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On the L_p Minkowski Problem for Polytopes*

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Abstract. Two new approaches are presented to establish the existence of polytopal solutions to the discrete-data L_p Minkowski problem for all $p > 1$.

Introduction

As observed by Schneider [23], the Brunn–Minkowski theory springs from joining the notion of ordinary volume in Euclidean d-space, $\mathbb{R}^d$, with that of *Minkowski combinations* of convex bodies. One of the cornerstones of the Brunn–Minkowski theory is the classical *Minkowski problem*. For polytopes the problem asks for the necessary and sufficient conditions on a set of unit vectors $u_1, \ldots, u_n \in S^{d-1}$ and a set of real numbers $\alpha_1, \ldots, \alpha_n > 0$ that guarantee the existence of a polytope in $\mathbb{R}^d$ with n facets whose outer unit normals are $u_1, \ldots, u_n$ and such that the facet whose outer unit normal is u_i has *area* (i.e., $(d-1)$-dimensional volume) α_i. This problem was completely solved by Minkowski himself (see [23] for reference): if the unit vectors do not lie on a closed hemisphere of S^{d-1}, then a solution (i.e., polytope) exists if and only if

$$\sum_{i=0}^{n} \alpha_i u_i = 0.$$

In addition, the solution is unique up to a translation.

In the middle of the last century, Firey (see [23] for references) extended the notion of a Minkowski combination of convex bodies and for each real $p > 1$ defined what

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are now called *Firey–Minkowski L_p combinations* of convex bodies. A decade ago, in [12], Firey–Minkowski L_p combinations were combined with volume and the result was an embryonic L_p Brunn–Minkowski theory—often called the Brunn–Minkowski–Firey theory. During the past decade various elements of the L_p Brunn–Minkowski theory have attracted increased attention (see, e.g., [3]–[5], [8]–[20], [22], and [24]–[29]).

A central problem within the L_p Brunn–Minkowski theory is the L_p Minkowski problem. A solution to the L_p Minkowski problem when the *data* is even was given in [12]. This solution turned out to be a critical ingredient in the recently established L_p affine Sobolev inequality [18].

Suppose the real index p is fixed. The L_p Minkowski problem for polytopes asks for the necessary and sufficient conditions on a set of unit vectors $u_1, \dots, u_n \in S^{d-1}$ and a set of real numbers $\alpha_1, \dots, \alpha_n > 0$ that guarantee the existence of a polytope in $\mathbb{R}^d$ containing the origin in its interior with n facets whose outer unit normals are $u_1, \dots, u_n \in S^{d-1}$ and such that if the facet with outer unit normal u_i has area a_i and distance from the origin h_i, then for all i,

$$h_i^{1-p} a_i = \alpha_i.$$

Obviously, the case $p = 1$ is the classical problem. For $p > 1$ uniqueness was established in [12]. The L_p Minkowski problem for polytopes is the *discrete-data* case of the general L_p Minkowski problem (described below).

In the discrete *even-data* case of the problem, outer unit normals $u_1, u_{-1}, \dots, u_m, u_{-m}$ are given in antipodal pairs, where $u_{-i} = -u_i$, and $\alpha_{-i} = \alpha_i$. With the exception of the case $p = d$, existence (and uniqueness) for the even problem was established in [12] for all cases (where the unit vectors do not lie in a closed hemisphere of S^{d-1}). A *normalized* version (discussed below) of the problem was proposed and completely solved for $p > 1$ and even data in [19]. For $d = 2$, the important case $p = 0$ of the discrete-data L_p Minkowski problem was dealt with by Stancu [26], [27].

A solution to the L_p Minkowski problem for $p > d$ was given by Guan and Lin [8] and independently by Chou and Wang [5]. The work of Chou and Wang [5] goes further and solves the problem for polytopes for all $p > 1$.

The works of Guan and Lin [8] and Chou and Wang [5] focus on existence and regularity for the L_p Minkowski problem. Both works make use of the machinery of the theory of PDEs. The classical Minkowski problem has proven to be of interest to those working in both discrete and computational geometry. It is likely that the L_p extension of the problem will in time prove to be of interest to those working in these fields as well. An approach accessible to researchers in convex, discrete, and computational geometry appears to be desirable. This article presents two such approaches.

We begin by recalling the formulation of the L_p Minkowski problem in full generality. For a convex body K let $h_K = h(K, \cdot)\colon \mathbb{R}^d \to \mathbb{R}$ denote the *support function* of K; i.e., for $x \in \mathbb{R}^d$, let $h_K(x) = \max_{y \in K} \langle x, y \rangle$, where $\langle x, y \rangle$ is the standard inner product of x and y in $\mathbb{R}^d$. The induced norm is denoted by $|\cdot|$. We write $V(K)$ for the d-dimensional volume of the convex body K in $\mathbb{R}^d$.

