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Covering with Fat Convex Discs*

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Abstract. According to a theorem of L. Fejes Tóth [4], if non-crossing congruent copies of a convex disc K cover a convex hexagon H, then the density of the discs relative to H is at least area $K/f_K(6)$ where $f_K(6)$ denotes the maximum area of a hexagon contained in K. We say that a convex disc is *r-fat* if it is contained in a unit circle C and contains a concentric circle c of radius r. Recently, Heppes [7] showed that the above inequality holds without the non-crossing assumption if K is a 0.8561-fat ellipse. We show that the non-crossing assumption can be omitted if K is an r_0-fat convex disc with $r_0 = 0.933$ or an r_1-fat ellipse with $r_1 = 0.741$.

1. Introduction

By a *convex disc* we mean a compact convex set with non-empty interior and by an *n-gon* we mean a polygon with at most n sides in the Euclidean plane. We denote the interior, boundary, convex hull, and area of a disc K by int K, bd K, conv K, and area K, respectively. Further, let $f_K(n)$ denote the maximum area of an n-gon contained in K. To simplify our notation, we omit the subscript if K is a unit circle. Thus

$$f(n) = \frac{n}{2} \sin \frac{2\pi}{n}$$

stands for the area of a regular n-gon inscribed in a unit circle.

We consider coverings of a convex hexagon H by congruent copies $K_1, \ldots, K_N$ of a convex disc K. The density of these discs relative to H is defined as $\sum_{i=1}^{N} \text{area} K_i / \text{area} H$. We say that two convex discs *cross* if removing their intersection from them each disc becomes non-connected.

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According to a theorem of L. Fejes Tóth [4], (see also p. 167 in [5]) if N non-crossing congruent copies of a convex disc K cover a convex hexagon H, then the density of the discs relative to H is at least

$$\text{area}\, K / f_K(6).$$

The bound is exact for $N = 1$. If K is centrally symmetric, then it is also asymptotically sharp, or in other words, it is sharp for the whole plane.

The assumption that the discs do not cross seems to be superfluous. However, as examples by Heppes and Wegner show, we cannot simply exclude the possibility that crossing pairs occur. Wegner [11] constructed for all $N > 1$ a convex hexagon H and a convex disc K such that N congruent copies of K cover H but they cannot be rearranged to obtain a crossing-free covering of H.

The use of crossing pairs in a covering is particularly wasteful if K is close to a circle. L. Fejes Tóth suggested that it might be possible to eliminate the crossing-free condition at least for "round" convex discs. A first step in this direction was recently achieved by Heppes [7]. He called ellipses with half-axis $a = 1$ and $1 \geq b \geq b_0 = 0.8561$ fat, and showed that if a convex hexagon is covered by congruent fat ellipses, then the density of the ellipses relative to the hexagon is at least $2\pi/\sqrt{27}$. Quite against the convention, here and throughout the paper we use the term ellipse to denote a convex disc, rather than a curve. Extending the notion of fatness, we say that a convex disc K is *r-fat* if it is contained in a unit circle C and contains a concentric circle c of radius r. We call the set $C \backslash c$ containing bd K an *annulus associated to K*. The common center of the circles C and c is referred to as the *center of K*. Our main result is the following

Theorem 1. *Let $r_0 = 0.933$. If congruent copies of an r_0-fat convex disc K cover a convex hexagon H, then the density of the discs relative to H is at least*

$$\frac{\text{area}\, K}{f_K(6)}.$$

We also weaken a little the fatness condition in Heppes's result for ellipses:

Theorem 2. *Let $r_1 = 0.741$. If congruent r_1-fat ellipses cover a convex hexagon H, then the density of the ellipses relative to H is at least*

$$\frac{\pi}{f(6)} = \frac{2\pi}{\sqrt{27}}.$$

If a countable system of convex discs covers the whole plane, their density can be defined by a suitable limit (see, e.g., [6]). Recall that the *covering density* $\vartheta(K)$ of a convex disc K is defined as the infimum of the densities of all coverings of the whole plane by congruent copies of K. Before Heppes's result, besides the trivial examples of space-fillers, the circle was the only convex body for which $\vartheta(K)$ was known. Theorem 1 implies the following:

Corollary. *We have*

$$\vartheta(K) = \frac{\text{area}\, K}{f_K(6)}$$

for all centrally symmetric r_0-fat convex discs.

