

New Upper Bound on the Transversal Width of $T(3)$ -Families of Discs*

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Abstract. A result of Eckhoff [3] implies that to every finite $T(3)$ -family of pairwise disjoint copies of a closed disc of unit diameter there exists a strip of width 1 meeting all members of the family. Our goal is to generalize this result giving a stricter upper bound by proving that the narrowest transversal strip has width < 0.65 .

1. Introduction, Definitions, Results

Throughout this paper the term *disc* is used for a closed solid circle of diameter 1 and *unit disc* means a closed solid circle of radius 1. $D(X)$ and $U(Y)$ denote the disc centered at X and the unit disc centered at Y .

A *line transversal* to a family of discs is a straight line having a non-empty intersection with every member of the family. Alternatively, we also say that this family has property T . A family of at least k discs is a $T(k)$ -family if every k -member subset of it has property T .

Problems concerning line transversals of families of discs, especially Helly-type questions, have been studied by many authors. (For an overview of the different results the reader is advised to consult the survey paper [4].) Disjoint translates of a convex compact set soon became a favorite field of research. For the special case when the domains are disjoint congruent circles Danzer [2] proved

Theorem D. *Every finite $T(5)$ -family of disjoint congruent discs in the plane has property T .*

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He also points out that if the disjointness hypothesis or the assumption that the discs are congruent is dropped, then the claim of Theorem D does not hold anymore even if property $T(5)$ is replaced by property $T(r)$ for any fixed r .

Grünbaum [5] conjectured, however, that property $T(5)$ implies the existence of a transversal if the family consists of disjoint translates of a compact convex set.

It was Tverberg [11] who finally verified this famous old conjecture proving:

Theorem T. *Every finite $T(5)$ -family of disjoint translates of a compact convex domain in the plane has property T .*

Although the statement of Theorem T does not hold anymore if property $T(5)$ is relaxed to property $T(4)$, Katchalski and Lewis found [10] that even property $T(3)$ implies the existence of a line which is “almost” a transversal to the family in the following sense:

Theorem KL. *There exists a universal integer constant g such that to every finite $T(3)$ -family of disjoint translates of a compact convex domain, a line can be found intersecting all but g members of the family.*

Katchalski and Lewis expect g to be a small number. Constructions given for discs by Bezdek [1] show that in general g cannot be smaller than 2. Holmsen [6]–[8] proved that $g \leq 22$ and found that g is 4 if the domains are translated copies of a parallelogram. In the case of discs, $g \leq 12$ is the best known result, due to Kaiser [9].

In the present paper the term “almost” is used in a different sense. Instead of counting the exceptional translates which the best line fails to meet we suggest a quantitative approach and measure the failure by the maximal distance of the best line from the translates, or by the double of this: i.e. by the width of the narrowest strip intersecting all members.

The *transversal strip* of a family of translates is a closed (parallel) strip intersecting all members of the family and the width of the narrowest transversal strip is the *transversal width* of the family.

Using this term the special case of an often cited much more general result of Eckhoff [3] can be reformulated in the following way:

Theorem E. *To every finite $T(3)$ -family of disjoint discs of diameter 1 there exists a transversal strip of width 1.*

Our main goal in this paper is to improve this bound by establishing the following:

Theorem. *To every finite $T(3)$ -family of disjoint discs of diameter 1 there exists a transversal strip of width < 0.65 .*

Eckhoff expects that the minimal value w_5 is attained for a regular arrangement of five discs (Fig. 1). This expectation can be expressed in the following:

Conjecture. *To every finite $T(3)$ -family of disjoint discs of diameter 1 there exists a transversal strip of width $\leq w_5 = 2 \sin(\pi/10) = 0.618\dots$*

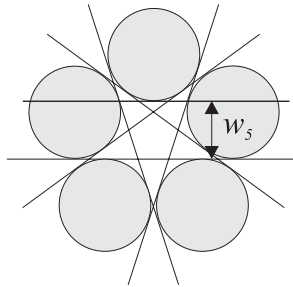


Fig. 1

Notice that the configuration displayed in Fig. 1 is of property $T(4)$, i.e. the truth of this conjecture would imply that even property $T(4)$ cannot guarantee a narrower transversal strip than property $T(3)$ does.

