

Sums, Projections, and Sections of Lattice Sets, and the Discrete Covariogram*

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Abstract. Basic properties of finite subsets of the integer lattice \mathbb{Z}^n are investigated from the point of view of geometric tomography. Results obtained concern the Minkowski addition of convex lattice sets and polyominoes, discrete X-rays and the discrete and continuous covariogram, the determination of symmetric convex lattice sets from the cardinality of their projections on hyperplanes, and a discrete version of Meyer's inequality on sections of convex bodies by coordinate hyperplanes.

1. Introduction

Geometric tomography is the area of mathematics concerning the retrieval of information about an unknown geometric object from data concerning its sections by lines or planes or projections onto lines or planes. When the object is a convex body, many results from convex geometry come into play. Examples are when the data consists of X-rays (Hammer's X-ray problem), or areas of projections (Aleksandrov's projection theorem, Shephard's problem), or areas of central sections (Funk's section theorem, the Busemann–Petty problem). See [14] for more information on all these topics.

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The unknown object may also be a finite set. Then discrete X-rays measure line counts (see Section 2 for formal definitions), and early work of Rényi on projections of probability distributions and of Ryser on reconstruction of binary matrices from row and column sums can be regarded as contributions to a discrete version of Hammer's X-ray problem; see Note 2.2 of [14]. This topic only took on a life of its own much later, however, with the introduction in 1994 of the term *discrete tomography* by Shepp. When the unknown finite set is restricted to a lattice, it models the atoms in a crystal; the invention of new methods in electron microscopy that can effectively measure discrete X-rays of crystals was in fact the major motivation behind discrete tomography. Further details can be found in [23] and in the references given below.

Summarizing, we have by now a rich theory of geometric tomography, involving not only X-rays but many other types of data, and a thriving new field called discrete tomography that more or less concerns only discrete X-rays. The purpose of this paper is to lay the groundwork for an extension of discrete tomography that will bring it more in line with geometric tomography. While one can certainly consider arbitrary finite subsets of \mathbb{R}^n , there seems little doubt that the most interesting problems and applications concern finite subsets of \mathbb{Z}^n , and we focus on these here.

Corresponding to compact convex sets in \mathbb{R}^n are the convex lattice sets in \mathbb{Z}^n and an important subclass, the convex polyominoes. Section 3 is a systematic investigation into the behavior of convexity in the lattice setting under Minkowski addition, and it turns out that this fundamental operation is far less predictable than in the continuous framework. For example, the Minkowski sum of convex polyominoes in \mathbb{Z}^n is a convex polyomino for $n = 2$ but not $n \geq 3$, and while the Minkowski sum of convex lattice sets need not in general be convex, it is true that if A is a convex lattice set in \mathbb{Z}^2 , then $A + A$ and $A + (-A)$ are convex.

In Section 4 we study X-rays and the covariogram. We show that the discrete version of the covariogram is related to the continuous version, and present an example for the discrete covariogram that also provides an example for the continuous covariogram that is in a sense optimal.

Section 5 is motivated by the intriguing possibility, one that apparently has not been considered before, of discrete versions of Aleksandrov's projection theorem: When is an origin-symmetric convex lattice set determined by the cardinalities of its projections on hyperplanes? We establish a connection with the discrete covariogram, and find several examples, one of which shows that a discrete Aleksandrov projection theorem must have additional hypotheses. In Section 6 we turn from projections to sections and find a best-possible discrete version in \mathbb{Z}^2 of Meyer's inequality on sections of convex bodies by coordinate hyperplanes.

The results of this paper, some positive and some negative, indicate that while the envisioned extension of discrete tomography appears feasible, much more work will have to be done before it reaches the same stage of development as geometric tomography. Several of the basic tools of convex geometry have discrete analogues, but the discrete setting appears to render these far less reliable. Nevertheless, the program seems to us eminently worthwhile. Lattice sets and polyominoes are fundamental objects of interest, not only in mathematics but also in physics and computer science; several remarks indicating connections and applications are scattered throughout this paper. We have also included several fascinating open problems whose solutions will be valuable contributions.

2. Definitions and Preliminaries

As usual, S^{n-1} denotes the unit sphere and o the origin in Euclidean n -space \mathbb{R}^n . If $u \in S^{n-1}$, we denote by u^\perp the $(n-1)$ -dimensional subspace orthogonal to u . The standard orthonormal basis for \mathbb{R}^n will be $\{e_1, \dots, e_n\}$.

If A is a set, we denote by $|A|$, $\text{int } A$, and $\text{conv } A$ the *cardinality*, *interior*, and *convex hull* of A , respectively. The *dimension* of A is the dimension of its affine hull $\text{aff } A$, and is denoted by $\dim A$. The notation for the usual orthogonal *projection* of A on a subspace S is $A \mid S$.

If A and B are subsets of \mathbb{R}^n , their *vector* or *Minkowski sum* is

$$A + B = \{a + b : a \in A, b \in B\},$$

and if $r \in \mathbb{R}$, then

$$rA = \{ra : a \in A\}.$$

Thus $-A$ is the reflection of A in the origin. We also write $DA = A - A = A + (-A)$ for the *difference set* of A .

We write V_k for the k -dimensional Lebesgue measure in \mathbb{R}^n , where $k = 1, \dots, n$, and where we identify V_k with the k -dimensional Hausdorff measure. If K is a k -dimensional convex body in \mathbb{R}^n , then $V(K)$ is its *volume* $V_k(K)$. The notation dz will always mean $dV_k(z)$ for the appropriate $k = 1, \dots, n$.

A set is *origin symmetric* if it is centrally symmetric, with center at the origin.

Let K be a *convex body* in \mathbb{R}^n , that is, a compact convex set with nonempty interior. In this case $DK = K - K$ is usually referred to as the *difference body* of K . We denote by

$$h_K(x) = \max\{x \cdot y : y \in K\}$$

its *support function*, by

$$w_K(u) = h_K(u) + h_K(-u), \quad (1)$$

for $u \in S^{n-1}$, its *width function*, and by

$$b_K(u) = V(K \mid u^\perp),$$

for $u \in S^{n-1}$, its *brightness function*. The *projection body* of K is the origin-symmetric convex body ΠK defined by

$$h_{\Pi K} = b_K.$$

Aleksandrov's projection theorem (see, for example, Theorem 3.3.6 of [14]) states that if K and L are origin-symmetric convex bodies in \mathbb{R}^n such that $b_K = b_L$ (or, equivalently, $\Pi K = \Pi L$), then $K = L$. An introduction to the theory of projection bodies is provided in Chapter 4 of [14] and Section 3.5 of [31].

