Discrete Comput Geom 34:523–536 (2005)
DOI: 10.1007/s00454-005-1175-1

Discrete & Computational
Geometry

Non-Periodic Rhomb Substitution Tilings that Admit Order n Rotational Symmetry*

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Abstract. For each $n \in \mathbb{N}$ we construct a substitution rule using the set of rhombs with angles $(\pi k)/n$. These substitution rules generate a local isomorphism class of tilings closed under rotation of order $2n$, and also admit singular tilings fixed under a rotation of order n. The scaling factors for this set of substitution rules includes algebraic numbers of every rank.

1. Introduction

Substitution rules have emerged as one of the most important methods of generating ordered non-periodic structures. In one dimension they are studied in computer science and combinatorics, for example in the study of the combinatorics of words [L]. In higher dimensions substitution tilings are linked to quasiperiodic or model sets [BM1]. They also provide important examples in dynamical systems and C^*-algebras [W], [KP].

Three of the most famous examples of substitution rules are the Ammann–Beenker, Penrose and Socolar substitution rules ([G], [S] and references therein). These substitution rules use the set of rhombs with angles $(\pi k)/n$ with $n = 4, 5$ and 6 respectively, and $k \in \{1, \ldots, n\}$. The tilings generated by these substitution rules may be considered to have rotational symmetry of orders 8, 10 and 12. It is therefore natural to ask what other rotational symmetries are possible.

In this paper we generalise an example of Goodman-Strauss [GS1] for the rhombs with $n = 7$ in (Fig. 1). This generalisation gives a substitution rule for all n (for $n = 4$, 5 and 6, the substitution rule is different to those mentioned above). For each n, the set of finite patches of the tilings generated remain invariant by a rotation of order $2n$. For each n, the substitution rule also admits at least one singular tiling that is fixed under a rotation of order n. A singular tiling is a tiling that appears in the complete tiling space for the substitution rule but contains finite patches that are not generated by iteration of the substitution rule on a single tile.

* This work was partly funded through a UK EPSRC Ph.D. studentship.

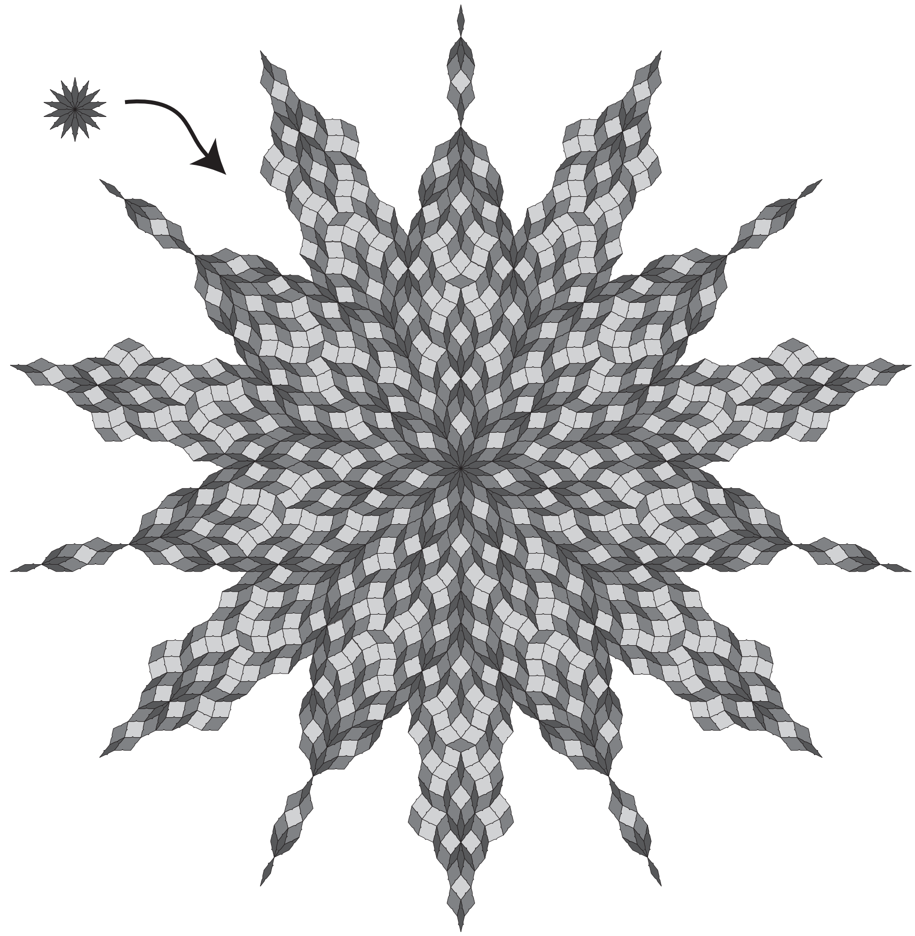

Fig. 1. Singular substitution tiling fixed under a rotation of order 7.

