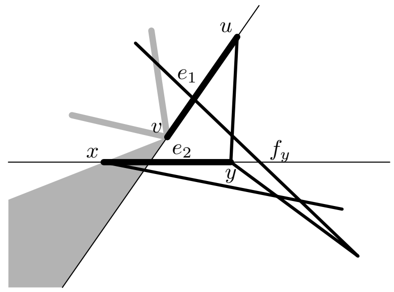
Where can edge e_u lie? Possible cases according to the angle with edge e_1 are in the above figure on the right.

(a) If the ray determined by e_u crosses neither edge e_y nor edge e_2 we have three disjoint edges—e_a, e_2 and f_y. In the case when f_y leads to vertex u, we have to consider edges e_u and e_y which are disjoint because of the Z argument and edge e_v which lies in the opposite halfplane determined by edge e_1.

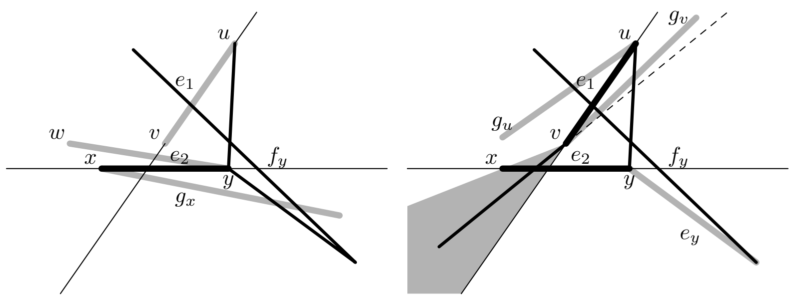
(b) If the ray crosses edge e_y or passes through vertex z then edges e_1 and e_2 lie in one halfplane determined by edge e_u. Moreover, edge f_u lies in the opposite halfplane. This yields three disjoint edges.

(c) Vertex y is cyclic in G, because edge e_2 does not cross edge f_y, but its supporting line crosses it between the crossings with the other edges incident to y. Note that both edges $e_c, e_y \in E_1$ so vertex y is cyclic in G_1 too.

If edge e_v is disjoint with edge xy, we have three disjoint edges (see the following figure): Either e_v is disjoint with edge f_y and we take the edges e_v, f_y and xy, or e_v crosses edge f_y and then we take e_v, uy and g_x.



Otherwise e_v crosses edge xy and lies in the grey area in the figure. In that case vertex v is either cyclic in G or edge g_v lies between edge e_1 and the dashed line which is a continuation of e_v.



In the former case (above figure on the left) there must be an edge yw to the left of segment yv and to the right of edge yx, because there are two cyclic vertices y and v in G, y is cyclic in G_1 and the edge at vertex y to the left of segment yv was deleted to get graph G_2. Then there are two disjoint edges e_1 and yw in one halfplane determined by edge e_2. In the opposite halfplane there is edge g_x. These edges are pairwise disjoint.

In the latter case (previous figure on the right) edges g_u, g_v and e_y are pairwise disjoint.

(d) If the ray crosses edge e_2 there are two disjoint edges e_y and e_u in one halfplane determined by edge e_1. In the opposite halfplane, there is edge e_v. Again we have three disjoint edges.

In all possible cases we found three disjoint edges. That contradicts the assumption that G does not have three disjoint edges. $\square$

Corollary 1. *Graph G_2 has no two disjoint edges. Therefore each vertex is pointed in G_2 or all edges of G_2 form a star with a cyclic vertex in G_2 as a root.*

Corollary 2. *Any geometric graph G on n vertices with no three pairwise disjoint edges has at most $3n - \gamma$ edges.*