Sets S_1 and S_2 are said to be *separated* if a straight line λ disjoint from S_1 and S_2 , exists such that every line segment connecting a point of S_1 with a point of S_2 is intersecting λ . We say that S_1 and S_2 are *widely separated* if a strip σ of unit width, disjoint from S_1 and S_2 , exists such that every line segment connecting a point of S_1 with a point of S_2 is intersecting σ .

The union of all straight lines intersecting two disjoint discs $D(X)$ and $D(Y)$ is a simply connected closed unbounded domain, the boundary of which consists of parts of the four lines tangent to both discs (Fig. 2(a)). This unbounded domain is called the *sheaf* belonging to the two discs and is denoted by $\Sigma(X, Y)$. Clearly, a third disc has a common transversal with $D(X)$ and $D(Y)$ if and only if it intersects the sheaf $\Sigma(X, Y)$.

The outer parallel domain of radius $\frac{1}{2}$ of sheaf $\Sigma(X, Y)$, which is the locus of the *centers* of the discs intersecting $\Sigma(X, Y)$, is called the *center sheaf* belonging to X and Y and is denoted by $\Sigma^c(X, Y)$. Two of the (at most) six lines generating the boundary of the center sheaf $\Sigma^c(X, Y)$ are non-separating tangent lines to both unit discs $U(X)$ and $U(Y)$ and the four other lines are tangent to one of the two unit discs and pass through the center of the other one (see Fig. 2(b)).

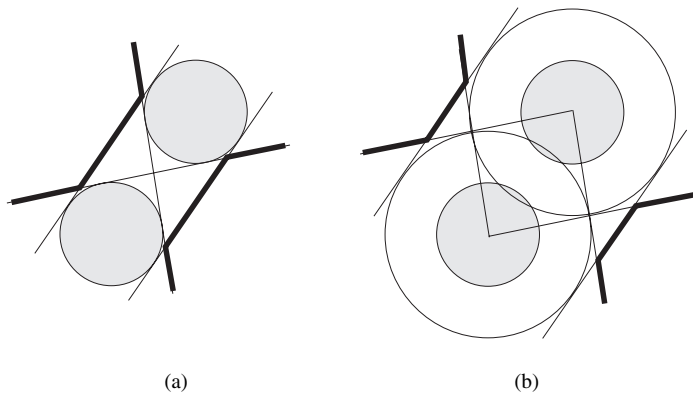


Fig. 2

The *generalized center sheaf* belonging to two orthogonal segments also play an important role in the proof. This is the union of all center sheaves belonging to a pair of discs, one of them centered in the first segment and the other centered in the second one. The generalized center sheaf is a domain bounded by parts of at most six straight lines—like in the case of the common center sheaf, however, the generalized center sheaf is usually not symmetric. $\Sigma^c(\overline{X_1X_2}, \overline{Y_1Y_2})$ denotes the generalized center sheaf belonging to the segments $X \in \overline{X_1X_2}$ and $Y \in \overline{Y_1Y_2}$, where $\overline{X_1X_2}$ and $\overline{Y_1Y_2}$ denote the segments on the x - and y -axis connecting X_1 and X_2 and Y_1 and Y_2 , respectively. The notation $\lambda(XY)$ is used for the line connecting the points X and Y . $\lambda^\ell(XY)$ and $\lambda^r(XY)$ denote the common non-horizontal tangent lines of sets X and Y on the left and on the right, respectively.

2. The Proof

1. Contrary to our claim suppose that a counterexample exists. Following Tverberg's reduction method [11] it can be assumed that the centers are in general position, i.e. no three centers form the vertices of a right-angled triangle.