Let E be a bounded Lebesgue measurable subset of \mathbb{R}^n . The function

$$g_E(x) = V_n(E \cap (E + x)),$$

for $x \in \mathbb{R}^n$, is called the *covariogram* of E . Analogously, we define the *discrete covariogram* of a finite subset A of \mathbb{R}^n by

$$g_A(x) = |A \cap (A + x)|,$$

for $x \in \mathbb{R}^n$.

A *convex polytope* is the convex hull of a finite subset of \mathbb{R}^n . We sometimes refer to a finite subset of the n -dimensional integer lattice \mathbb{Z}^n as a *lattice set*. A *convex lattice set* is a finite subset of \mathbb{Z}^n such that $A = \text{conv } A \cap \mathbb{Z}^n$. If A is a convex lattice set, we denote the set $A + A + \cdots + A$ (m summands) by $A[m]$. (Note that whereas for convex bodies K we have $K + K + \cdots + K = K[m] = mK$, it is not true for finite sets that $A[m] = mA$.)

A *polyomino* is a finite subset A of \mathbb{Z}^n such that the union $A + [0, 1]^n$ of lattice unit cubes has a connected interior. We also refer to the set $A + [0, 1]^n$ (itself called a polyomino by many authors) as the *animal* of the polyomino A . A *convex polyomino* is a polyomino that is also a convex lattice set.

One can also consider weaker forms of convexity. Call a vector $u \in \mathbb{Z}^n$ *primitive* if the line segment $[o, u]$ contains no lattice points other than o and u . Then a finite subset A of \mathbb{Z}^n is called *u -convex* if the intersection of A with a line L parallel to u consists of consecutive lattice points in L . When $n = 2$ and $u = e_1$ (or e_2), then *u -convex* sets have also been called *horizontally convex* (or *vertically convex*, respectively). Such properties and connectivity of finite sets in \mathbb{Z}^2 were studied by Daurat [10].

Let A be a finite subset of \mathbb{Z}^n and let $u \in \mathbb{Z}^n \setminus \{o\}$. The *discrete X-ray of A parallel to u* is the function $X_u A$ defined by

$$X_u A(v) = |A \cap (L_u + v)|,$$

for each $v \in u^\perp$, where L_u denotes the line through o and u . The function $X_u A$ is in effect the projection, counted with multiplicity, of A on u^\perp . For an introduction to the many known results on discrete X-rays and their applications, see [6], [15], [16], and [23].

3. Minkowski Addition and Convexity

Certain basic topics such as Helly's theorem have been studied in the context of convex lattice sets (see, for example, [1] and other papers arising from the work of Doignon [13]). There is also a large body of work concerning the lattice-point enumerator, some of which involves obtaining upper or lower bounds for the number of lattice points in the Minkowski sum of two convex polytopes; see, for example, [17], [20], and the references given in these papers. Despite this, we are only aware of one very recent paper, that of Danilov and Koshevoy [9], that considers convexity properties of Minkowski addition in a lattice setting, and this has almost no overlap with our investigation in this section.

When the two convex lattice sets on the left of Fig. 1 are summed by Minkowski addition, the resulting set, on the right of Fig. 1, is not convex. (The authors of [9], who call convex lattice sets "pseudo-convex," were aware of such examples.) Note that these lattice sets are even origin symmetric if placed with their centers at the origin.

Polyominoes are of fundamental interest in many areas, and their convexity properties have also received some attention; see, for example, [12]. The Minkowski sum of two

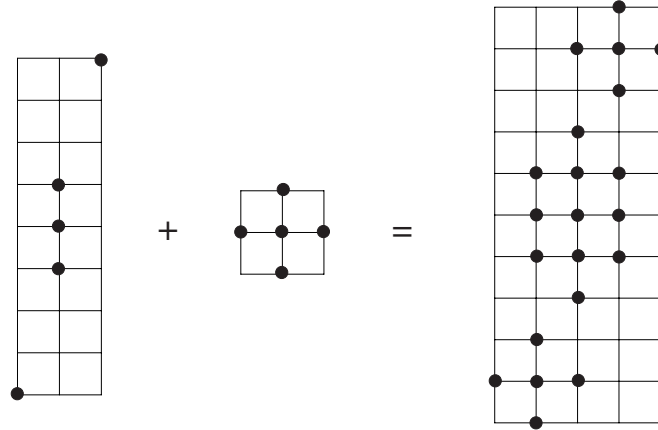


Fig. 1. Minkowski sum does not preserve convexity of lattice sets.

polyominoes is clearly a polyomino, but in three or more dimensions, Minkowski addition of convex lattice sets fails to preserve convexity even for important special cases. For example, let

$$Q = \text{conv}\{(-1, 1, 0), (1, -1, 0), (1, 2, 0), (-1, -2, 0)\} \cap \mathbb{Z}^3$$

and let

$$A = \text{conv}\{Q, (0, 0, 2), (0, 0, -2)\} \cap \mathbb{Z}^3. \tag{2}$$

Then A is an origin-symmetric convex polyomino in \mathbb{Z}^3 consisting of the 11 points in Q and the points $(0, 0, \pm 1), (0, 0, \pm 2)$. (To see this, note that since Q contains no points with coordinates that are all even, $(0, 0, 1)$ is the only point in A with the last coordinate 1.) Now clearly $x = (-1, 1, 2) \in A + A$ and $y = (1, 3, 0) \in A + A$, but $(x + y)/2 = (0, 2, 1) \notin A + A$, so $A + A$ is not convex. Moreover, since $-A = A$, $DA = A + (-A)$ is also not convex.

The following construction provides further examples by lifting sets into one dimension higher. Let A and B be convex lattice sets in \mathbb{Z}^n , and choose origin-symmetric convex polyominoes C and D such that $A \subset C$ and $B \subset D$. In \mathbb{Z}^{n+1} take

$$X = (\{-1\} \times -A) \cup (\{0\} \times C) \cup (\{1\} \times A)$$

and

$$Y = (\{-1\} \times -B) \cup (\{0\} \times D) \cup (\{1\} \times B).$$

Then X and Y are convex polyominoes in \mathbb{Z}^{n+1} , and the intersection of $X + Y$ with the hyperplane $x_1 = 2$ is precisely $A + B$. If we let A be the origin-symmetric convex polyomino from the previous paragraph, take $B = A$, and apply this construction repeatedly, we obtain origin-symmetric convex polyominoes X in \mathbb{Z}^n , $n > 3$, such that $X + X = DX$ is not convex. Furthermore, if we apply this construction to the sets A and B in Fig. 1, we obtain origin-symmetric convex polyominoes X and Y in \mathbb{Z}^3 such that $X + Y$ is not even convex with respect to a coordinate direction.