It is known that the scaling factor (defined below) of every substitution rule is an algebraic number [T]. For the rhomb substitution rules shown here, the algebraic rank of the scaling factor generally increases with n. As the algebraic rank increases every rank is included. We are therefore able to show that there exist rhomb substitution rules with every given algebraic rank. This is a new result for rhomb tilings.

1.1. *Definitions*

In this section we give an informal presentation of the concepts discussed in this paper. More formal definitions are given, for example, in [PS], [GS2], [H].

In this paper we consider tilings of the plane by rhombic tiles. This means that the plane is covered by an infinite number of tiles of a finite number of shapes, only intersecting at their edges. A *patch* of a tiling is a finite number of tiles that cover a connected region.

Two tilings $\mathcal{T}, \mathcal{T}'$ are *locally isomorphic* if a translation of every patch of $\mathcal{T}$ occurs in $\mathcal{T}'$ and vice versa. Local isomorphism gives an equivalence relationship on the set of all tilings. The equivalence classes under this relation are called *local isomorphism classes*. A tiling is *locally derivable* from another if there is a rule replacing some finite set of patches of the second tiling that gives the first tiling. Two tilings are *mutually locally derivable* if each is locally derivable from the other.

A *substitution rule* is a way of generating tilings. We start with a finite collection of labelled shapes, the *prototiles*. A substitution rule acts by first expanding a prototile by a constant, called the *scaling factor*. A *replacement rule* then replaces every instance of a particular, expanded, labelled prototile by a patch of tiling. This process can be applied to any patch. If the tiles after substitution lie over the edges of the original tile, we must ensure that two neighbouring tiles do not substitute to give tiles that intersect beyond their edges. We can then repeatedly apply the rule to obtain arbitrarily large patches of tilings. Note that the set of prototiles can contain congruent shapes with different labels. Thus the same tile shape can substitute in several ways. This definition considers two tilings to be identical if they are related by an isometry of the plane. In some cases one might wish to introduce a specific origin and basis to distiguish these tilings.

A substitution rule on the plane is *volume hierarchic* if the replacement rule simply divides up the expanded tiles. Such substitution rules are also described as self-similar (where the tilings which are not volume hierarchic are called pseudo-self-similar), “with amalgamation” and inflation rules in the literature. Note that the term self-similar has also been used in many contexts so should be treated with caution.

The *predecessor* P for a tiling T under a substitution rule σ is a tiling such that $\sigma(P) = T$. That is the tiling that substitutes to T. A *substitution tiling* T is a tiling that has a infinite sequence of predecessor tilings P_n such that $\sigma^n(P_n) = T$. A *non-singular substitution tiling* is a substitution tiling all of whose patches are subsets of the patches generated by a substitution rule from the individual prototiles. The set of non-singular substitution tilings for a substitution rule forms a local isomorphism class.

A substitution rule is *primitive* if for some iteration of the rule, the rule for each labelled tile contains instances of every other labelled tile. If a substitution rule is primitive, the substitution tilings for that substitution rule form a translation local isomorphism class.

A scaling factor is *Pisot–Vijayaraghavan* (*PV*) if it is a real algebraic number greater than 1, all of whose algebraic conjugates have absolute value less that 1.

In this paper we deal with rhombs. Recall that a rhomb is a quadrilateral with all four sides of equal length and two pairs of parallel sides. The angles at the two pairs of opposite corners are therefore equal and if the smaller pair have angle a, the larger pair have angle $\pi - a$. The general notation T^n_p will be used for the rhomb with angles $p\pi/n$ and $(n-p)\pi/n$. Note that $T^n_p = T^n_{n-p}$ as rhombs. This break in the symmetry will be used in showing the substitution rule for the rhombs. In addition some further labels will be introduced for tiles that substitute in different ways.

1.2. *Overview of Paper*

Section 2 states and discusses the main result of this paper, stating that there are rhomb substitution rules with arbitrary rotational symmetry. Some examples of the substitution rules are given in Section 3. In Section 4 these examples are generalised to prove the main result. Finally, some properties of the substitution rules described here are considered in Section 5.

2. Main Result and Discussion

The main result of this paper is the following theorem.