Tacitly the following well-known fact will also be used:

Lemma. *The disjoint sets S_1 , S_2 and S_3 have no common transversal if and only if each can be separated from the union of the other two sets.*

Let σ^c denote the narrowest strip covering all centers of the family. Assume that σ^c is horizontal, and its upper boundary line is the x -axis. Then we can assume that there exists a center $A(0, -a)$ on the lower boundary line $y = -a$ of σ^c and two centers $B(b, 0)$ and $C(-c, 0)$ on the upper one (the x -axis) such that $1.65 \leq a \leq 2$ and $0 < b \leq c$. We are going to show that these assumptions lead to contradiction: there exist three discs in the family without a line transversal.

The sketch of the proof is the following. It is a direct consequence of the $T(3)$ -property of the discs about A , B and C that to any given value of a there belongs a feasibility interval for b and for any given a and b a feasibility interval for c . As center C must be in $\Sigma^c(A, B)$ our assumption $b \leq c$ implies

$$b \in \left(0, \frac{1}{\sqrt{4 - 1/a^2}} \right]. \quad (1)$$

Since the upper end of the interval is a decreasing function of a and $1/\sqrt{4 - 1/1.65^2} = 0,5246\dots$ the feasibility interval of b is a subinterval of the interval $[0, 0.525]$.

In our proof every pair of values $a \in [1.65, 2]$ and $b \in [0, 0.525]$ will be considered although not all of these combinations are feasible in a $T(3)$ -family. Technically, the intervals will be cut into subintervals: $a \in [1.65, 2]$ into five subintervals:

$$a \in [1.65, 1.67], [1.67, 1.70], [1.70, 1.75], [1.75, 1.8], [1.8, 2], \quad (2)$$

and, independently, $b \in [0, 0.525]$ into six subintervals:

$$b \in (0, 0.1], [0.1, 0.3], [0.3, 0.42], [0.42, 0.48], [0.48, 0.51], [0.51, 0.525]. \quad (3)$$

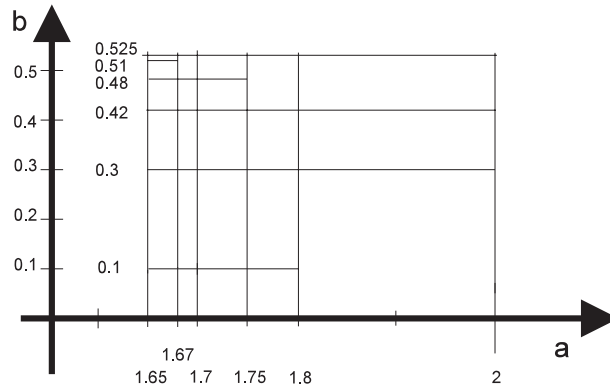


Fig. 3. The “parameter boxes.”

This gives 30 “parameter boxes”. For some a -intervals several b -intervals have been united to reduce the calculations. Finally, we have 23 parameter boxes. (For a graphical overview see Fig. 3.)

It will be shown that none of these parameter boxes contains a feasible solution, i.e. a counterexample to the claim of the theorem. Except for the numerical calculations the proof follows the same line for each box therefore it will be carried out for an arbitrary parameter box.

The values $a_1 < a_2$ and $b_1 < b_2$ denote the limits of the selected parameter intervals and $A_i(0, -a_i)$ and $B_i(0, b_i)$, $i = 1, 2$, denote the endpoints of the assigned intervals.

2. It is easily seen that in our cases, when the inequalities

$$1.6 \leq a \leq 2, \quad 0 \leq b_1 < 1, \quad 0.5 \leq c_1 < 1 \quad (4)$$

hold, the generalized center sheaf $\Sigma^c(\overline{B_1 B_2}, \overline{A_1 A_2})$ defined by the segments $\overline{A_1 A_2}$ and $\overline{B_1 B_2}$ is lying between two polygonal lines (see Fig. 4), one—on the left—consisting of parts of lines

$$\lambda^\ell(U(B_1)A_1), \lambda^\ell(U(B_1)U(A_1)), \lambda^\ell(B_2U(A_1)), \quad (5)$$

and the other boundary—on the right—consisting of parts of lines

$$\lambda^r(U(B_2)A_1), \lambda^r(U(B_2)U(A_2)), \lambda^r(B_1U(A_1)). \quad (6)$$