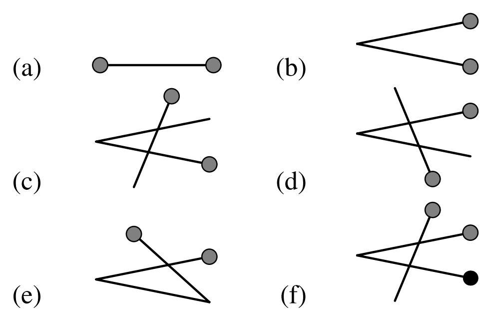
Proof. We have deleted at most $2n-\gamma$ edges of G to get graph G_2 with no two disjoint edges. By Theorem 1 graph G_2 has at most n edges. Thus, graph G has at most $3n-\gamma$ edges. □

For each vertex v which is neither isolated nor cyclic in G_1 we say that the second endpoint of the edge $e_v \in E_1 - E_2$ is the *partner vertex* $\bar{v}$ of vertex v. The other vertices do not have any partner vertex. We say that two vertices *share* their partner if they have a common partner vertex.

The idea of the proof is to show that partner vertices are isolated in G_2. Then if the partner vertices are not shared, the number of vertices contained in some edge of G_2 is less than or equal to the number of vertices that are either isolated in G_2 or cyclic in G. That would yield the bound $n/2$ on the number of edges in G_2.

In the following lemma we show the cases in which the partner vertices are not shared. Its proof is rather technical and is based on the case study.

Lemma 5 (Sharing of Partner Vertices). *For any edges in graph G_2 which are in one of the positions* (a)–(e), *the grey vertices do not share their partner. In position* (f) *the grey vertices do not share their partner or the black vertex's partner is isolated in G_2 or the black vertex is cyclic in G.*



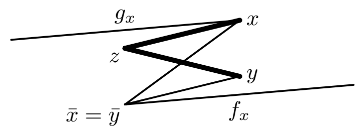
Lemma 6 (Auxiliary). *Let $e = zx$ and $f = zy \in E_2$ be two adjacent edges. Edges e_x and e_y are not disjoint.*

Proof. If e_x and e_y are disjoint, then consider these two edges and edge e_z. These edges are pairwise disjoint in G. □

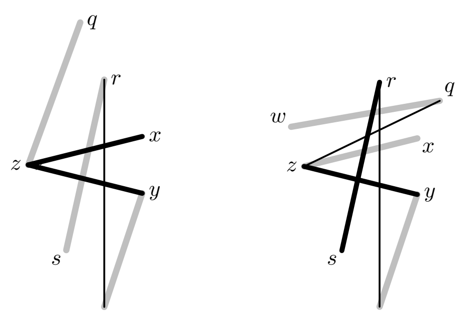
Proof of Lemma 5. (By contradiction) The basic scheme of the proof for positions (b)–(f) is the following: If one of the grey vertices has no partner vertex, the two grey vertices cannot share a partner vertex. Assume that in each position the two grey vertices have a partner vertex and they share it. Then we find three disjoint edges in G and that is the contradiction.

(a) If there is an edge xy connecting two vertices in G_2 then edges e_x and e_y lie in the opposite halfplanes determined by edge xy.

(b) If vertices x and y share their partner $\bar{x} = \bar{y}$, then there are three disjoint edges g_x, zy and f_x (disjoint because of the Z argument). It is not necessary that all edges which cross in the figure must cross (i.e. e_x and zy), but the same choice of disjoint edges works.

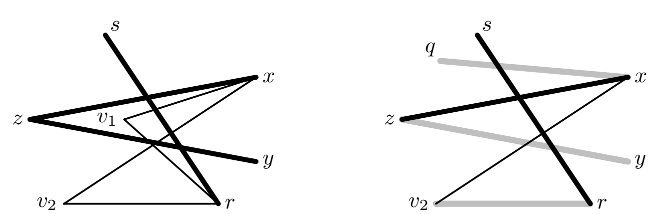


(c) Edges xz, zy and rs are the edges of position (c). Assume that r and y share their partner vertex.