Theorem 2.1. *Let $n \in \mathbb{N}$ be a natural number. Let $\mathcal{T}^n$ be the set of rhombs with angles in $\{\pi k/n | k \in \{1, \dots, n-1\}\}$. There is a primitive substitution rule on the tiles in $\mathcal{T}^n$. The translation local isomorphism class of tilings for this substitution rule are closed under a rotation of order $2n$.*

Proof. We prove this theorem by showing how to construct substitution rules for each n in the following sections. □

Theorem 2.1 shows that there is no restriction on the order of rotation for substitution rules with rhombs. Rhomb tilings with rotational symmetry have previously been considered [WW], [P]. These papers consider tilings constructed by the canonical projection method which projects a slice of a higher-dimensional lattice to a plane. However, canonical projection tilings with a substitution rule cannot have rotational symmetry of order 7, 9, 11 or greater than 12. This is because the scaling for tilings with these rotational symmetries would have to be an algebraic number of rank at least 3 and the canonical projection tilings with a substitution rule have quadratic scaling [P], [H].

The construction given here is different from the construction of triangle substitution rules with general rotational symmetry given in [ND] and those for trapezia given in [F]. It is not known if the tilings given here are mutually locally derivable from these tilings.

The scaling factors for the substitution rules show that there is no bound on the degree of the scaling factor for a rhomb substitution. It is known that the scaling factor of a substitution rule is an algebraic Perron number [T]. A Perron number is an algebraic number whose absolute value is strictly greater than the absolute value of its algebraic conjugates. Kenyon showed in [Ke] that there is a substitution rule on the plane whose tiles have fractal edges for every such scaling factor. One may also consider substitution rules on sets of squares and rectangles constructed by composing one-dimensional substitution rules. Less is known, however, about what can happen on the plane with polygonal tiles outside this narrow setting. The examples constructed in this paper show that for rhomb tilings there is no limit on the degree of the scaling factor of substitution rules with rhomb tiles.

The substitution rules constructed here do not have a PV scaling factor with the exception of the substitution rules for $n = 4$ or 6. Thus they cannot be considered within the cut and project formalism ([GLJJ], [BM2] and references therein).

The only rhomb tilings with PV scaling known to the author have integer or quadratic algebraic scaling. It has been shown that a PV rhomb substitution rule with cubic or greater scaling will not have a polytope window [P], [H]. It would be interesting, however, to construct windows for such substitution rules in a manner analogous to the Arnoux–Ito method used to construct the Rauzy fractals for one-dimensional substitution rules [AI]. Note that if the substitution rule is not unimodular then internal space used for the projection will include some p-adic part [BMS], [BS].

3. Examples

In this section we give some substitution rules for the sets of rhombs with angles $(\pi k)/n$ for various n. These substitution rules are generalisations of an example of Goodman-Strauss [GS1] shown in Fig. 2.

Figure 3 shows a substitution rule for $n = 4$, along with the second iteration of the rule for one of the tiles. This substitution rule is applied to the same set of rhombs as the Ammann–Beenker substitution [B], [AGS]. This substitution rule, however, does not generate the same tilings as the Ammann–Beenker substitution rule. Patches of the tiling generated here do not occur in the Ammann–Beenker tilings. One such patch is shown in Fig. 3 in white. This patch has rhomb tiles pointing to both left and right of a run of parallel edges. The tiling generated does not satisfy the alternating condition of the Ammann–Beenker tiling, which says that the two types of rhomb tile should alternate [Ka]. The tiling given here is therefore not locally isomorphic to the Ammann–Beenker tilings. In fact this tiling is not locally derivable from the Ammann–Beenker tilings either. This is because the Ammann–Beenker tilings are canonical projection tilings. Thus any locally derived tiling will also be a projection tiling with a polygonal window. The tiling described in this