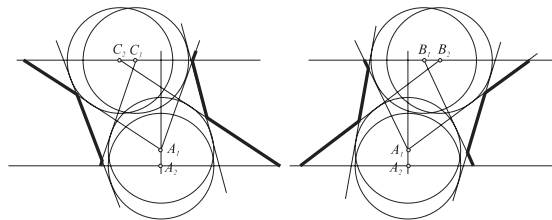


Fig. 4. Generalized center sheaves.

Analogously, the boundary of $\Sigma^c(\overline{C_1 C_2}, \overline{A_1 A_2})$ consists of parts of lines

$$\lambda^\ell(U(C_2)A_1), \lambda^\ell(U(C_2)U(A_2)), \lambda^\ell(C_1 U(A_1)), \quad (7)$$

and parts of lines

$$\lambda^r(U(C_1)A_1), \lambda^r(U(C_1)U(A_1)), \lambda^r(C_2 U(A_1)). \quad (8)$$

3. To every box of $a_1 \leq a \leq a_2$ and $b_1 \leq b \leq b_2$ a feasibility interval $[-c_2, -c_1]$ can be calculated for the Cartesian x -coordinate of $C(-c, 0)$ based on the condition $b \leq c$, on the disjointness hypothesis and on the $T(3)$ property of the family. Since the lower bound for c is a decreasing function of b and the upper bound is a decreasing function of both a and b we have

$$c \geq c_1 = \max(1 - b_2, 0.5) \quad (9)$$

and, from $\Sigma^c(\overline{A_1 A_2}, \overline{B_1 B_2})$,

$$c \leq c_2 = \frac{a_1 \sqrt{a_1^2 + b_1^2 - 1} - a_1^2 b_1}{a_1^2 - 1}. \quad (10)$$

$C_i(0, -c_i)$, $i = 1, 2$, denote the endpoints of the feasibility segment for C .

4. By the $T(3)$ property of the family the three centers A , B and C can always be covered with a strip of width 1, thus the family, contradicting our claim, contains further discs. All further centers must lie in the bounded polygonal domain which is the intersection of strip σ^c , center sheaf $\Sigma^c(A, B)$ defined by the discs about A and B and center sheaf $\Sigma^c(A, C)$ defined by the discs about A and C (Fig. 5(a)), moreover, the centers must lie, by the disjointness condition, outside of the unit discs $U(A)$, $U(B)$ and $U(C)$ (Fig. 5(b)).

5. Let $X(AB)$ and $X(AC)$ be the images of the x -axis received by reflection on the line $\lambda(AB)$ and $\lambda(AC)$, and let $X^*(AB)$ and $X^*(AC)$ be the lines through A parallel to $X(AB)$ and $X(AC)$, respectively. Let $\sigma(AB)$ ($\sigma(AC)$) denote the strip bounded by $X(AB)$ and $X^*(AB)$ (by $X(AC)$ and $X^*(AC)$). Strips σ^c , $\sigma(AB)$ and $\sigma(AC)$ have, of course, the same width (Fig. 6).