Principally, this enables us to determine the covering density of all r_0-fat convex discs. Of course, we still face the problem of calculating $f_K(6)$, which is a difficult task in general, but it can be done in special cases. We give some examples in Section 5.

We prove Theorems 1 and 2 in Section 4. Their proof is prepared in the next section by stating some known results which we shall need in the proof and in Section 3 by proving a lemma claiming that crossing fat discs are close.

2. Some Auxiliary Results

In this section we gather some known results needed in the proof of Theorems 1 and 2.

Proposition 1 [7]. *If E_1 and E_2 are ellipses with half-axes $a_1 \geq b_1$ and $a_2 \geq b_2$ such that*

$$\min_{i=1,2} \frac{a_i^2}{b_i} \leq \max_{i=1,2} \frac{b_i^2}{a_i},$$

then E_1 and E_2 do not cross.

The special case when one of the ellipses is a circle is stated on p. 480 in [7]. The proof of the general case is similar.

Proposition 2 [7]. *Let q be a point on the boundary of an ellipse of half-axes a and b, $a \geq b$, let l_1 be a line touching the ellipse at q and let l_2 be the line orthogonal to l_1 and containing the center of the ellipse. Then the maximum of the distance from q to l_2 is $a - b$.*

This is Lemma 1 in [7] (with the notation slightly changed in order to fit to ours).

Proposition 3 [7], [1]. *Let $K_1, \ldots, K_N$ be convex discs covering a convex hexagon H. Suppose that no pair of the discs $K_1, \ldots, K_N$ cross and no proper subset of them covers H. Then it is possible to construct convex polygons $D_1, \ldots, D_N$ with the number of sides $n_1, \ldots, n_N$ such that*

$$D_i \subset K_i \cap H \qquad \textit{for} \quad i = 1, \ldots, N,$$

$$\bigcup_{i=1}^{N} D_i = H,$$

$$(\operatorname{int} D_i) \cap (\operatorname{int} D_j) = \emptyset \qquad \textit{for} \quad i, j = 1, \ldots, N, \quad i \neq j,$$

and

$$\sum_{i=1}^{N} n_i \leq 6N.$$

The construction of polygons with the above properties was described first in [4], however, the presentation there is very vague. A more thorough treatment can be found on p. 170 in [5]. A very detailed description of the construction is given in [1]. A more general statement is proved in [2].

Proposition 4 [10]. *We have, for all convex discs K and for all integers $n \geq 3$,*

$$\frac{\text{area } K}{\pi} f(n) \leq f_K(n).$$

For a proof see [10], [6, pp. 36–37], or [9, pp. 14–15].

Proposition 5 [3]. *The sequence $f_K(n)$ is concave for all convex discs K:*

$$f_K(n+1) - f_K(n) \leq f_K(n) - f_K(n-1) \qquad \textit{for} \quad n \geq 4.$$

See [3], [5, p. 169], [6, pp. 34–35], or [9, pp. 11–13] for a proof.

3. Crossing Fat Discs are Close

Intuitively it is clear that if two fat discs cross, then they are close in some sense. The following lemma describes this precisely for different types of crossing fat discs.

Lemma.

(i) *If two r-fat discs cross, then the distance between their centers is at most*

$$2\sqrt{1 - r^2}.$$

(ii) *If two ellipses with half-axes a_1, b_1 and a_2, b_2 ($a_1 \geq b_1, a_2 \geq b_2$) cross, then the distance between their centers is at most*

$$\sqrt{(a_1 + a_2 - b_1 - b_2)^2 + \left(\max_{i=1,2} a_i - \min_{i=1,2} b_i\right)^2}.$$

(iii) *Let K be an r-fat disc and let E be an ellipse with half-axes a and b such that $b \leq a \leq b^2$. If K and E cross, then the distance between their centers is at most*

$$\sqrt{\left(a - b + \sqrt{1 - r^2}\right)^2 + (a - r)^2}.$$

Part (ii) of the lemma is due to Heppes. The proposition in [7] refers to the special case when $a_1 = a_2 = 1$ and $b_i \geq b_0$, however he actually proves claim (ii) of our lemma.