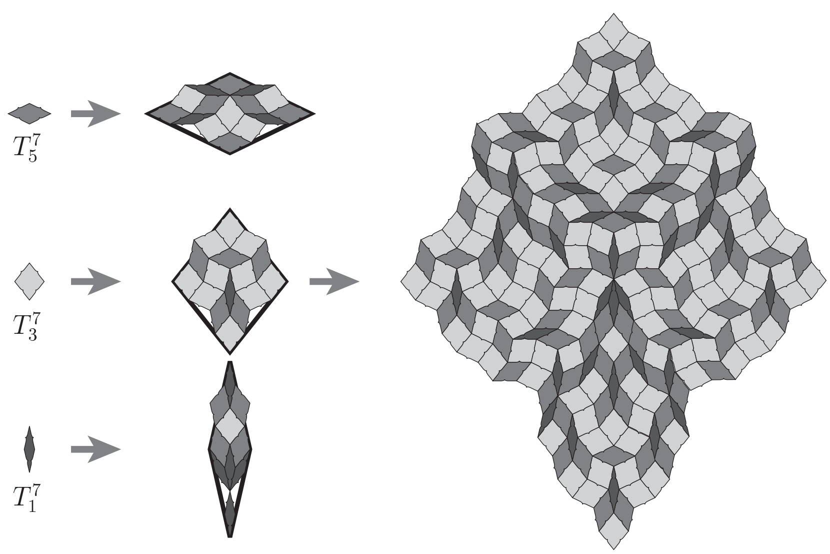

Fig. 2. A substitution rule for the rhombs with angles $(\pi k)/7$.

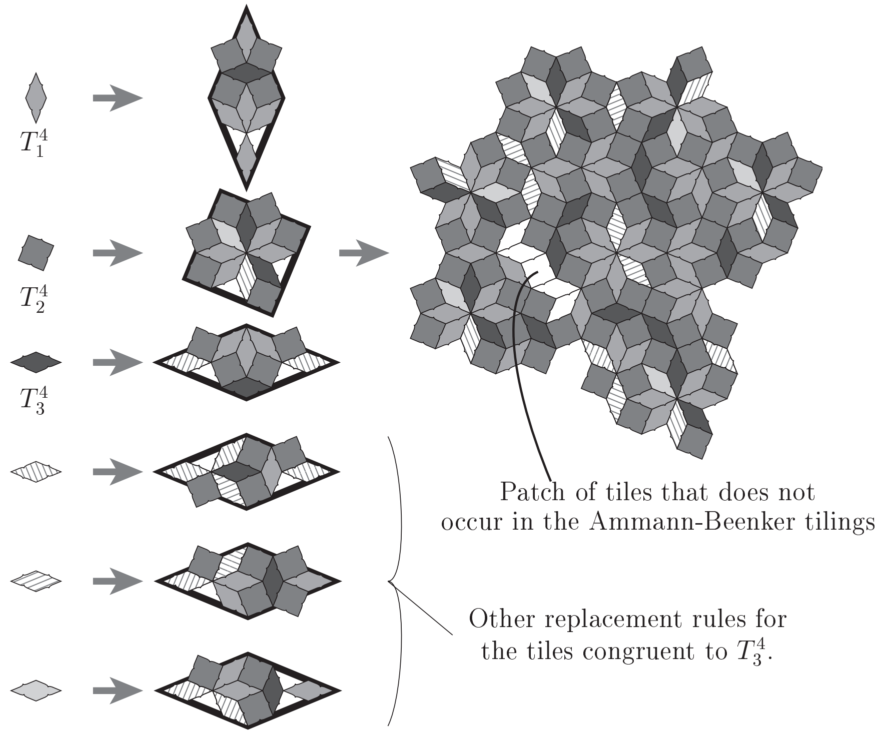

Fig. 3. A substitution rule for the rhombs with angles $(\pi k)/4$. This substitution rule has the same tiles as the Ammann–Beenker tiling. Note, however, that an Ammann–Beenker tiling is not generated. (See the text for details.)

paper cannot be a projection tiling with a polygonal window as the scaling is not a unit [H].

The scaling factor for this tiling is the largest solution of the following equation, which is the characteristic polynomial of a matrix describing the way the edges substitute:

$$2 - 4\lambda + \lambda^2 = 0. \tag{1}$$

A final example shown in Fig. 4 shows a substitution rule related to the sevenfold example. The replacement rules for the tiles T_3^9 and T_5^9 are closely related to the replacement rule for the tile T_3^7 in the sevenfold example. The substitution rule is essentially the same in each case. If one considers the substitution rule as a collection of rods joined by hinges at the points one can get from one rule to another by changing the bottom angle and letting the system hinge together.

Similarly, the replacement rules for the tiles T_1^9 and T_7^9 are related, in the sevenfold example, to the replacement rules for the tiles T_1^7 and T_5^7, respectively. In the next section we show how this principle can be extended to give substitution rules for every set of rhombs.