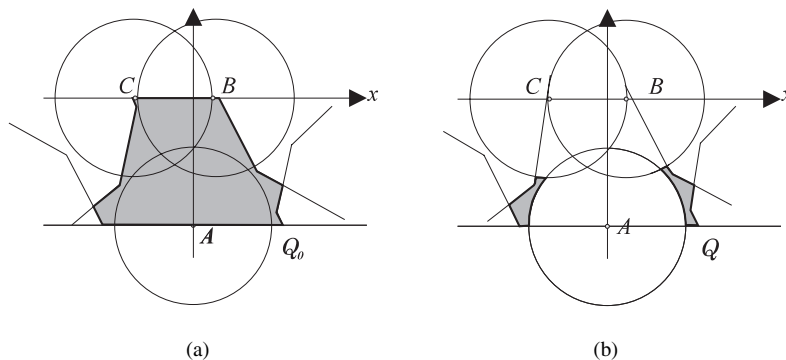


Fig. 5

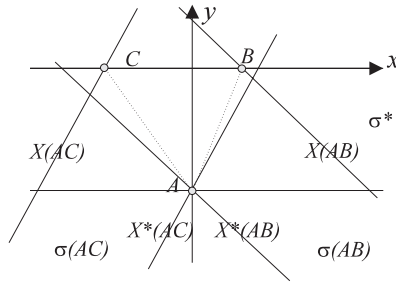


Fig. 6

Since there is no center below the lower boundary line $y = -a$ of the narrowest transversal strip σ^c the following holds:

Proposition 1. *All centers in the open half-plane $x > 0$ are strictly above $X^*(AB)$ and all centers in the open half-plane $x < 0$ are strictly above $X^*(AC)$ for any $a \in [1.65, 2]$ and $b \in [0, 0.525]$.*

The proof is left to the reader. (Remember: no triple of centers is supposed to form a right-angled triangle, hence line $X^*(AB)$ is strictly descending and line $X^*(AC)$ is strictly ascending.)

Proposition 2. *Center C is an inner point of strip $\sigma(AB)$ and center B is an inner point of strip $\sigma(AC)$ for any $a \in [1.65, 2]$ and $b \in [0, 0.525]$.*

The proof is left to the reader. (Hint: consider the intervals $b \leq 0.2$ and $b \geq 0.2$ separately.)

6. If centers A , B and C are allowed to vary in segments $\overline{A_1A_2}$, $\overline{B_1B_2}$ and $\overline{C_1C_2}$, respectively, then the role of the center sheaves is taken over by the suitable generalized center sheaves. Exploiting the condition that any other center is disjoint from $U(A)$, $U(B)$ and $U(C)$ for any choice in the parameter box, a further reduction of this polygonal domain

$$Q_0 = \sigma^c \cap \Sigma^c(\overline{A_1A_2}, \overline{B_1B_2}) \cap \Sigma^c(\overline{A_1A_2}, \overline{C_1C_2})$$

is possible. Those points which lie outside of $U(A)$ for at least one $a \in [a_1, a_2]$ are the points of the complement of the set of inner points of $U(A)$ for all $a \in [a_1, a_2]$, which is the intersection $U(A_1) \cap U(A_2)$. Similarly, a point of Q_0 is not feasible if it is not outside of $U(B)$ for at least one value of $b \in [b_1, b_2]$ and outside of $U(C)$ for at least one value of $c \in [c_1, c_2]$. Thus the feasibility domain of further centers reduces—in each box—to

$$Q = Q_0 \setminus ((U(A_1) \cap U(A_2)) \cup (U(B_1) \cap U(B_2)) \cup (U(C_1) \cap U(C_2))). \quad (11)$$

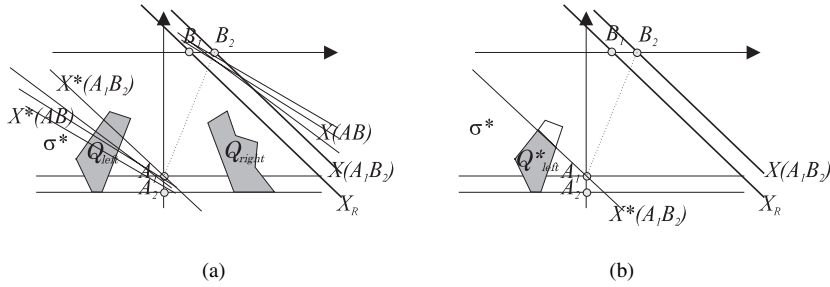


Fig. 7

(Lying in Q is, of course only a necessary condition to be a potential center and it is far from sufficient.) Q_{left} and Q_{right} denote the parts of set Q in the half-planes $x \leq 0$ and $x \geq 0$, respectively.