For the proof of (i) we consider two r-fat discs K_1 and K_2 with associated annuli $C_i \backslash c_i$ centered at p_i, $i = 1, 2$ (see Fig. 1). We introduce Cartesian coordinates so that the coordinates of p_1 and p_2 are $(0, -a)$ and $(0, a)$, respectively. Suppose that

$$a > \sqrt{1 - r^2}. \tag{1}$$

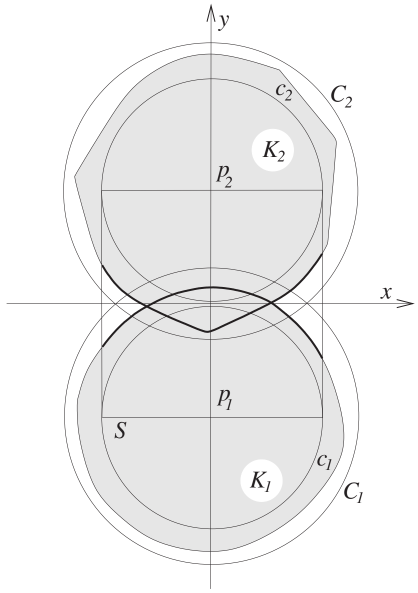

Fig. 1

We shall show that then K_1 and K_2 do not cross.

Let

$$S = \{(x, y) | -r < x < r,\ -a < y < a\}$$

and observe that (1) implies, on one hand, that

$$(\text{bd}\, K_1) \cap (\text{bd}\, K_2) \subset (C_1 \backslash c_1) \cap (C_2 \backslash c_2) \subset S,$$

and, on the other hand, that bd $K_1 \cap S$ is the graph of a concave function $f(x)$ and bd $K_2 \cap S$ is the graph of a convex function $g(x)$. It is then clear that (bd K_1) $\cap$ (bd K_2), if not empty, consists of a single point, of two points, or a line segment. In neither case can K_1 and K_2 cross. This settles part (i) of the lemma.

Consider now an r-fat disc K with associated annulus $C \backslash c$ centered at p, and an ellipse E with half-axes a and b, $b \le a \le b^2$ (see Fig. 2). Since $b^2/a \ge 1$, Proposition 1 implies that E and C do not cross. Suppose that K and E cross. Then the set $E \backslash K$ consists of at least two connected components. As E and C do not cross, one of these components is contained in C (actually in $C \backslash c$).

Let M be a connected component of $E \backslash K$ such that $M \subset C$. Let q_1 and q_2 be two points on bd M dividing bd M into two arcs, one of which is contained in bd K and the other one in bd E. Let l_1 be the line parallel to $q_1 q_2$ tangent to bd $M \cap$ bd E at a point, say q. Let l_2 be the line orthogonal to l_1 through the center of E and let l_3 be the line parallel to l_2 through p. According to Proposition 2 the distance from q to l_2 is at most $a - b$. The distance from q to l_3 is at most $\sqrt{1 - r^2}$, hence the distance between the lines l_2 and l_3 is at most $a - b + \sqrt{1 - r^2}$. Now (iii) follows by noting that the difference of the distances of the centers of E and K from l_1 is at most $a - r$.