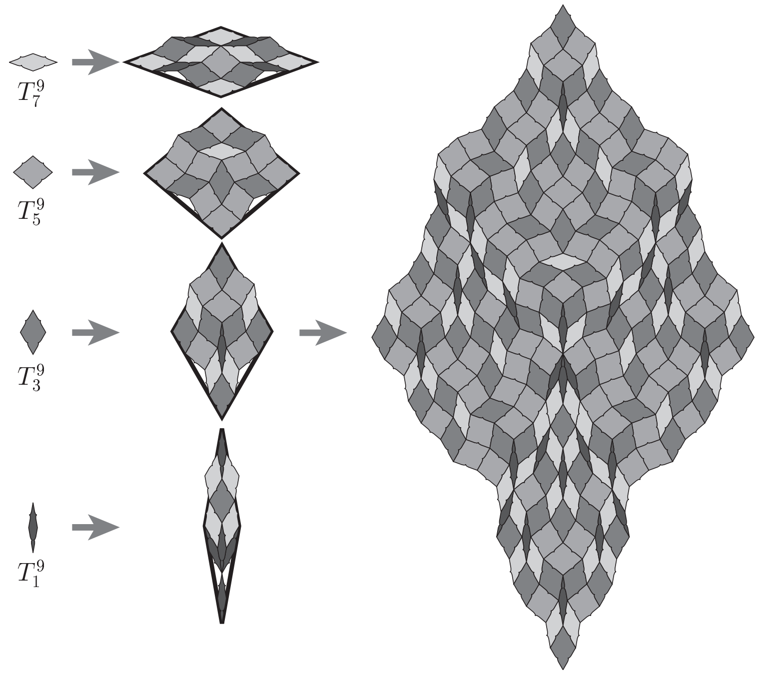

Fig. 4. A substitution rule for the rhombs with angles $(\pi k)/9$. This substitution rule is a generalisation of the substitution rule in Fig. 2.

4. Construction of Substitution Rules

In this section we show how to construct a substitution rule for the set of rhombs with angles $\pi k/n$, where $n \in \mathbb{N}$ and $k \in \{1, \ldots, n-1\}$, and thus prove Theorem 2.1. The first step (Section 4.1) in the construction considers how the edges of the tiles will substitute. We then give a generalised substitution rule that can be used for most of the tiles in Section 4.2. In Section 4.3 we give substitution rules for the other tiles. Finally, in Sections 4.4 and 4.5 we show how these substitution rules for individual tiles can be brought together to give the substitution rules for all the tiles.

The scaling used for these substitution rules seems to give many other possibilities for replacement rules. For example, the tiles within the replacement rules below can be exchanged without changing the behaviour at the edge. Many of the different substitution rules created will generate different substitution tilings. In this paper we show how one such substitution rule can be generated. The question of enumerating the possible ways that an expanded tile can be replaced and the number of ways that these can be combined into substitution rules is left open.

4.1. *Edge Substitution Rule*

The substitution rules created here are not volume hierarchic, they thus require a simple matching rule to ensure that no tiles overlap. Each edge will be marked with a central

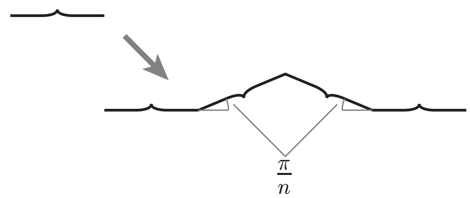

Fig. 5. The substitution rule for a dimpled edge.

"dimple", which is symmetric under the reflection which fixes the edge (the reflection in the line orthogonal to the edge which passes through its midpoint). The substitution rule for a dimpled edge is shown in Fig. 5.

When considered as the edge of a tile, a dimpled edge can either point in to the tile or out from it. For each rhomb there are several possible edge markings. For most of this paper we use only the marking in which the dimples for the two edges on each side of a vertex go in, and the other two go out, as shown in Fig. 6. The edge dimples on the tile T_p^n are those shown in Fig. 6. Note that with these edge markings the congruent tiles T_p^n and T_{n-p}^n are different unless $n = 2p$ and the tile is a square.

4.2. *The Replacement Rule for the Tile T_p^n with $p \in \{3, \ldots, n-2\}$*

Figure 7 shows a general substitution rule which can be adapted to any rhomb T_p^n, where $2 \leq p \leq n-2$. The angles of each vertex in the diagram add up to 2π for any $n \in \mathbb{N}$ and $2 \leq p \leq n-2$. In addition the four edges follow the edge substitution shown in Fig. 5.

The replacement rule replaces the tile T_p^n by the tiles T_p^n, T_{p+2}^n, T_{p-2}^n, T_{n-p+1}^n, T_{n-p-1}^n. In the case where $p = n-2$, the tile T_{p+2}^n is a straight line, but the dimples will still fit together.

This rule, however, does not give rules for T_1^n, T_2^n and T_{n-1}^n. In the T_2^n case this is because the dimples on the straight line given for T_{2-2}^n will not fit together. In the other two cases it would force the tiles to intersect. In the next section we give replacement rules for these tiles.