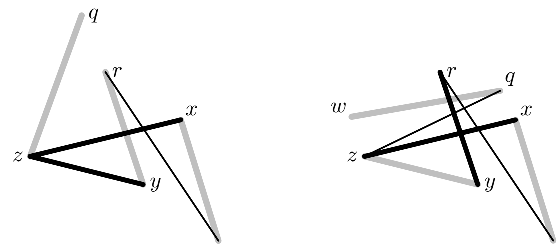
Consider the edge $zq = e_z$. Edges zq and rs are either disjoint or not. In the first case, which is on the left, there are three disjoint edges zq, rs and $y\bar{y}$. The second case, which includes the case $r = q$, is on the right. There are three disjoint edges wq, zx and $y\bar{y}$ in G.

(d) There are two possible positions of the shared vertex—v_1 and v_2 (see the following figure on the left):



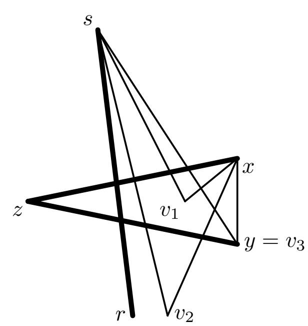
In the first case (shared vertex v_1), use Lemma 6, which says that this cannot happen otherwise we have three disjoint edges. In the second case consider edges g_x, zy and e_r (previous figure on the right). They are pairwise disjoint.

(e) Edges xz, zy and yr are the edges of position (e). Assume that r and x share their partner vertex.



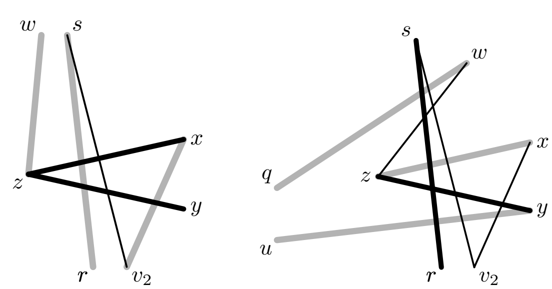
Consider the edge $zq = e_z$. Edges zq and ry are either disjoint or not. In the first case, which is on the left, there are three disjoint edges zq, ry and $x\bar{x}$. The second case, which includes the case $r = q$, is on the right. There are three disjoint edges wq, zy and $x\bar{x}$ in G.

(f) There are three possible positions of the shared vertex $\bar{x} = \bar{y}$. Denote them by v_1, v_2 and v_3. See the following figure:

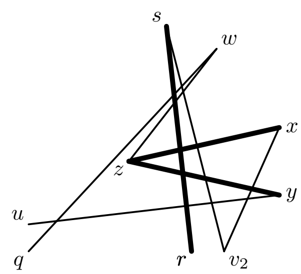


(v_1) Use Lemma 6, which yields three disjoint edges.

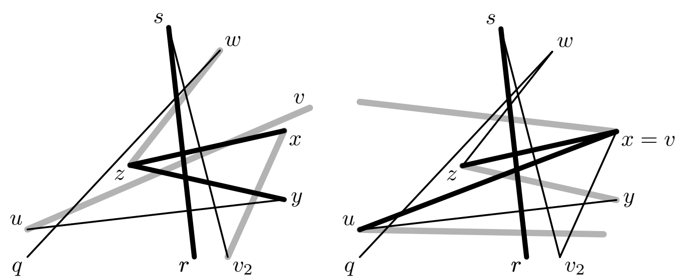
(v_2) Assume that vertex y is not cyclic in G otherwise we are done. We start in the following figure on the left. Consider the edge $zw = e_z$. If edges wz and rs are disjoint, we have three disjoint edges. Otherwise look at the figure on the right. The case $s = w$ is handled in the same way. Vertex y is not cyclic in G hence it has a partner vertex u. Consider the edges $wq = f_z$ and $yu = e_y$. If they are disjoint, we again have three disjoint edges.