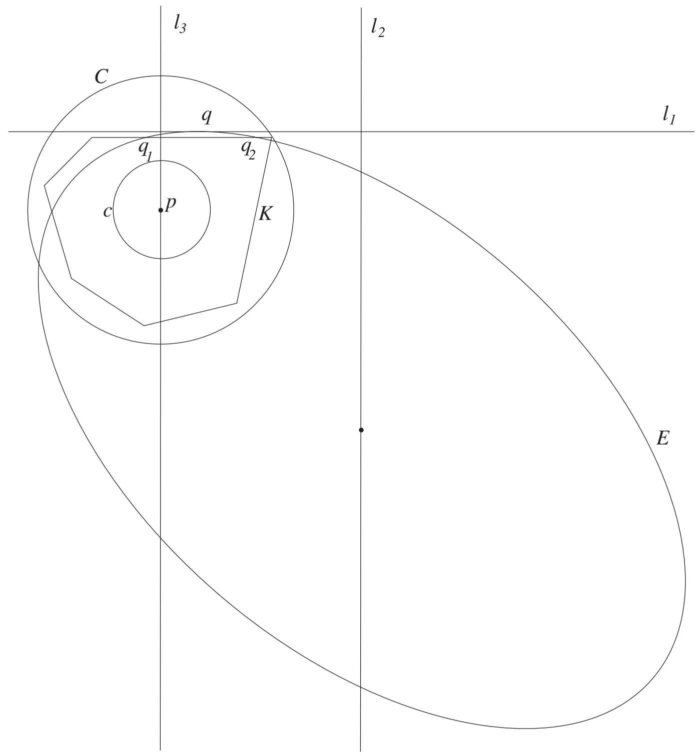

Fig. 2

4. Proof of the Theorems

We recall that the proof of Theorem 1 in the case when there are no crossing pairs is based on Proposition 3. The construction mentioned there cannot be carried out if crossings occur. We shall construct a new, crossing-free covering by successively replacing crossing pairs by other discs.

Let $K_1, \dots, K_N$ be congruent copies of an r_0-fat convex disc K covering the hexagon H. It will help to understand some steps in the proof if we recall the proof for the case when crossings do not occur. Then, considering the polygons D_i described in Proposition 3 and using Proposition 5, we have

$$\text{area } H = \sum_{i=1}^{N} \text{area } D_i \le \sum_{i=1}^{N} f_K(n_i) \le \sum_{i=1}^{N} (f_K(6) + (n_i - 6)(f_K(6) - f_K(5))) \le N f_K(6),$$

which is exactly, what we have to prove.

Suppose now that two of the discs, say K_1 and K_2 cross. Let $C_i \setminus c_i$ with center p_i be an annulus associated to K_i $(i = 1,\ 2)$. According to the lemma, the distance between p_1 and p_2 is at most $2\sqrt{1 - r_0^2}$.

Let E_1 be an ellipse with half-axes

$$a_1 = 1 + \sqrt{1 - r_0^2} = 1.3598\ldots$$

and

$$b_1 = \frac{r_0^2}{a_1}\left(1 + \frac{f(6)}{\pi}\right) = 1.1695\ldots. \tag{2}$$

Then a congruent copy of E_1, placed so that its center is the midpoint of the segment $p_1 p_2$ and its longer axis lies on the line $p_1 p_2$, covers $\mathrm{conv}(C_1 \cap C_2)$, hence it covers $K_1 \cap K_2$. To see this it is enough to observe that the minimum radius of curvature of E_1,

$$\underline{\varrho}_1 = b_1^2/a_1 = 1.0057\ldots,$$

is greater than 1. For later reference we note that the maximum radius of curvature of E_1 is

$$\overline{\varrho}_1 = a_1^2/b_1 = 1.5812\ldots.$$

The choice of a_1 is natural. We also need the restriction that the minimum radius of curvature of E_1 is at least 1. Another condition specifying the particular choice of b_1 is the following. Our goal is to construct a crossing-free covering of H in which a copy of E_1 substitutes two copies of K. We apply Proposition 3 to this new covering. If to a copy of E_1 a polygon D with n sides is associated, we would like to have that its area is at most $f_K(n) + f_K(6)$. This is guaranteed by (2). Indeed, (2), Proposition 4, and the obvious inequalities $f(n) < \pi$ and area $K \geq \pi r_0^2$ imply

$$\text{area}\, D \leq f_{E_1}(n) = a_1 b_1 f(n) = r_0^2 f(n)\left(1 + \frac{f(6)}{\pi}\right) < f_K(n) + f_K(6).$$

Replacing the discs K_1 and K_2 by a congruent copy of E_1 covering their union we obtain again a covering of H. We repeat this process until no two discs from the original covering cross. Of course, it can occur that now two ellipses or an ellipse and one of the original discs cross. We consider first the latter case.