In the rest of the propositions it is assumed—without mentioning it repeatedly—that the parameters a and b belong to one of the 23 parameter boxes defined above.

Let X_R be the line parallel to $X(A_1 B_2)$ and passing through B_1 (Fig. 7(a)) and let X_L be the line parallel to $X(A_1 C_2)$ and passing through C_1 . Then the following holds:

Proposition 3. *Inside the quarter plane $x > 0, y < 0$, line $X(AB)$ (the upper boundary of $\sigma(AB)$) runs above X_R for any A and B of the box and inside the quarter plane $x < 0, y < 0$ line $X(AC)$ (the upper boundary of $\sigma(AC)$) runs above X_L for any A and C of the box.*

Proof. On one hand the steepness of line $X(AB)$ is a decreasing function of the steepness of segment \overline{AB} , and (within a parameter box) $X(AB)$ is never steeper than $X(A_1 B_2)$. On the other hand $X(AB)$ intersects segment $\overline{B_1 B_2}$. \square

We also have

Proposition 4. *Domain Q_{right} lies strictly below X_R (Fig. 7(a)) and Q_{left} lies strictly below X_L .*

Proof. This is easily verified by checking that every vertex of $\text{conv}(Q_{\text{right}})$, a polygon, is strictly below X_R and every vertex of $\text{conv}(Q_{\text{left}})$ is strictly below X_L . \square

Then the feasibility domain Q_{right} has no point above $X(AB)$ and Q_{left} has no point above $X(AC)$ for any choice of parameters in the parameter box.

Combining Propositions 1, 3 and 4 we have

Proposition 5. *For any choice of parameters in a given box, all centers of the $x > 0$ half-plane lie in the interior of $\sigma(AB)$ except for B , which is on its boundary and all centers of the $x < 0$ half-plane lie in the interior of $\sigma(AC)$ except for C , which is on its boundary.*

By Propositions 5 and 2 the set of centers A , B and C and those finite number of further centers which might be in the interior of $\sigma(AB)$ can be covered by a strip narrower than σ^c . Consequently, then we have

Corollary E. *There exists a center E_0 in the half-plane $x \leq 0$, i.e. in domain Q_{left} , which is not an inner point of strip $\sigma(AB)$.*

As the lower boundary $X^*(AB)$ of strip $\sigma(AB)$, intersecting segment $\overline{A_1A_2}$, is never steeper than line $X^*(A_1B_2)$ the following holds:

Proposition 6. *Line $X^*(AB)$ either coincides with line $X^*(A_1B_2)$ or runs below it in the $x < 0, y < 0$ quarter plane for any choice of parameters of the box.*

Hence center E_0 must lie below $X^*(A_1B_2)$ or on it. The part of Q_{left} consisting of points below or on line $X^*(A_1B_2)$ will be denoted by Q_{left}^* (Fig. 7(b)).

By a similar argument we have

Corollary F. *There exists a center F_0 in the half-plane $x \geq 0$, i.e. in domain Q_{right} , which is not an inner point of strip $\sigma(AC)$.*

In what follows Q_{right}^* denotes the domain which is the rest of Q_{right} having the part above line $X^*(A_1C_2)$ removed.

7. Some special points play a key role in the rest of the proof. Points $E_i(x_{E_i}, y_{E_i}), i = 1, \dots, 7$, defined below, are selected so that the convex hull of them is covering Q_{left}^* in each case. This simplifies the required calculations.