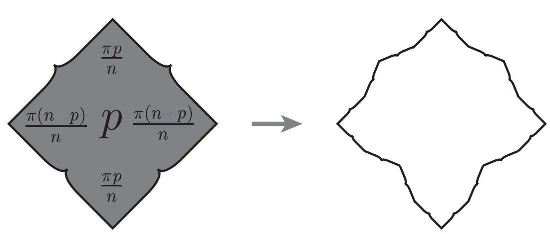

Fig. 6. The angles and dimples for the tile T_p^n, and the boundary of the tile after the substitution rule.

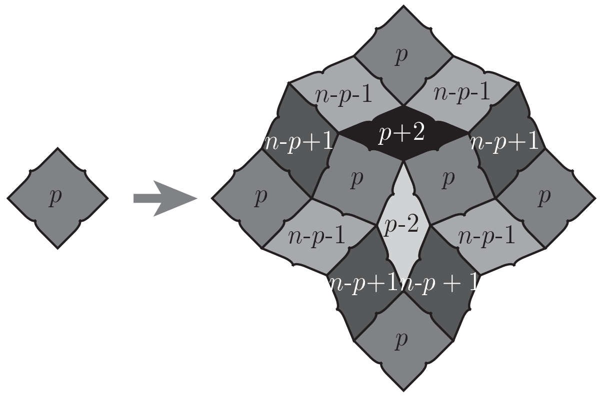

Fig. 7. Substitution rule for the tile T_p^n.

4.3. *The Substitution Rules for the Tiles T_1^n, T_2^n and T_{n-1}^n*

The substitution rule for the tile T_1^n is shown in Fig. 8. Again the angles at each vertex add up to 2π radians for any n, so we can construct a substitution rule for any n.

Any attempt to construct a replacement rule for the tile T_{n-1}^n presents a problem. If we restrict ourselves to the tiles T_p^n, we cannot give the correct edge substitution rules. This is because we are forced to use the tile T_{n-1}^n at the sharp vertex, followed by the tile T_{n-2}^n along the edge for the dimple that sticks out of the tile. To give the correct edge substitution rule the dimples for the edge that lies at the intersection of these two tiles would have to go out for both tiles.

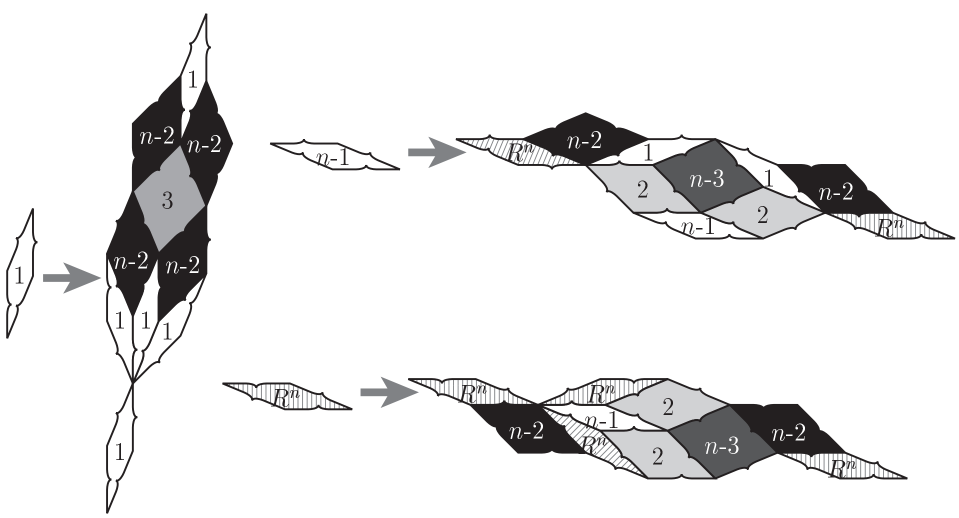

Fig. 8. Substitution rules for the tiles T_1^n, T_{n-1}^n and R^n (see the text for details).