In view of these somewhat surprising facts, it is necessary to supply proofs of two theorems supplying positive results in the planar case, that may otherwise seem rather obvious.

Theorem 3.1. *If A and B are convex polyominoes in \mathbb{Z}^2 , then $A + B$ is also a convex polyomino.*

Proof. Let A and B be convex polyominoes in \mathbb{Z}^2 , and let $P = \text{conv } A$ and $Q = \text{conv } B$. Without loss of generality, suppose that o is a vertex of both P and Q . Label the vertices of P clockwise around the boundary by $v_1 = o, v_2, \dots, v_m$, and let $E_i = [v_i, v_{i+1}]$, $i = 1, \dots, m$, be the edges of P , where the indices are taken modulo m . Since $P + Q$ is the union of P and the sets $Q + x$, where x is a boundary point of P , it is easy to see that $P + Q = P \cup F \cup G$, where

$$F = \bigcup_{i=1}^m (Q + v_i),$$

$$G = \bigcup_{i=1}^m G_i,$$

and $G_i = E_i + [o, u_i]$ is a parallelogram with u_i a suitable vertex of Q . (Specifically, u_i is chosen so that the line parallel to E_i through u_i supports Q on the same side as the line containing E_i supports P .) Since $(Q + v_i) \cap \mathbb{Z}^2 \subset A + B$ for $i = 1, \dots, m$, we have $F \cap \mathbb{Z}^2 \subset A + B$, so to prove the theorem it will suffice to show that $G \cap \mathbb{Z}^2 \subset A + B$.

To this end, fix $i \in \{1, \dots, m\}$. Since E_i has endpoints v_i and v_{i+1} and A is a polyomino, there is a simple polygonal arc $C_i = \bigcup_{j=1}^p [s_j, s_{j+1}] \subset P$, where $s_1 = v_i$, $s_{p+1} = v_{i+1}$, each line segment $[s_j, s_{j+1}]$ is of length one, and each $s_j \in A$. Moreover, we can ensure that if R_i is the open region bounded by C_i and E_i , then $R_i \cap \mathbb{Z}^2 = \emptyset$. Similarly, there is a simple polygonal arc $D_i = \bigcup_k U_k \subset Q$, where each U_k is a line segment of length one whose endpoints lie in B , such that D_i has endpoints o and u_i and the open region bounded by D_i and $[o, u_i]$ contains no lattice points. Hence, we conclude that

$$\begin{aligned} G_i \cap \mathbb{Z}^2 &\subset (C_i + [o, u_i]) \cap \mathbb{Z}^2 = \bigcup_{j=1}^p ([s_j, s_{j+1}] + [o, u_i]) \cap \mathbb{Z}^2 \\ &\subset \bigcup_{j=1}^p ([s_j, s_{j+1}] + D_i) \cap \mathbb{Z}^2 = \bigcup_{j=1}^{p+1} (s_j + D_i) \cap \mathbb{Z}^2 \subset A + B. \end{aligned}$$

Therefore $G \cap \mathbb{Z}^2 \subset A + B$, as required. \square

In Theorem 4 of [28], where planar convex polyominoes are called 4-connected convex digital sets, it is proved that a planar lattice set A is a convex polyomino if and only if it satisfies the *strong chord property*: Given distinct points $a, b \in A$ and $t \in \mathbb{R}$, there exist distinct points $c, d \in A$ and $r, s \in \mathbb{R}$ such that $c - d$ has length one, $c \in r[-1, 1]^2 + (1-t)a + tb$, $d \in s[-1, 1]^2 + (1-t)a + tb$, and $r + s < 2$. Theorem 3.1 asserts that the strong chord property is preserved by Minkowski addition.

It is natural to consider a generalization of Theorem 3.1 whereby convexity is replaced by u -convexity for a given primitive $u \in \mathbb{Z}^2$. However, the resulting statement is false. To see this, let $u = (2, 5)$, let $A = [(0, 0), (4, 0)] \cup [(4, 0), (4, 8)] \cap \mathbb{Z}^2$, and let $B = \{(0, 0), (0, 1), (0, 2)\}$. Then A and B are u -convex polyominoes, but $A + B$ is not u -convex, since $(0, 0) \in A + B$ and $(4, 10) \in A + B$ but $(2, 5) \notin A + B$.

Despite the previous example, the following version of Theorem 3.1 holds for vertical and horizontal convexity.

Theorem 3.2. *If A and B are vertically convex (or horizontally convex) polyominoes in \mathbb{Z}^2 , then $A + B$ is also a vertically convex (or horizontally convex, respectively) polyomino.*

Proof. Suppose that A and B are vertically convex polyominoes; the proof for the horizontally convex case is similar. We may assume that $o \in A$ and that A is contained in the closed half-plane lying to the left of the y -axis. Suppose that $a_i \in A$ and $b_i \in B$, $i = 1, 2$, are such that $a_1 + b_1$ lies below $a_2 + b_2$ on the same vertical line L . It suffices to show that each lattice point on L between $a_1 + b_1$ and $a_2 + b_2$ belongs to $A + B$.

From our assumptions it follows that b_i lies on or to the right of L , $i = 1, 2$. Since A is a polyomino, there is a simple polygonal arc D composed of line segments of length one whose endpoints lie in A , such that D has endpoints o and a_2 . Moreover, by the vertical convexity of A , we may assume that D lies in the closed strip bounded by the vertical lines through o and a_2 . Similarly, there is a simple polygonal arc E composed of line segments of length one whose endpoints lie in B , such that E has endpoints b_1 and b_2 , and by the vertical convexity of B , we may assume that E lies in the closed strip bounded by the vertical lines through b_1 and b_2 .

Without loss of generality, we may assume that the x -coordinate of b_2 is greater than or equal to that of b_1 , so that $a_2 + b_1$ lies on or to the left of L . Since the polygonal arc $D + b_1$ has endpoints b_1 and $a_2 + b_1$, it must meet L at some lattice point, $a_3 + b_1$, say. Now for each lattice point x on E , all the lattice points on the polygonal arc $D + x$ lie in $A + B$, and $D + x$ meets L . It follows that all the lattice points on L between $a_3 + b_1$ and $a_2 + b_2$ belong to $A + B$. Since A is vertically convex, all the lattice points on L between $a_1 + b_1$ and $a_3 + b_1$ also belong to $A + B$. It follows that each lattice point on L between $a_1 + b_1$ and $a_2 + b_2$ belongs to $A + B$. \square

The next lemma gives a formula for the difference body of a polygon that may be of some independent interest. It will find application in the proof of Theorem 3.4, which, together with the set A defined by (2), shows that it does not hold when $n \geq 3$.