The first three are points of line $X^*(A_1B_2)$. E_1 is on the upper half of the boundary of disc $U(A_2)$, E_2 is on line $\lambda^\ell(U(B_1)U(A_1))$ and E_3 is on line $\lambda^\ell(B_2U(A_1))$. In some cases—like, e.g. in case of box $a \in [-1.8, -1.75], b \in [0.42, 0.525]$ — E_2 is a vertex and E_3 is not, while in other cases—like, e.g. in box $a \in [-1.75, -1.7], b \in [0, 0.1]$ — E_3 is a vertex and E_2 is not (see Fig. 8). E_4 is the point of intersection of lines $\lambda^\ell(B_2U(A_1))$

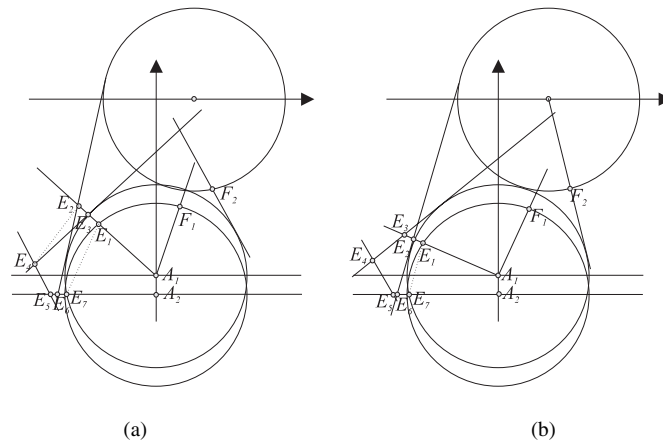


Fig. 8

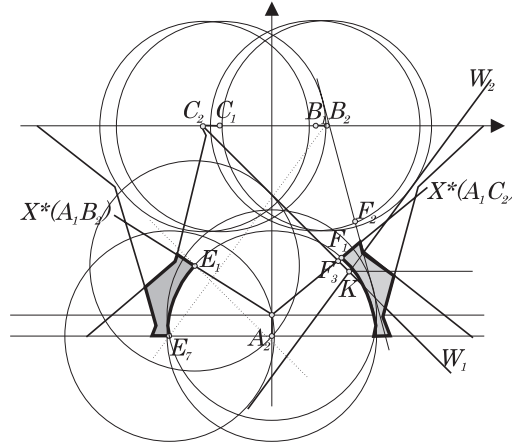


Fig. 9

and $\lambda^\ell(C_2U(A_2))$. E_5, E_6 and E_7 are points of line $y = -a_2$ on one hand; and lying on $\lambda^\ell(U(C_2)U(A_2))$, on $\lambda^1(U(B_1)U(A_1))$ and on the $x < 0$ half of the boundary of the disc $U(A_1)$ on the other hand. (We mentioned that the convex hull of Q_{left}^* is always a pentagon. That one of E_2 and E_3 which has the smaller x -coordinate and E_1 are vertices of the convex hull of Q_{left}^* . That one of E_5 and E_6 which has the smaller x -coordinate and E_7 are also vertices of the convex hull of Q_{left}^* .)

In the other half-plane let $F_1(x_{F_1}, y_{F_1})$ be the point of intersection of $X^*(A_1C_2)$ with the upper half of the boundary of $U(A_2)$, and let $F_2(x_{F_2}, y_{F_2})$ be the point of intersection of $\lambda^1(U(C_1)U(A_1))$ and the lower half of the boundary of unit disc $U(B_1)$.

8. Consider now the upper tangent line W_1 of unit disc $U(E_1)$ passing through C_2 and the lower tangent line W_2 of both unit discs $U(E_7)$ and $U(B_2)$ and denote the point of intersection of W_1 and W_2 by $K(x_K, y_K)$ (Fig. 9).

Simple substitution based on equations of $X^*(A_1B_2)$ and W_1 shows that W_1 is steeper than $X^*(A_1B_2)$ (in each box). Hence E_1 is the point of Q_{left}^* nearest to W_1 , thus Q_{left}^* is widely separated from the points above W_1 . This implies

Proposition 7. *If $F_0 \in Q_{\text{right}}^*$ is above W_1 then E_0 is widely separated from $C \cup F_0$.*

Similarly, it is easy to check that W_2 is a lower support line of the outer parallel domain of radius 1 of Q_{left}^* (the critical point being E_7) hence the following holds:

Proposition 8. *If $F_0 \in Q_{\text{right}}^*$ is below W_2 then F_0 is widely separated from $B \cup E_0$.*

Let $F_3(x_{F_3}, y_{F_3})$ be the point of intersection of W_1 and $X^*(A_1C_2)$. Simple substitutions yield, in each parameter box:

Proposition 9. *K is an inner point of $U(A_2)$.*

Proposition 10. F_3 is an inner point of $U(A_2)$.