If a copy of E_1 and a copy of K cross, then according to part (iii) of the lemma the distance between their centers is at most

$$\sqrt{\left(a_1 - b_1 + \sqrt{1 - r_0^2}\right)^2 + (a_1 - r_0)^2} = 0.6964\ldots.$$

Let E_2 be an ellipse with half-axes

$$a_2 = \tfrac{1}{2}\left(a_1 + 1 + \sqrt{\left(a_1 - b_1 + \sqrt{1 - r_0^2}\right)^2 + (a_1 - r_0)^2}\right) = 1.5281\ldots$$

and

$$b_2 = \frac{r_0^2}{a_2}\left(1 + \frac{2f(6)}{\pi}\right) = 1.5118\ldots. \tag{3}$$

We note that the extreme values of the radius of curvature of E_2 are

$$\underline{\varrho}_2 = b_2^2/a_2 = 1.4956\dots \quad \text{and} \quad \overline{\varrho}_2 = a_2^2/b_2 = 1.5446\dots.$$

As $\underline{\varrho}_2 > a_1$, the union of a unit circle and a circle of radius a_1 whose centers are at a distance of $\sqrt{(a_1 - b_1 + \sqrt{1-r_0^2})^2 + (a_1 - r_0)^2}$ apart can be covered by a congruent copy of E_2. Therefore, the union of a crossing pair of copies of E_1 and K can be covered by a copy of E_2. Using this, we successively eliminate all such crossing pairs by replacing them by an appropriate copy of E_2.

In the covering obtained thus, four different types of crossings can occur: Two copies of E_1 or two copies of E_2, as well as a copy of E_1 and a copy of E_2 can cross. Finally, a copy of E_2 and a copy of K can cross. Using the lemma and considerations similar to the above, one can see that the union of any of these types of crossing pairs of discs can be covered by a circle of radius

$$R = r_0\sqrt{1 + \frac{3f(6)}{\pi}} = 1.7407\dots. \tag{4}$$

We continue to replace all crossing pairs of the system by circles of radius R. We end up with a covering of H consisting of four types of discs, in which, as can be easily checked, crossings can occur only among an r_0-fat disc from the original covering and a circle of radius R. If such a crossing occurs, say a copy of K with associated annulus $C\backslash c$ centered at p and a circle $\overline{C}$ of radius R centered at q cross, then the distance between p and q is at most

$$d = \sqrt{1 - r_0^2 + (R - r_0)^2} = 0.8842\dots.$$

An easy computation shows that then the length of the common chord of $\overline{C}$ and C is at most

$$\frac{1}{d}\sqrt{4R^2d^2 - (R^2 + d^2 - 1)^2} = 1.4168\dots < 2r_0$$

and, moreover, $\overline{C}$ contains the greater of the two segments of C determined by this common chord. Thus $C\backslash\overline{C}$ can be covered by a circle of radius r_0. Now, in our covering whenever a circle of radius R and a copy of K cross, we discard the latter and introduce a circle of radius r_0 to cover the part of the discarded copy of K that was left uncovered by the circle of radius R.

A circle of radius r_0 introduced thus cannot cross an ellipse or a circle. Still, it can occur that such a circle and one of the original discs cross. We observe that a circle of radius r_0 is itself an r_0-fat disc, therefore the union of such a crossing pair can be covered by a congruent copy of E_1, and we can start doing the whole procedure over again. Since the number of the original discs decreases in each cycle, the procedure ends in finitely many steps.

The final result of the procedure is a new covering of H consisting of, besides some of the original discs, ellipses congruent to E_1 or E_2, as well as of circles of radius r_0 and R. It might occur that one of these discs is contained in the union of the others. If this happens, we discard it.