Otherwise look at the following figure:



The partner vertex u is either isolated in G_2 as we claim in the lemma or there must be an edge $uv \in E_2$. All edges in G_2 must intersect (Lemma 4). All possible positions of this edge are in the following figures:

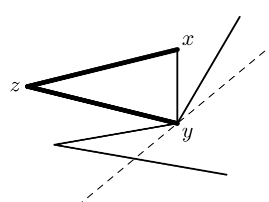


In the case on the left, there are three disjoint edges zw, uv and xv_2. On the right is the case when $x = v$. If vertex u is pointed in G then there is an edge to the right of edge uy because edge uy belongs to graph G_1. Graph G_1 was obtained from G by deleting the rightmost edge at each pointed vertex. Edges g_u, zy and g_x are three disjoint edges.

If vertex u is cyclic in G then there exists an edge $\bar{e}$ below the supporting line of edge uy (by the definition of a cyclic vertex). Edge $\bar{e}$ either intersect edge xv_2 and then edges $\bar{e}$, zy and g_x are pairwise disjoint or $\bar{e}$ does not intersect xv_2 and then edges $\bar{e}$, xv_2 and zw are pairwise disjoint.

In all possible cases we have three disjoint edges.

(v_3) The vertex $y = v_3$ is either cyclic in G or we have three disjoint edges f_y, xz and g_y. Edge g_y lies to the right of edge xy because $xy \in E_1$ and to get graph G_1 the rightmost edge was deleted at vertex y.



□

In the following two corollaries we summarize the results on sharing partner vertices.

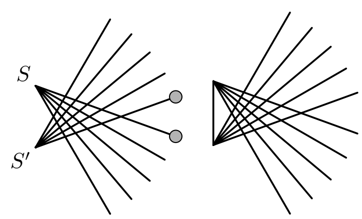
Corollary 3. *In each star $S \subseteq G_2$ there are no two vertices $v_i, v_j \in S$ which share their partner $\bar{v}_i = \bar{v}_j$.*

Proof. Apply Sharing Lemma 5(b) to each pair of leaves. A leaf and the root of the star cannot share their partner either, because they are joined by an edge (Lemma 5(a)). □

Corollary 4. *Let S and S' be two different stars in G_2.*

(i) *If the root vertices of these stars are not joined by an edge in G_2, then there are at most two leaves $u \in S$ and $v \in S'$ that can share their partner $\bar{u} = \bar{v}$. Vertex u is the leftmost leaf in one star and vertex v is the rightmost leaf in the second star (grey vertices in the following figure on the left). Moreover, if vertices u and v share their partner, then all the remaining leaves in star S are cyclic in G or have a partner vertex isolated in G_2.*

(ii) *If the root vertices are joined by an edge in G_2 then no two vertices of stars S, S' can share a partner vertex (figure on the right).*



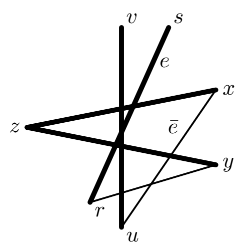
Proof. (i) Apply one of the Sharing Lemmas 5(b)–(d) to each other pair of leaves to show the first part. Apply Lemma 5(f) to the grey vertices and another leaf of S to show that this leaf is cyclic in G or it has a partner vertex isolated in G_2. (ii) Use the result of case (i) and for the other pairs apply Lemma 5(e) or 5(a). □

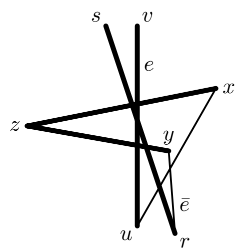
Now we show that there are many isolated vertices in G_2.

Lemma 7. *Let $zx,\ zy \in G_2$ be two adjacent edges. At least one vertex of x or y has a partner vertex isolated in G_2 or is cyclic in G.*

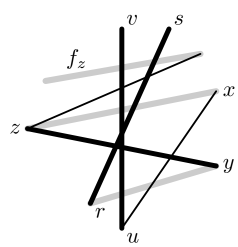
Proof. (By contradiction) The edges $xu = e_x$ and $yr = e_y$ are neither disjoint by Lemma 6 nor lead to a common vertex by Lemma 5(b).