Lemma 3.3. *Let P be a convex polygon in \mathbb{R}^2 with vertices v_1, \dots, v_m . Then*

$$DP = \left(\bigcup_{i=1}^m (P - v_i) \right) \cup \left(\bigcup_{i=1}^m (-P + v_i) \right). \quad (3)$$

Proof. By the definition of DP , we have

$$\left(\bigcup_{i=1}^m (P - v_i) \right) \cup \left(\bigcup_{i=1}^m (-P + v_i) \right) \subset DP, \quad (4)$$

so it suffices to prove the reverse containment.

Suppose first that P has no pair of parallel edges. Then each oriented edge of DP is either a translate of an oriented edge of P , or a translate of an oriented edge of $-P$. (This can be seen by noting that in \mathbb{R}^2 the surface area measure of DP is the sum of the surface area measures of P and $-P$, and recalling that in \mathbb{R}^2 the surface area measure of a convex polygon is essentially the set of outer normals to its edges, each weighted by the length of the corresponding edge; see, for example, Section A.1 of [14].) Therefore we can partition DP into triangles as follows:

$$DP = \bigcup_{i=1}^{2m} T_i, \quad (5)$$

where $T_i = \text{conv}\{E_i, o\}$ shares the oriented edge E_i with DP and has o as a vertex. Fix i and suppose, without loss of generality, that E_i is a translate of an oriented edge F_i of P . (The argument for the other case is similar.) Let $v \in E_i$ have outer unit normal u . Then $v \in DP$, so there are $p_i \in P$, $i = 1, 2$, such that $v = p_1 - p_2$. Since v is in the boundary of DP , p_1 and p_2 must be in the boundary of P and moreover have outer unit normals u and $-u$, respectively. The fact that P has no pair of parallel edges implies that $p_1 \in F_i$ and $p_2 = v_i$, say, is the vertex of P opposite to F_i . As $v \in E_i$ was arbitrary, we conclude that $E_i = F_i - v_i$ and therefore $T_i \subset P - v_i$. In view of (5) we have proved the reverse containment to (4), as required.

Now suppose that P has at least one pair of parallel edges. We can write $P = \lim_k P_k$, where each P_k is a convex polygon with the same number of vertices as P but with no pair of parallel edges. Since (3) holds for each P_k , it also holds for P . \square

Theorem 3.4. *Let A be a convex lattice set in \mathbb{Z}^2 and let $m \in \mathbb{N}$. If $A[m] = A + A + \cdots + A$ (m summands) and $DA = A + (-A)$, then $A[m] = (m \text{ conv } A) \cap \mathbb{Z}^2$ and $DA = D(\text{conv } A) \cap \mathbb{Z}^2$; in particular, $A[m]$ and DA are also convex lattice sets.*

Proof. Let A be a convex lattice set in \mathbb{Z}^2 and let $m \in \mathbb{N}$. Let x be a vertex of $P = \text{conv } A$, and let $\{T_1, \dots, T_k\}$ be a partition of P into lattice triangles with one vertex at x and the other two at adjacent vertices of P . Define $A_i = T_i \cap \mathbb{Z}^2$ and let $y \in mP \cap \mathbb{Z}^2$. Then $y \in mT_i \cap \mathbb{Z}^2$ for some $i \in \{1, \dots, k\}$, and if $A_i[m] = mT_i \cap \mathbb{Z}^2$, then $y \in A_i[m] \subset A[m]$ and $A[m] = mP \cap \mathbb{Z}^2$.

Therefore to prove $A[m] = mP \cap \mathbb{Z}^2$ we may assume that $P = \text{conv } A$ is a lattice triangle. The proof is by induction on the number r of lattice points in $\text{int } P$. Suppose that $r = 0$ and without loss of generality let $P = \text{conv}\{o, p, q\}$. Note that mP can be partitioned into translates of P of the form $P + ip + jq$, $i, j \in \{0, \dots, m-1\}$, $i + j \leq m-1$, and translates of $-P$ of the form $-P + ip + jq$, $i, j \in \{0, \dots, m-1\}$, $1 \leq i + j \leq m$. Let $z \in mP \cap \mathbb{Z}^2$. Since $\text{int } P$ contains no lattice points, z must actually belong to a translate $P + ip + jq$ of P for some $i, j \in \{0, \dots, m-1\}$, $i + j \leq m-1$. Therefore $z \in A + ip + jq \subset A[m]$, and the case $r = 0$ is finished. Suppose that the inductive hypothesis holds for $r < s$ and $\text{int } P$ contains exactly s lattice points. Let v be

one of these points, and partition P into three lattice triangles $P_1, P_2,$ and $P_3,$ each with a vertex at v and two others at vertices of P . Let $A_i = P_i \cap \mathbb{Z}^2, i = 1, 2, 3,$ and note that $P_i = \text{conv } A_i$ and $\text{int } P_i$ has less than s lattice points. Now if $w \in mP \cap \mathbb{Z}^2,$ then $w \in mP_i \cap \mathbb{Z}^2$ for some i and so $w \in A_i[m] \subset A[m]$. This completes the proof that $A[m] = (m \text{ conv } A) \cap \mathbb{Z}^2$.

To see that $DA = D(\text{conv } A) \cap \mathbb{Z}^2,$ let $P = \text{conv } A$ have vertices $v_1, \dots, v_m,$ and note that $DP = \text{conv } DA$. Let $x \in DP \cap \mathbb{Z}^2$. By Lemma 3.3, either $x \in P - v_i$ for some i or $x \in -P + v_i$ for some i . Since $A - v_i = (P - v_i) \cap \mathbb{Z}^2$ and $-A + v_i = (-P + v_i) \cap \mathbb{Z}^2,$ we have $x \in DA,$ as required. \square

4. X-Rays, the Covariogram, and Their Discrete Analogs

Let E be a bounded Lebesgue measurable set in \mathbb{R}^n . It is a well-known fact that E is determined, up to a set of measure zero, among all bounded Lebesgue measurable sets by its X-rays $X_u E$ in all directions $u \in S^{n-1}$; see, for example, Theorem C.1.1 of [14] and the references given there. In particular, if two convex bodies K and L in \mathbb{R}^n have equal X-rays in all directions, they must be equal. (For an introduction to X-rays of convex bodies, see Chapter 1 of [14].) The statement that the covariograms of K and L are equal provides weaker information. Indeed, it can be shown that $g_K = g_L$ if and only if for each $u \in S^{n-1},$ their X-rays $X_u K$ and $X_u L$ parallel to u are rearrangements of one another. We omit the proof since the main interest here is in finite sets, for which we prove the analogous statement below in Theorem 4.1.