As the new covering is crossing-free, we can now apply Proposition 3 to them. Let $D_{11}, \ldots, D_{1N_1}, D_{21}, \ldots, D_{2N_2}, D_{31}, \ldots, D_{3N_3}, D_{41}, \ldots, D_{4N_4}$, and $D_{51}, \ldots, D_{5N_5}$ be the polygons associated to the original discs, the circles of radius r_0, the ellipses congruent to E_1, the ellipses congruent to E_2, and the circles of radius R, respectively. Denoting by n_{ij}, $1 \leq i \leq N_j$, $1 \leq j \leq 5$, the number of sides of D_{ij} we have

$$\sum_{i=1}^{5} \sum_{j=1}^{N_i} n_{ij} \leq 6 \sum_{i=1}^{5} N_i. \tag{5}$$

We note that ultimately in the new covering a copy of E_1 replaces two original discs, a copy of E_2 replaces three copies of K, and, finally a circle of radius R replaces at least four discs from the original covering. Hence,

$$N_1 + N_2 + 2N_3 + 3N_4 + 4N_5 \leq N. \tag{6}$$

Our next goal is to give upper bounds for the areas of polygons contained in different types of discs. We have, by definition,

$$\text{area}\, D_{1j} \leq f_K(n_{1j}) \quad \text{for} \qquad 1 \leq j \leq N_1. \tag{7}$$

Obviously, area $K \geq r_0^2\pi$, hence in view of Proposition 4

$$r_0^2 f(n) \leq f_K(n). \tag{8}$$

Therefore

$$\text{area}\, D_{2j} \leq r_0^2 f(n_{2j}) \leq f_K(n_{2j}) \qquad \text{for} \quad 1 \leq j \leq N_2. \tag{9}$$

Further, it follows from (2), (3), (4), (8), and the obvious inequality $f(n) < \pi$, that

$$\text{area}\, D_{3j} \leq f_{E_1}(n_{3j}) = a_1 b_1 f(n_{3j}) \leq f_K(n_{3j}) + f_K(6) \qquad \text{for} \quad 1 \leq j \leq N_3, \tag{10}$$
$$\text{area}\, D_{4j} \leq f_{E_2}(n_{4j}) = a_2 b_2 f(n_{4j}) \leq f_K(n_{4j}) + 2 f_K(6) \qquad \text{for} \quad 1 \leq j \leq N_4, \tag{11}$$

and

$$\text{area}\, D_{5j} \leq R^2 f(n_{5j}) \leq f_K(n_{5j}) + f_K(6) \qquad \text{for} \quad 1 \leq j \leq N_5. \tag{12}$$

Inequalities (7), (9), (10), (11), and (12) imply that

$$\text{area}\, H \leq \sum_{i=1}^{5} \sum_{j=1}^{N_i} \text{area}\, D_{ij} \leq \sum_{i=1}^{5} \sum_{j=1}^{N_i} f_K(n_{ij}) + (N_3 + 2N_4 + 3N_5) f_K(6). \tag{13}$$

By Proposition 5 and inequality (5) it follows that

$$\sum_{i=1}^{5} \sum_{j=1}^{N_i} f_K(n_{ij}) \leq \sum_{i=1}^{5} N_i f_K(6). \tag{14}$$

Combining (13) and (14) and taking into account (6) we get

$$\text{area}\, H \leq N f_K(6).$$

Multiplying both sides by area K and rearranging we get the claim of Theorem 1. □

The proof of Theorem 2 is similar. We introduce two ellipses and a circle. In their definition the function

$$g(k,n) = \frac{kf(6) + (n-6)(f(6) - f(5))}{f(n)}, \tag{15}$$

defined for integers $k \geq 1$ and $n \geq 3$, plays an important role. It is easy to check that

$$\min_{n\geq 3} g(2,n) = g(2,7) = 1.9745\dots, \tag{16}$$

$$\min_{n\geq 3} g(3,n) = g(3,8) = 2.9115\dots, \tag{17}$$

and

$$\min_{n\geq 3} g(4,n) = g(4,9) = 3.8214\dots. \tag{18}$$

Let E_1 be an ellipse with half-axes

$$a_1 = 1 + \frac{\sqrt{5}}{2}(1 - r_1) = 1.2895\dots$$

and

$$b_1 = \frac{r_1}{a_1} g(2,7) = 1.1374\dots. \tag{19}$$

Let E_2 be an ellipse with half-axes

$$a_2 = \tfrac{1}{2}\left(1 + a_1 + \sqrt{(1 + a_1 - r_1 - b_1)^2 + (a_1 - r_1)^2}\right) = 1.4875\dots$$