Let us assume that x and y are not cyclic in G and that both partners of vertices x and y are not isolated in G_2. So there must be edges in G_2 containing partner vertices. These edges must intersect all the other edges of G_2 (Lemma 4), so except for the special cases, when some vertices are equal, there are only two possible positions. They are on the following figures:

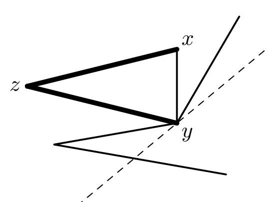


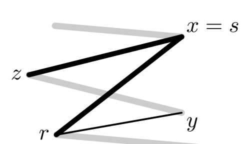


We choose one edge from rs and uv and one edge from xu and yr in such a way that we have two disjoint edges. Denote the first chosen edge by e and the second chosen edge by $\bar{e}$. Then consider edge e_z. If it is disjoint with edge e then we have three disjoint edges e, $\bar{e}$ and e_z. Otherwise consider edge f_z and take three disjoint edges f_z, zx and ry (see the following figure):



There are some special cases when some of the vertices are equal. In case $v = s$, the former approach works. In the following figure on the left we show the case when $u = y$. Vertex y is either cyclic in G or we have three disjoint edges f_y, g_y zx. Edge g_y lies to the right of edge xy because $xy \in E_1$ and to get graph G_1 the rightmost edge was deleted at vertex y. The same approach will work also for the cases $v = z$ and $s = z$. On the right is the last case when $s = x$. We again have three disjoint edges g_x, zy and g_r.





□

Corollary 5. *In any star $S \subseteq G_2$ all but at most one leaf are cyclic in G or have a partner vertex isolated in G_2.*

Proof. Apply Lemma 7 to the pair of leaves that do not satisfy the condition yet. □

Lemma 8. *Graph G_2 has at most $(n + 2\gamma)/2$ edges.*

Proof. We use the discharging method. Give \$1 to each vertex and an additional \$2 to each vertex cyclic in G. We need to pay for all edges of G_2, where each edge costs \$2 (\$1 per each end). So each vertex v must pay \$ $\deg v \cdots$ \$1 for each edge incident to it. How will the vertices pay for all the edges?

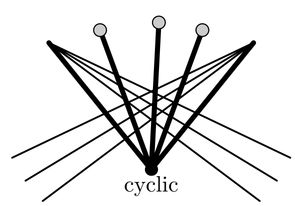
- Isolated vertices do not pay anything. Someone can borrow from them.
- Vertices of degree one pay for themselves.
- Vertices with $\deg v \geq 2$ borrow \$$(\deg v - 1)$ from the partners of their neighbors or cyclic neighbors and pay for one edge themselves. If possible, they pay for the leftmost edge on their own and borrow for the other edges.

It remains to show that this works. Let us look at the vertex of $\deg v \geq 2$. Corollary 5 says that in each star in G_2 all but at most one leaf have either partner vertex isolated in G_2 or are cyclic in G. So v always has someone to ask to borrow.

We must also show that the partner vertices are not lending to more than one vertex and that cyclic vertices in G are not lending to more than two vertices. However, the first condition would mean that the partner vertex is shared by some vertices.

In the case when edges of G_2 form only one star (Lemma 4), use Corollary 3, which says that no two vertices can share their partner vertex. Otherwise look at the vertices which borrowed from the shared partner vertex. These vertices together with their neighbors form two maximal stars S and S' in G_2. Use Corollary 4, which says that the only leaves of two maximal stars S and S' which can share a partner vertex are the leftmost in S and the rightmost in S'. Moreover, this holds only in the case when the roots of considered stars are not joined by an edge. If these two leaves share a partner then we can apply the second part of Corollary 4(i) which says that all the other leaves of the star S are cyclic in G or have a partner vertex isolated in G_2. So the root vertex of S can borrow from them and pay for the leftmost edge on his own.