The covariogram of a bounded Lebesgue measurable set is clearly unchanged by a translation or a reflection in the origin. If K is a convex body in $\mathbb{R}^n,$ then the support of g_K is DK . Therefore, if K and L are convex bodies such that $g_K = g_L,$ then $DK = DL$. It follows that a centrally symmetric convex body K is determined up to translation, among all convex bodies, by its covariogram. (As Bianchi [4] remarks, this requires also the observation that $g_K(o) = V(K)$ and the fact that in the class of all convex bodies with a given difference body, the unique origin-symmetric member has maximal volume, a consequence of the Brunn–Minkowski inequality.) Matheron [25, p. 86] (or see [18]) observed that $g_K = g_L$ also implies $\Pi K = \Pi L$.

Let A be a finite subset of \mathbb{R}^n . Again, A is determined among all finite sets by its discrete X-rays $X_u A$ in all directions $u \in S^{n-1}$. Indeed, one can say much more. By a result of Rényi [27], as extended by Heppes [22], if $|A| = m$ then A is determined by any set of $m + 1$ discrete X-rays in mutually nonparallel directions.

The discrete covariogram g_A of a finite subset A of \mathbb{R}^n is also unchanged by a translation or a reflection in the origin, and the support of g_A is the difference set $DA = A + (-A)$. By Theorem 3.4, we have for $n = 2$ the convenient fact that when A is a convex lattice set, g_A is supported by the convex lattice set DA . Note that

$$g_A(x) = |\{y \in A: y - x \in A\}|,$$

the number of chords in A that are translates of the line segment $[o, x]$. Thus the covariogram can be identified with the *multiset* $A + (-A)$, that is, the set DA where each element is repeated with multiplicity. In particular, $g_A = g_B$ if and only if A and B have the same set of chords, each repeated with multiplicity, and this is true if and only if the multisets $A + (-A)$ and $B + (-B)$ are equal.

Finite multisets A and B such that the multisets $A + (-A)$ and $B + (-B)$ are equal are sometimes called *homometric*. The structure of homometric multisets turns out to be important in X-ray crystallography via the Patterson function, by which the difference set of the atoms in a crystal can be determined from its X-ray diffraction pattern; see, for example, [19]. Rosenblatt and Seymour [29], using algebraic techniques, find an algorithm for reconstructing all multisets A with a given difference multiset $A + (-A)$. The one-dimensional case of this method is applied by Skiena and Sundaram [35] to the so-called partial digest problem in the mapping of DNA.

Let $u \in \mathbb{Z}^n$, let F be a finite subset of \mathbb{Z}^n , and let j be a nonnegative integer. Define

$$a_F(j, u) = |\{v \in u^\perp : X_u F(v) \geq j\}|.$$

If A and B are convex lattice sets in \mathbb{Z}^n , we say, by analogy with the usual meaning of the term, that their X-rays $X_u A$ and $X_u B$ are *rearrangements* of each other if $a_A(j, u) = a_B(j, u)$ for each j .

Theorem 4.1. *Let A and B be convex lattice sets in \mathbb{Z}^n . Then $g_A = g_B$ if and only if for each $u \in \mathbb{Z}^n$, the X-rays $X_u A$ and $X_u B$ of A and B parallel to u are rearrangements of each other.*

Proof. Let $u \in \mathbb{Z}^n$ be primitive, let F be a convex lattice set in \mathbb{Z}^n , and let j be a nonnegative integer. If $v \in u^\perp$ and $X_u F(v) = k \geq j + 1$, then there are exactly $k - j$ chords of F that are translates of $[o, ju]$ lying in the line through v parallel to u . Therefore

$$\begin{aligned} g_F(ju) &= \sum_{k=j+1}^{\infty} (k-j) |\{v \in u^\perp : X_u F(v) = k\}| \\ &= \sum_{k=j+1}^{\infty} (k-j)(a_F(k, u) - a_F(k+1, u)) \\ &= \sum_{k=j+1}^{\infty} a_F(k, u) \end{aligned} \tag{6}$$

for each j . (This is a discrete version of the final equation on p. 512 of [18].) The result follows immediately. \square

The previous theorem shows that, as in the continuous case, the statement that two finite sets have equal discrete covariograms is formally weaker than the statement that all their X-rays are equal. Below, we examine in more detail what information is given by the discrete covariogram. The following connection between continuous and discrete covariograms will be useful.

Lemma 4.2. *Let A be a finite subset of \mathbb{R}^n with discrete covariogram g_A , and let C be a bounded Lebesgue measurable set in \mathbb{R}^n with continuous covariogram g_C . Then for all $x \in \mathbb{R}^n$,*

$$\sum_{a_1, a_2 \in A} V_n((C + a_1) \cap (C + a_2 + x)) = \sum_{z \in DA} g_C(z + x) g_A(z).$$

Proof. Let $x \in \mathbb{R}^n$. We have

$$\begin{aligned} \sum_{a_1, a_2 \in A} V_n((C + a_1) \cap (C + a_2 + x)) &= \sum_{a_1, a_2 \in A} V_n(C \cap (C + a_2 - a_1 + x)) \\ &= \sum_{a_1, a_2 \in A} g_C(a_1 - a_2 + x) \\ &= \sum_{z \in DA} g_C(z + x) g_A(z). \quad \square \end{aligned}$$

Theorem 4.3. *Let A and B be finite subsets of \mathbb{R}^n with equal discrete covariograms. If C is a bounded Lebesgue measurable set such that*

$$V_n(A + C) = |A|V_n(C) \quad \text{and} \quad V_n(B + C) = |B|V_n(C), \quad (7)$$

then $A + C$ and $B + C$ have equal continuous covariograms.

Proof. The assumption $V_n(A + C) = |A|V_n(C)$ says that there are no overlaps in the Minkowski sum $A + C$, except for sets of measure zero. Specifically, the assumption implies that for all $a_1, a_2 \in A$ and almost all $c_1, c_2 \in C$, if $a_1 + c_1 = a_2 + c_2$, then $a_1 = a_2$ and $c_1 = c_2$. Using this fact, it is easy to see that

$$V_n((A + C) \cap (A + C + x)) = \sum_{a_1, a_2 \in A} V_n((C + a_1) \cap (C + a_2 + x)),$$

for all $x \in \mathbb{R}^n$. By Lemma 4.2, we obtain

$$g_{A+C}(x) = \sum_{z \in DA} g_C(z + x) g_A(z),$$

for all $x \in \mathbb{R}^n$. The conclusion of the theorem now follows from the hypotheses (7) and $g_A = g_B$, and the consequence $DA = DB$ of the latter equation. \square

The following is a result of Cabo and Janssen [8]. Since it appeared only in a technical report, we provide a proof (somewhat shorter than that in [8]).