and

$$b_2 = \frac{r_1}{a_2} g(3,8) = 1.4503\dots. \tag{20}$$

Finally, let E_3 be a circle of radius

$$R = \sqrt{r_1 g(4,9)} = 1.6827\dots. \tag{21}$$

We note that the extreme values of the radius of curvature of E_1 are

$$\underline{\varrho}_1 = b_1^2/a_1 = 1.0032\dots \quad \text{and} \quad \overline{\varrho}_1 = a_1^2/b_1 = 1.4620\dots$$

and those for E_2 are

$$\underline{\varrho}_2 = b_2^2/a_2 = 1.4140\dots \quad \text{and} \quad \overline{\varrho}_2 = a_2^2/b_2 = 1.5257\dots.$$

Let E be an r_1-fat ellipse with half-axes 1 and b, $r_1 \leq b \leq 1$. Using the lemma it can be checked that if two congruent copies of E cross, then their union can be covered by a copy of E_1, and if a copy of E and a copy of E_1 cross, then their union can be covered by a copy of E_2. Further, if any two copies of E, E_1, or E_2 cross, then their union can be covered by a copy of E_3.

Consider now N congruent copies of E covering a convex hexagon H. We construct a new, crossing-free covering of H consisting of congruent copies of E, E_1, E_2, and E_3 as follows. We start by replacing step-by-step all crossing pairs of copies of E by appropriate copies of E_1 covering their union. We continue by successively replacing all crossing pairs of a copy of E and a copy of E_1 by appropriate copies of E_2. If there are still crossing pairs of ellipses in the covering, we replace them, again step-by-step, by copies of E_3. Since R is greater than the maximum radius of curvature of any of the ellipses E, E_1, or E_2, no copy of E_3 can cross a copy of these ellipses. Thus, the resulting system is crossing-free. Finally, we reduce the covering by discarding any disc which is contained in the union of some others.

To the resulting covering of H, consider the convex polygons described in Proposition 3. Let $D_{11}, \ldots, D_{1N_1}$, $D_{21}, \ldots, D_{2N_2}$, $D_{31}, \ldots, D_{3N_3}$, and $D_{41}, \ldots, D_{4N_4}$ be the polygons associated to the copies of E, E_1, E_2, and E_3, respectively. Denoting by n_{ij}, $1 \le i \le N_j,\ 1 \le j \le 4$, the number of sides of D_{ij} we have

$$\sum_{j=1}^{4}\sum_{i=1}^{N_j} n_{ij} \le 6\sum_{j=1}^{4} N_j. \tag{22}$$

The construction of the new covering readily implies that

$$N_1 + 2N_2 + 3N_3 + 4N_4 \le N. \tag{23}$$

We continue to give upper bounds for the areas of the polygons D_{ij}. The definition of $f_E(n)$, together with Proposition 4, yields that

$$\text{area}\, D_{1j} \le f_E(n_{1j}) \le f_E(6) + (n_{1j} - 6)(f_E(6) - f_E(5)) \qquad \text{for} \quad 1 \le j \le N_1. \tag{24}$$