Vertices cyclic in G cannot lend more than they have either. Each cyclic vertex has \$2 to lend directly to the vertices which need help. There cannot be more than two such vertices. See the following figure:



All neighbors of cyclic vertex in G_2 except for the outer ones must have degree one. Otherwise there are two disjoint edges in graph G_2. Vertices with degree one pay for themselves and do not need any help.

Altogether vertices have \$$(n+2\gamma)$; we paid for all edges so we have at most $(n+2\gamma)/2$ edges. □

We have deleted at most $2n-\gamma$ edges to get graph G_2. Graph G_2 has at most $(n+2\gamma)/2$ edges. Thus graph G has at most $2.5n$ edges. That yields the bound $d_2(n) \leq \lfloor 2.5n \rfloor$.

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References

1. P. K. Agarwal, B. Aronov, J. Pach, R. Pollack, and M. Sharir: Quasi-planar graphs have a linear number of edges, *Combinatorica*, **17** (1997), 1–9.
2. J. Akiyama and N. Alon: Disjoint simplices and geometric hypergraphs, in *Combinatorial Mathematics*, G. S. Blum et al., eds., Annals of the New York Academy of Sciences, vol. 555, New York Academy of Sciences, New York, 1989, pp. 1–3.
3. N. Alon and P. Erdős: Disjoint edges in geometric graphs, *Discrete Comput. Geom.*, **4** (1989), 287–290.
4. P. Erdős: On the set of distances of n points, *Amer. Math. Monthly*, **53** (1946), 248–250.
5. S. Felsner: *Geometric Graphs and Arrangements*, Advanced Lectures in Mathematics, Vieweg Verlag, Braunschweig, 2004.
6. W. Goddard, M. Katchalski, and D. J. Kleitman, Forcing disjoint segments in the plane, *European J. Combin.*, **17** (1996), 391–395.
7. Y. S. Kupitz: *Extremal Problems in Combinatorial Geometry*, Aarhus University Lecture Notes Series 53, Aarhus University, Aarhus, 1979.
8. Y. S. Kupitz and M. Perles: Extremal theory for convex matchings in convex geometric graphs, *Discrete Comput. Geom.*, **15** (1996), 195–220.
9. Z. Mészáros: Geometrické grafy, Diploma work, Charles University, Prague, 1998 (in Czech).
10. J. Pach: Geometric graph theory, in *Surveys in Combinatorics*, 1999 (*Canterbury*). London Mathematical Society Lecture Note Series, 267, Cambridge University Press, Cambridge, 1999, pp. 167–200.
11. J. Pach and P. K. Agarwal: *Combinatorial Geometry*, Wiley Interscience, New York, 1995.
12. J. Pach and J. Törőcsik: Some geometric applications of Dilworth's theorem, *Discrete Comput. Geom.*, **12** (1993), 1–7.
13. G. Tóth: Note on geometric graphs, *J. Combin. Theory Ser. A*, **80** (2000), 126–132.
14. G. Tóth and P. Valtr: Geometric graphs with few disjoint edges, *Discrete Comput. Geom.*, **22** (1999), 633–642 (also *Proc. Ann. Symp. Comput. Geom.*, 1998, pp. 181–194).
15. P. Valtr: Graph drawings with no k pairwise crossing edges, in *Graph Drawing* (*Rome*), Lecture Notes in Computer Science, vol. 1353, Springer, Berlin, 1997, pp. 205–218.
16. P. Valtr: On geometric graphs with no k pairwise parallel edges, *Discrete Comput. Geom.*, **19** (1998), 184–191.
17. P. Valtr: Generalizations of Davenport–Schinzel sequences, in *Contemporary Trends in Discrete Mathematics*, R. Graham et al., eds., DIMACS, Series in Discrete Mathematics & Theoretical Computer Science, vol. 49, AMS, Providence, RI, 1999, pp. 349–389.

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