By Propositions 9 and 10 the segment of W_2 between $y = -a_2$ and $y = y_K$ runs inside $U(A_2)$ and segments $\overline{KF_3}$ and $\overline{F_3F_1}$ run inside $U(A_2)$ as well. In addition, the part of $U(A_2)$ above $y = y_{F_1}$ is to the left of $X^*(A_1C_2)$. Hence no point of Q_{right}^* lies to the left of W_1 above line $y = y_K$ and no point of Q_{right}^* lies to the left of W_2 below line $y = y_K$. Consequently, F_0 lies either above W_1 or below W_2 .

Let us now decompose domain Q_{right}^* into two parts by line $y = y_K$. Q^{upper} and Q^{lower} denote the closures of the upper and lower parts of Q_{right}^* , respectively. Using this notation Proposition 7 implies

Proposition 11. E_0 is widely separated from $C \cup F_0$ if $F_0 \in Q^{\text{upper}}$.

It is easy to check the truth of the next proposition for each box of parameters:

Proposition 12. F_0 is widely separated from $E_0 \cup C$ if $F_0 \in Q^{\text{upper}}$.

Proof. Consider the line $\lambda(C_1E_1)$. The substitutions show (in each box) that corner E_7 is to the left of the line, consequently $\lambda(C_1E_1)$ is the support line to the whole domain Q_{left}^* , where E_0 is supposed to lie. Point F_1 is on the right of $\lambda(C_1E_1)$ and its distance from the line is larger than 1. As $\lambda(C_1E_1)$ is steeper than $X^*(A_1C_2)$ the line parallel to it and passing through F_1 is a support line of $F_1 \cup Q^{\text{upper}}$, thus every point F_0 of it is widely separated from $E_0 \cup C$. \square

We need also

Proposition 13. C is widely separated from $E_0 \cup F_0$ if $F_0 \in Q^{\text{upper}}$.

Proof. Since the shape of Q^{upper} depends on the choice of the parameter box, this proposition requires more, however simple, calculation to verify that the distance of C and $Q_{\text{left}}^* \cup Q^{\text{upper}}$ is larger than 1 (in each box) as claimed. \square

Based on Propositions 11–13 we have

Corollary U. The discs about C , E_0 and F_0 have no common transversal if $F_0 \in Q^{\text{upper}}$, thus F_0 cannot lie in the upper part of Q_{right}^* .

Analogously, since Q^{lower} is below line W_2 , Proposition 8 implies

Proposition 14. F_0 is widely separated from $E_0 \cup B$ if $F_0 \in Q^{\text{lower}}$.

The last two steps are rather simple to prove:

Proposition 15. E_0 is widely separated from $B \cup F_0$ if $F_0 \in Q^{\text{lower}}$.

Proof. The strip of unit width between lines $y = -4x$ and $y = -4x - \sqrt{17}$ widely separates E_1 and K while the lines through E_1 and K and parallel to $y = -4x$ are support lines of Q_{left}^* and $K \cup Q_{\text{lower}}$, respectively. Hence the claim is proved. \square

Proposition 16. *B is widely separated from $E_0 \cup F_0$ if $F_0 \in Q_{\text{lower}}$.*

Proof. This is a simple consequence of the fact that the topmost point of Q_{left}^* (E_2 or E_3) as well as point K lie below line $y = -1$. \square

Propositions 14–16 imply

Corollary L. *The discs about B , E_0 and F_0 have no common transversal if $F_0 \in Q_{\text{lower}}$, thus F_0 cannot lie in the lower part of Q_{right}^* .*

Corollaries E, F, U and L together yield contradiction and this concludes the proof to the theorem. \square

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