Using relations (15)–(21) we get

$$\begin{aligned} \text{area}\, D_{2j} &\le f_{E_1}(n_{2j}) = a_1 b_1 f(n_{2j}) = r_1 f(n_{2j}) g(2,7) \le b f(n_{2j}) g(2, n_{2j}) \\ &= 2 f_E(6) + (n_{2j} - 6)(f_E(6) - f_E(5)) \qquad \text{for} \quad 1 \le j \le N_2, \end{aligned} \tag{25}$$

$$\begin{aligned} \text{area}\, D_{3j} &\le f_{E_2}(n_{3j}) = a_2 b_2 f(n_{3j}) = r_1 f(n_{3j}) g(3,8) \le b f(n_{3j}) g(3, n_{3j}) \\ &= 3 f_E(6) + (n_{3j} - 6)(f_E(6) - f_E(5)) \qquad \text{for} \quad 1 \le j \le N_3, \end{aligned} \tag{26}$$

and

$$\begin{aligned} \text{area}\, D_{4j} &\le f_{E_3}(n_{4j}) = R^2 f(n_{4j}) = r_1 f(n_{4j}) g(4,9) \le b f(n_{4j}) g(4, n_{4j}) \\ &= 4 f_E(6) + (n_{4j} - 6)(f_E(6) - f_E(5)) \qquad \text{for} \quad 1 \le j \le N_4. \end{aligned} \tag{27}$$

The combination of inequalities (22)–(27) readily yields

$$\text{area}\, H = \sum_{i=1}^{4}\sum_{j=1}^{N_i} \text{area}\, D_{ij} \le f_E(6) \sum_{j=1}^{4}\sum_{i=1}^{N_j} j + (f_E(6) - f_E(5)) \sum_{j=i}^{4}\sum_{j=1}^{N_i} (n_{ij} - 6) \le N f_E(6).$$

This completes the proof of Theorem 2. □

5. Remarks

In our theorems we gave r_0 and r_1 to three decimals. In fact, these are rounded up values of the solutions of the equations

$$\left(1+\frac{f(6)}{\pi}\right)^2 r_0^4 = \left(1+\sqrt{1-r_0^2}\right)^3$$

and

$$(g(2,7))^2 r_1^2 = \left(1+\frac{\sqrt{5}}{2}(1-r_1)\right)^3,$$

respectively, with which the theorems still hold. More accurate values are

$$r_0 = 0.93242333\ldots \quad \text{and} \quad r_1 = 0.74039619\ldots .$$

The fatness condition in Theorem 1 can be weakened a little further. It is easy to see that the bound for the distance of the centers of crossing r-fat discs in part (i) of the lemma is best possible. However, it can also be seen that if two congruent copies of an r-fat disc K cross and the distance between their centers is $d \le 2\sqrt{1-r^2}$, then area $K \ge rd/2 + r^2(\pi - \arctan(d/2r))$. In the proof of Theorem 1 we used the lower bound πr_0^2 for area K. Using instead the information above, we can show that the density bound of Theorem 1 holds for 0.93125458-fat convex discs. We omit the proof, since it is more involved and the improvement it gives is very slight.

Let P_n denote a regular n-gon inscribed in a unit circle. P_n is r_0-fat for $n \ge 9$. The regular octagon P_8 is not r_0-fat, and it misses the fatness-bound 0.93125458, as well. However, repeating the argument of the proof of Theorem 1 and taking into account that the area of P_8 is considerably greater than the area of the circle inscribed into it, we can see that the density-bound of the theorem holds for P_8 without the assumption that the octagons do not cross. It is easy to find the hexagon of maximum area contained in P_n. Its vertices are vertices of P_n and the difference between the lengths of its sides is as small as possible under this condition. Using this we get that

$$\vartheta(P_{6k}) = \frac{k\sin(\pi/3k)}{\sin(\pi/3)}$$

and

$$\vartheta(P_{6k\pm 2}) = \frac{(3k\pm 1)\sin(\pi/(3k\pm 1))}{2\sin(k\pi/(3k\pm 1)) + \sin((k\pm 1)\pi/(3k\pm 1))}$$

for all $k \ge 1$.

Mount and Silverman [8] gave an algorithm which determines the value of $f_K(6)$ in $O(n)$ time if K is a convex n-gon. Their algorithm can now be applied to determine the covering density of a centrally symmetric r_0-fat n-gon in $O(n)$ time.

Finally we note, that using the method developed by Böröczky Jr. in [2], a slight modification of our proof yields that the bounds for the density in Theorems 1 and 2 hold if H is an arbitrary convex disc, provided that the number of discs is sufficiently large.

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