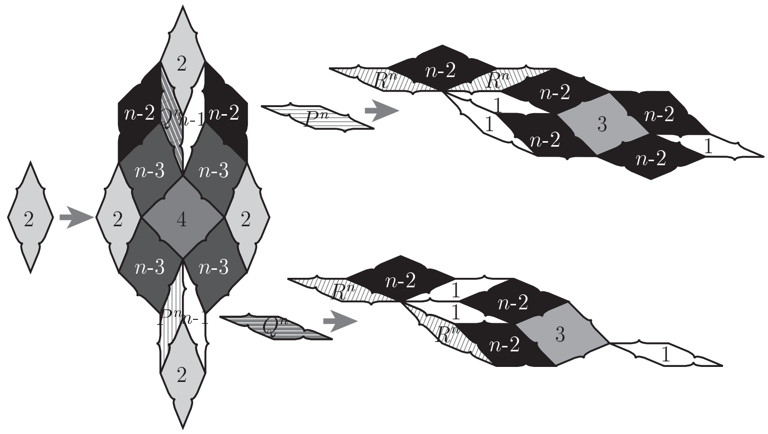

Fig. 9. Substitution rules for the tiles T_2^n, P^n and Q^n (see the text for details).

To give a replacement rule we must introduce the tile where the dimples of the two opposite edges are different. Denote this tile R^n. The replacement rules for T_{n-1}^n and R^n are shown in Fig. 8.

Figure 9 shows the substitution rule for the tile T_2^n. We must also introduce two new tiles: the tile with three dimples out and one in, and the tile with three in and one out. The tile with three dimples out is denoted P^n and the tile with three dimples in is denoted Q^n. The substitution rules for P^n and Q^n are given in Fig. 9. Note that, like the tile R^n, these tiles have two orientations under reflection. In this case, however, only one of the orientations is used.

The tile types after applying a substitution rule are shown for all tiles in Table 1.

4.4. *Substitution Rules for n Odd*

Recall that the general rule takes T_p^n to the tiles shown in Table 1. If p is odd, all the tiles after the substitution will also be T_q^n with odd q. Similarly, the tile T_1^n substitutes

Table 1. Tile types after substitution.

Tile	Tiles after substitution
T_p^n	$T_p^n(6)$, $T_{p+2}^n(1)$, $T_{p-2}^n(1)$, $T_{n-p+1}^n(4)$, $T_{n-p-1}^n(4)$
T_1^n	$T_1^n(5)$, $T_3^n(1)$, $T_{n-2}^n(4)$
T_2^n	$P^n(1)$, $Q^n(1)$, $T_2^n(4)$, $T_4^n(1)$, $T_{n-3}^n(4)$, $T_{n-2}^n(2)$, $T_{n-1}^n(2)$
T_{n-1}^n	$R^n(2)$, $T_1^n(2)$, $T_2^n(2)$, $T_{n-3}^n(1)$, $T_{n-2}^n(2)$, $T_{n-1}^n(1)$
R^n	$R^n(4)$, $T_2^n(2)$, $T_{n-3}^n(1)$, $T_{n-2}^n(2)$, $T_{n-1}^n(1)$
P^n	$R^n(2)$, $T_1^n(3)$, $T_3^n(1)$, $T_{n-2}^n(5)$
Q^n	$R^n(2)$, $T_1^n(3)$, $T_3^n(1)$, $T_{n-2}^n(3)$

to the tiles T_1^n, T_3^n and T_{n-2}^n, which all have odd coefficient. Each tile shape substitutes in a unique way. We may therefore define a substitution rule acting on the $(n-1)/2$ rhombic prototiles. This substitution rule uses the replacement rules defined for the tiles T_p^n above.

To show that this substitution rule is primitive, recall that the substitution rule takes each tile T_p^n to the tiles T_{p-2}^n and T_{p+2}^n if they exist. The tiling is therefore primitive, as repeating the addition and subtraction of 2 will eventually give all the tiles.

In addition, every orientation of every tile will occur. To see this consider the substitution rule for T_1^n, which includes the tile T_{n-2}^n. The substitution rule for T_{n-2}^n will include the tile T_1^n, and this tile will lie rotated π/n from the original T_1^n after applying both substitution rules. Thus because the substitution rule is primitive every orientation of every tile shape will appear in a sufficiently large patch of the tiling.

4.5. *Substitution Rules for n Even*

In the case where n is even, the general rule takes any tile to tiles with both even and odd values of p. We must therefore consider two substitution rules for most of the tile shapes, that is, the rule for T_p^n and the rule for T_{n-p}^n. There are two exceptions: the square $T_{n/2}^n$, where there is only one rule by symmetry, and the tile shape with angles π/n and $\pi(n-1)/n$, where five tiles T_1^n, T_{n-1}^n, R^n, P^n and Q^n are required. The substitution rule therefore acts on $n+2$ prototiles.