Proposition 4.4. *If C and D are bounded, origin-symmetric, regular (equal to the closure of their interiors), compact subsets of \mathbb{R}^n with equal continuous covariograms, then $C = D$.*

Proof. It is well known (see, for example, [4]) that for any Lebesgue measurable set E , $g_E = 1_E * 1_{-E}$, the convolution of the characteristic functions of E and $-E$. The origin symmetry implies $1_C * 1_C = 1_D * 1_D$, and taking Fourier transforms, we obtain $(\widehat{1_C})^2 = (\widehat{1_D})^2$. Therefore $\widehat{1_C}(x) = \pm \widehat{1_D}(x)$, for each $x \in \mathbb{R}^n$. Since the Fourier transform is analytic, and any analytic function is determined by its values on a set with a limit point, we conclude that $\widehat{1_C} = \pm \widehat{1_D}$. Fourier inversion yields $1_C = 1_D$ almost everywhere, and since C and D are regular, we have $C = D$, as required. \square

Corollary 4.5. *If A and B are finite origin-symmetric subsets of \mathbb{R}^n with equal discrete covariograms, then $A = B$.*

Proof. Let C be an origin-symmetric n -dimensional ball of sufficiently small radius so that (7) holds. Then $g_{A+C} = g_{B+C}$, by Theorem 4.3. Since $A+C$ and $B+C$ are bounded, origin-symmetric, regular compact sets, Proposition 4.4 yields $A+C = B+C$, and it follows that $A = B$. \square

The *covariogram problem*, attributed to Matheron (compare p. 86 of [25]) asks whether g_K determines a convex body K , among all convex bodies, up to translation and reflection in the origin. The answer is affirmative when $n = 2$ and K is a polygon (a result of Nagel; see [3] for a complete proof) or when K is C_+^2 (see [4] and [5]), and is negative when $n \geq 4$ (proved by Bianchi [4]). Applications of the covariogram to stereology, image processing, and mathematical morphology are discussed in [7], [30], and [34]. Remarkably, the problem is still open when $n = 2$ and 3.

Let A be a finite set in \mathbb{R}^n . It is natural to ask whether g_A determines a convex lattice set A , among all convex lattice sets, up to translation and reflection in the origin. The authors of [11] use the algebraic method of Rosenblatt and Seymour [29] to give an algorithm for constructing a finite subset of \mathbb{Z}^n , if one exists, with a given covariogram. They also note, without giving specific examples, that sets not equal up to translation and reflection in the origin may have the same covariogram. The question raised above is also answered negatively in a strong way by the example depicted in Fig. 2. This displays convex polyominoes A and B that are not congruent, yet satisfy $g_A = g_B$.

Examples such as these have some relevance to the (continuous) covariogram problem. To see this, let A and B be finite subsets of \mathbb{Z}^n such that $g_A = g_B$. If $C = [0, 1]^n$, then by Theorem 4.3, the sets $K = A + [0, 1]^n$ and $L = B + [0, 1]^n$ have equal covariograms. In particular, if A and B are the convex polyominoes shown in Fig. 2, then the animals $A + [0, 1]^2$ and $B + [0, 1]^2$ have equal covariograms.

Since two convex bodies in \mathbb{R}^n have equal covariograms if and only if for each $u \in S^{n-1}$, their X-rays parallel to u are rearrangements of one another, this condition is much stronger than the property that the two bodies have equal chord length distributions; indeed, the distributions of lengths of chords parallel to any given direction must be equal. The previous example can therefore be viewed as related to an example of Mallows and Clark [24]. These authors answered an old question of Blaschke by exhibiting two noncongruent convex polygons with equal chord length distributions. Thus the polygons $A + [0, 1]^2$ and $B + [0, 1]^2$ have the stronger property that their covariograms are equal,

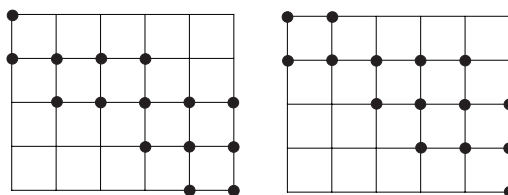


Fig. 2. Noncongruent convex polyominoes with equal covariograms.

but on the other hand the polygons are not convex. Indeed, they cannot be convex by Nagel's result mentioned above, but they are "close to convex" since they arise from convex polyominoes, and in this sense the example is probably optimal.

5. Projections of Finite Sets

To set the scene, we briefly describe some previous work on projections of finite sets. Motivated by a file management problem, Schwenk and Munro [33] consider the minimum of the geometric mean of the cardinality of projections of a finite subset A of \mathbb{R}^n onto coordinate subspaces of fixed dimension k . Their result in the case $k = n - 1$, for example, is that

$$\left(\prod_{j=1}^n |A|e_j^\perp \right)^{1/n} \geq |A|^{(n-1)/n}.$$

However, this follows immediately from (and is equivalent to) the Loomis–Whitney inequality (see, for example, p. 340 of [14]), as can be seen by replacing each point in A by a small cube whose facets are parallel to the coordinate hyperplanes. In [33] a more general inequality is given, involving projections onto all k -dimensional coordinate subspaces of \mathbb{R}^n , but this too follows from a corresponding generalization of the Loomis–Whitney inequality due to Hadwiger [21, Theorem 4.4.2]. Schwenk [32] tackles a variant of the problem above in which the geometric mean is replaced by the maximum, and solves this when $n = 3$ and $k = 2$, noting that the general case appears very difficult.

We begin our investigation with the following connection between projections of convex lattice sets and the discrete covariogram.

Lemma 5.1. *If A is a convex lattice set in \mathbb{Z}^n and $u \in \mathbb{Z}^n$ is primitive, then*

$$|A \mid u^\perp| = |A| - g_A(u). \quad (8)$$

Proof. By (6), we have

$$|A| - g_A(u) = g_A(0u) - g_A(1u) = a_A(1, u) = |\{v \in u^\perp : X_u A(v) \geq 1\}| = |A \mid u^\perp|. \quad \square$$

By analogy with the standard terminology for projection bodies (see, for example, Chapter 4 of [14]), we write $\Pi A = \Pi B$ if

$$|A \mid u^\perp| = |B \mid u^\perp|,$$

for all $u \in \mathbb{Z}^n$, and say that A and B have equal *projection counts*. (Note that ΠA and ΠB are not defined separately as sets.)

Corollary 5.2. *Let A and B be convex lattice sets in \mathbb{Z}^n . If $g_A = g_B$, then $\Pi A = \Pi B$.*

Proof. Note that $\Pi A = \Pi B$ if and only if $|A \mid u^\perp| = |B \mid u^\perp|$ for all primitive $u \in \mathbb{Z}^n$. Since $g_A(o) = |A|$, the implication $g_A = g_B \Rightarrow \Pi A = \Pi B$ follows immediately from Lemma 5.1. \square

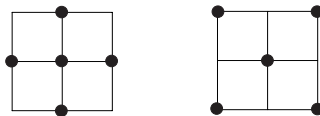


Fig. 3. Nonhomothetic lattice sets with equal projection counts.

By analogy with Aleksandrov's projection theorem, it is natural to ask whether, if centrally symmetric convex lattice sets have equal projection counts, they must be translates of each other. The symmetry assumption is necessary, since it follows from Corollary 5.2 that the convex lattice sets A_1 and B_1 shown in Fig. 2 satisfy $\Pi A_1 = \Pi B_1$, yet they are not translates of each other (nor even congruent). Convexity is clearly essential, but one can ask whether arbitrary finite lattice sets with equal projection counts must be homothetic. However, this is false, as the sets A_2 and B_2 in Fig. 3 demonstrate.

Surprisingly, there are actually noncongruent origin-symmetric convex lattice sets A and B in \mathbb{Z}^2 with $\Pi A = \Pi B$, as in Fig. 4. The two on the left in Fig. 4 are also convex polyominoes; while they are not equal up to a direct rigid motion, each is a reflection of the other in the line $y = x$. The convex lattice set on the right (contributed by a referee) in Fig. 4 is not a convex polyomino.

For any two of the sets in Fig. 4, A_3 and B_3 say, we have of course $DA_3 \neq DB_3$, since if $DA_3 = DB_3$, the origin symmetry of the sets and Theorem 3.4 would imply that $A_3 = B_3$. With the symmetry condition removed, however, it is possible to find noncongruent convex polyominoes with equal projection counts and equal difference sets. The following general construction can produce such examples. Let C be a finite origin-symmetric set in \mathbb{R}^n , and let $p \in C$ and $q \in C \setminus \{p, -p\}$ be such that there are points $r \in C \setminus \{p, q\}$ and $s \in C \setminus \{-p, q\}$ such that $r \in [p, q]$ and $s \in [-p, q]$. Let $A = C \setminus \{p, q\}$ and $B = C \setminus \{-p, q\}$. Then $\Pi A = \Pi B$. (Even more general constructions along these lines are possible.)

If we apply this construction with $C = \text{conv}\{p, -p, q, -q\} \cap \mathbb{Z}^2$, where $p = (-2, 1)$ and $q = (0, -1)$, we obtain the convex polyominoes A_4 and B_4 shown in Fig. 5. Note that these satisfy not only $\Pi A_4 = \Pi B_4$, but also $DA_4 = DB_4$. They also show that the converse of Corollary 5.2 is false, since $g_{A_4}(x) = 2 \neq 1 = g_{B_4}(x)$ when $x = (-2, 2)$.

Now consider the convex polyominoes A_5 and B_5 on the left and right of Fig. 6. Here we also have $DA_5 = DB_5$, but since $|A_5| \neq |B_5|$, we have $\Pi A_5 \neq \Pi B_5$ and hence $g_{A_5} \neq g_{B_5}$.

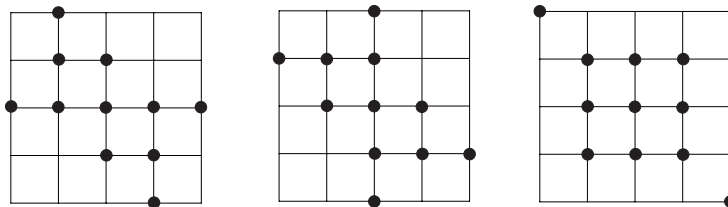


Fig. 4. Noncongruent origin-symmetric convex lattice sets with equal projection counts.

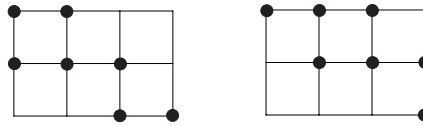


Fig. 5. Noncongruent convex polyominoes with equal projection counts and difference sets.

Despite the example shown in Fig. 3, it might be possible to find a conditional discrete version of Aleksandrov’s projection theorem, even in \mathbb{Z}^2 . The following question also remains open, even for origin-symmetric convex polyominoes.

Problem 5.3. Let $n \geq 3$, and let A and B be centrally symmetric convex lattice sets in \mathbb{Z}^n with $\dim A = \dim B = n$ such that for each $u \in \mathbb{Z}^n$, we have

$$|A \mid u^\perp| = |B \mid u^\perp|.$$

Is A a translate of B ?

6. Sections of Finite Sets

Dual to Aleksandrov’s projection theorem is Funk’s section theorem (see Theorem 7.2.6 of [14]), which implies that origin-symmetric convex bodies whose intersections with hyperplanes through the origin have equal volumes must be equal. It is easy to see that the discrete analog of this fact holds. Indeed, a stronger statement is true. Suppose that A and B are origin-symmetric convex lattice sets in \mathbb{Z}^n such that for each $u \in \mathbb{Z}^n$ we have

$$|A \cap u^\perp| \leq |B \cap u^\perp|.$$

Let $v \in \mathbb{Z}^n$ and let L_v be the line through o and v . Clearly we can choose $u \in \mathbb{Z}^n$ so that $L_v \subset u^\perp$, $A \cap u^\perp = A \cap L_v$, and $B \cap u^\perp = B \cap L_v$. Then $|A \cap L_v| \leq |B \cap L_v|$. It follows by convexity and origin symmetry that $A \subset B$. In particular, the dual question to Problem 5.3 has an affirmative answer.

Let A be a convex lattice set. For $u \in \mathbb{Z}^n$, let

$$m_A(u) = \max\{|A \cap (u^\perp + x)|: x \in \mathbb{Z}^n\}.$$

Suppose that $n = 2$. If $v \in \mathbb{Z}^2$ is primitive and parallel to u^\perp , then $kv \in DA$ if and only if there are $x, y \in A$ with $x - y = kv$ and hence if and only if $k \leq m_A(u) - 1$. It follows that convex lattice sets A and B in \mathbb{Z}^2 satisfy $m_A = m_B$ if and only if $DA = DB$. The noncongruent convex polyominoes shown in Fig. 6 therefore satisfy $m_A = m_B$.