To show that the substitution rule is primitive, first note that we can get from any even or odd tile to all the others, as in the case with n odd. The two sets are linked as every even tile has some odd tiles in its substitution rule and every odd tile has even tiles. The tile R^n is included because it substitutes to T_1^n and T_1^n substitutes to it. The tiles P^n and Q^n both substitute from T_2^n, to which all other tiles can be connected, and in turn they substitute to T_1^n which will connect to all other tiles. The substitution rule is therefore primitive.

5. Properties of the Substitution Rules

In this section we consider the properties of the substitution rules. We consider first some highly symmetric tilings that can be generated using the substitution rules in Section 5.1. Finally, in Section 5.2 we discuss the matrix that describes how the edges of the tiles substitute. This matrix is then used to consider the scaling factors of the substitution rules.

5.1. *Tiling with Order n Rotational Symmetry*

As shown in Fig. 1 (at the start of this paper), one can take the tiles T_1^n and place them round a point. This has symmetry of order n as the opposite tiles have the same orientation. Applying the substitution rule to this patch of tiling generates a tiling of the whole plane which is fixed under a rotation of order n. The patch used to generate this tiling does not appear in the patches generated by iterating the substitution rule on single tiles.

The tiling generated above also has reflectional symmetry. One can perform a similar trick using the tiles T^n_{n-2} to generate a tiling with the order n rotational symmetry, but without the reflectional symmetry.

5.2. *The Edge Substitution Matrices and Scaling Factors*

For substitution rules which only have a finite number of translation classes of prototiles, such as the ones described here, one can consider the matrix which describes the substitution rule on the edges of the tiles. In this case, each edge is replaced by two copies of the same edge and the edges rotated by π/n and $-\pi/n$ radians. We may therefore consider the general edge substitution matrices for the substitution rules.

A tiling with the rhombs with angles $k\pi/n$ will have n possible edge directions. After defining some edge to be at angle 0, we can consider the set of edges at angles $k\pi/n$, with $k \in \{0, \dots, n\}$. We can now give the edge substitution matrix, for this labelling of the edges. The (k, l)th entry of the matrix is the number of edges of angle $(l-1)\pi/n$ in the substitution for the edge at angle $(k-1)\pi/n$. As the rule replaces an edge by two copies of itself and the edges rotate by π/n and $-\pi/n$ radians, the matrix will be

$$\begin{pmatrix} 2 & 1 & 0 & \dots & 0 & 0 & -1 \\ 1 & 2 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 2 & 1 \\ -1 & 0 & 0 & \dots & 0 & 1 & 2 \end{pmatrix}. \tag{2}$$

The scaling factor of the substitution rule will be $2+2\cos(\pi/n)$. One may observe this from the edge substitution. This takes an edge of length 1 to two parallel edges of length 1, and two edges of length 1 rotated by $\pm\pi/n$. The total length will be $2+2\cos(\pi/n)$. This is the largest eigenvalue of the matrix given in (2). The other eigenvalues of the matrix are $2+2\cos((k\pi)/n)$ where k ranges over the odd integers less than or equal to n. If the matrix is of even rank all the eigenvalues are doubly degenerate. If it is of odd rank then all the eigenvalues are doubly degenerate apart from a single 0.

The value of $\cos(\pi/n)$ is an algebraic number for all n. Furthermore, there is such an algebraic number of every degree. The set of numbers $\cos(\pi/n)$ for $n \in \mathbb{N}$ will therefore contain instances of algebraic numbers with every degree. The algebraic conjugates of the scaling factor will be other eigenvalues of the matrix. They will be precisely the eigenvalues $2+2\cos(k\pi/n)$, where k is odd and coprime to n. If n is greater than 9, this set will include a k such that $\cos(k\pi/n)$ is positive. Thus the scaling factor will have an algebraic conjugate greater than 1, and therefore be non-PV. Calculations on $n < 9$ (see Table 2) show that the only substitution rules given here with PV scaling are the ones for $n = 4$ and $n = 6$. The scalings for both these substitution rules are quadratic algebraic numbers.

Table 2. Scaling factors for $n = 1 \cdots 9$, and their algebraic conjugates.

n	Scaling factor and algebraic conjugates
3	3
4	3.414, 0.586
5	3.618, 1.382
6	3.732, 0.268
7	3.802, 2.445, 0.753
8	3.848, 2.765, 1.235, 0.152
9	3.879, 1.653, 0.468

Acknowledgements

It is a pleasure to thank Chaim Goodman-Strauss for the initial example and support and encouragement in developing this paper and Jeroen Lamb for his support. The author also thanks the referees, including Ludwig Danzer, for their comments and suggestions.

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Received June 30, 2004, *and in revised form September* 23, 2004, *and January* 21, 2005.
Online publication July 28, 2005.