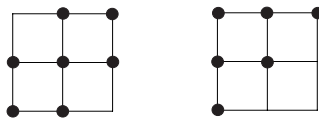


Fig. 6. Noncongruent convex polyominoes with equal difference sets but different cardinalities.

However, the following is a discrete version of the old and still-unsolved problem of Bonnesen (see Problem 8.10(i) of [14]).

Problem 6.1. Let A and B be convex lattice sets in \mathbb{Z}^n with $\dim A = \dim B = n$. If $n \geq 3$ and $m_A = m_B$, is $A = B$ up to translation and reflection in the origin?

For the remainder of this section, we consider a discrete version of the following dual Loomis–Whitney inequality proved by Meyer [26]. This states that if K is a convex body in \mathbb{R}^n , then

$$V(K)^{n-1} \geq d_n \prod_{i=1}^n V(K \cap e_i^\perp), \quad (9)$$

where

$$d_n = \frac{((n-1)!)^n}{(n!)^{n-1}}, \quad (10)$$

and equality holds if and only if K is a cross-polytope with vertices on the coordinate axes.

Problem 6.2. Find the best-possible constants c_n such that if A is a convex lattice set in \mathbb{Z}^n , then

$$|A|^{n-1} \geq c_n \prod_{i=1}^n |A \cap e_i^\perp|. \quad (11)$$

We claim that $c_n \leq d_n$, where d_n is given by (10). To see this, one can use the fact that for each convex body K in \mathbb{R}^n , the number $G(K)$ of lattice points in K satisfies

$$|G(mK) - V(mK)| = O(m^{n-1}), \quad (12)$$

for large m ; see, for example, [2]. Now let $K = \text{conv}\{\pm e_i, i = 1, \dots, n\}$ be the standard cross-polytope in \mathbb{R}^n , and let $\varepsilon > 0$ be given. Let $m \in \mathbb{N}$ and $A_m = mK \cap \mathbb{Z}^n$. Then, using (12) and the equality condition for (9), we have

$$\begin{aligned} |A_m|^{n-1} &\geq (V(mK) - O(m^{n-1}))^{n-1} = V(mK)^{n-1} - O(m^{n^2-n-1}) \\ &= d_n \prod_{i=1}^n V((mK) \cap e_i^\perp) - O(m^{n^2-n-1}) \\ &\geq d_n \prod_{i=1}^n (|A_m \cap e_i^\perp| - O(m^{n-2})) - O(m^{n^2-n-1}) \\ &= d_n \prod_{i=1}^n |A_m \cap e_i^\perp| - O(m^{n^2-n-1}) \\ &\geq (d_n - \varepsilon) \prod_{i=1}^n |A_m \cap e_i^\perp|, \end{aligned}$$

for sufficiently large m . This proves the claim.

Let $A = \{\pm e_i: 1 \leq i \leq n-1, \pm j e_n: 0 \leq j \leq m\} \subset \mathbb{Z}^n$. Then $|A| = 2m + 2n - 1$, $|A \cap e_n^\perp| = 2n - 1$, and $|A \cap e_i^\perp| = 2m + 2n - 3$. This shows that $c_n \leq 1/(2n - 1)$. Note that $d_3 = \frac{2}{9} > \frac{1}{5}$, while $d_n < 1/(2n - 1)$ for $n \geq 4$.

Theorem 6.3. *If A is a convex lattice set in \mathbb{Z}^2 , then*

$$|A| \geq \frac{1}{3} \prod_{i=1}^2 |A \cap e_i^\perp|, \quad (13)$$

and the constant $\frac{1}{3}$ is the best possible.

Proof. It will be convenient to set $x = |A \cap e_1^\perp|$ and $y = |A \cap e_2^\perp|$.

Suppose first that $o \in A$. Let $A \cap e_1^\perp = \{-a_2, -a_2 + 1, \dots, a_1\}$ and $A \cap e_2^\perp = \{-b_2, -b_2 + 1, \dots, b_1\}$. If $a_2 = b_2 = 0$, we have

$$|A| \geq (a_1 + 1)(b_1 + 1)/2 + 1. \quad (14)$$

Using (14) in each quadrant, we see that in general we have

$$\begin{aligned} |A| &\geq \left(\frac{(a_1 + 1)(b_1 + 1) + (a_1 + 1)(b_2 + 1) + (a_2 + 1)(b_1 + 1) + (a_2 + 1)(b_2 + 1)}{2} \right) \\ &\quad + 4 - (a_1 + a_2 + 1) - (b_1 + b_2 + 1) - 1 \\ &= \frac{(a_1 + a_2)(b_1 + b_2)}{2} + 3 = \frac{(x - 1)(y - 1)}{2} + 3. \end{aligned}$$

Now if $x, y \geq 3$, we have

$$\frac{(x - 1)(y - 1)}{2} + 3 \geq \frac{xy}{3} \quad \Leftrightarrow \quad \frac{(x - 3)(y - 3)}{6} + 2 \geq 0,$$

which is true. If $x \leq 2$, say, then $|A| \geq y + 1 \geq (xy)/2 + 1 \geq (xy)/3$.

Now suppose that $o \notin A$. Since A is convex, it suffices to consider the situation when $A \cap e_1^\perp = \{a_2, a_2 + 1, \dots, a_1\}$ and $A \cap e_2^\perp = \{b_2, b_2 + 1, \dots, b_1\}$, where $0 < a_2 \leq a_1$ and $0 < b_2 \leq b_1$. Using (14) again, we obtain

$$\begin{aligned} |A| &\geq \frac{(a_1 + 1)(b_1 + 1)}{2} + 1 - \left(\frac{(a_2 + 1)(b_2 + 1)}{2} - 1 \right) \\ &= \frac{(a_1 - a_2 + 1)(b_1 - b_2 + 1) + a_2(b_1 - b_2) + b_2(a_1 - a_2)}{2} \geq \frac{xy}{2}, \end{aligned}$$

an even better estimate in this case than the one required.

We have proved that $c_2 \geq \frac{1}{3}$. Let $A = \{(\pm 1, 0), (0, \pm i): 0 \leq i \leq m\}$. Then $|A| = 2m + 3$, $x = 3$, and $y = 2m + 1$, which shows that $c_2 \leq \frac{1}{3}$. \square

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