The *surface area measure*, $S(K, \cdot)$, of the convex body K is a Borel measure on the unit sphere, $S^{d-1} := \{x \in \mathbb{R}^d\colon |x| = 1\}$, such that

$$\lim_{\varepsilon \to 0^+} \frac{V(K + \varepsilon Q) - V(K)}{\varepsilon} = \int_{S^{d-1}} h_Q(u) S(K, du), \tag{1}$$

for each convex body Q. Here $K + \varepsilon Q$ is the *Minkowski combination* defined by

$$h(K + \varepsilon Q, \cdot) = h(K, \cdot) + \varepsilon h(Q, \cdot).$$

Existence of the surface area measure was shown by Aleksandrov and Fenchel and Jessen (see [23]). The limit on the left-hand side of (1) is also equal to the special mixed volume $dV_1(K, Q) := dV(K, \ldots, K, Q)$, with $d - 1$ copies of K, and hence

$$V_1(K, Q) = \frac{1}{d} \int_{S^{d-1}} h_Q(u) S(K, du); \tag{2}$$

see a special case of Theorem 5.1.6 in [23].

The classical Minkowski problem asks for necessary and sufficient conditions for a Borel measure μ on S^{d-1} (called the *data*) to be the surface area measure of a convex body K. The solution as obtained by Aleksandrov and independently by Fenchel and Jessen (see [23]) is: Corresponding to each Borel measure μ on S^{d-1} that is not concentrated on a closed hemisphere of S^{d-1}, there is a convex body K such that

$$S(K, \cdot) = \mu$$

if and only if

$$\int_{S^{d-1}} u \mu(du) = 0.$$

The uniqueness of K (up to translation) is a direct consequence of the *Minkowski mixed-volume inequality* (see (6.2.2) of [23]) which states that for convex bodies K, Q,

$$V_1(K, Q) \geq V(K)^{(d-1)/d} V(Q)^{1/d}, \tag{3}$$

with equality if and only if K is a dilate of Q (after a suitable translation).

Suppose $p > 1$ is fixed and K is a convex body that contains the origin in its interior. The *L_p surface area measure*, $S_p(K, \cdot)$, of K is a Borel measure on S^{d-1} such that

$$\lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot Q) - V(K)}{\varepsilon} = \frac{1}{p} \int_{S^{d-1}} h_Q^p(u) S_p(K, du),$$

for each convex body Q that contains the origin in its interior. Here $K +_p \varepsilon \cdot Q$ is the *Minkowski–Firey L_p combination* defined by

$$h(K +_p \varepsilon \cdot Q, \cdot)^p = h(K, \cdot)^p + \varepsilon h(Q, \cdot)^p.$$

Existence of the L_p surface area measure was established in [12] where it was also shown that

$$S_p(K, \cdot) = h_K^{1-p} S(K, \cdot).$$

It is easily seen that the surface area measure of a convex body (and hence also all the L_p surface area measures) cannot be concentrated on a closed hemisphere of S^{d-1}.

It turns out that if P is a polytope with outer unit facet normals $u_1, \ldots, u_n$, then $\{u_1, \ldots, u_n\}$ is the support of the measure $S(P, \cdot)$ and $S(P, \{u_i\}) = a_i$, where as before

a_i denotes the area of the facet of P whose outer unit normal is u_i. Thus, if P contains the origin in its interior, then

$$S_p(P, \{u_i\}) = h_i^{1-p} a_i,$$

where as before $h_i = h(P, u_i)$.

The L_p Minkowski problem asks for necessary and sufficient conditions for a Borel measure μ on S^{d-1} (the data for the problem) to be the L_p surface area measure of a convex body K; i.e., given a Borel measure μ on S^{d-1} that is not concentrated on a closed hemisphere of S^{d-1}, what are the necessary and sufficient conditions for the existence of a convex body K that contains the origin in its interior such that

$$S_p(K, \cdot) = \mu$$

or equivalently

$$h_K^{1-p} S(K, \cdot) = \mu?$$

The problem is of interest for all real p.

For $p > 1$, but $p \neq d$, the uniqueness of K is a direct consequence of the L_p *Minkowski mixed-volume inequality* (established in [12]) which states that if $p > 1$, then for convex bodies K, Q that contain the origin in their interior,

$$V_p(K, Q) \geq V(K)^{(d-p)/d} V(Q)^{p/d},$$

with equality if and only if K is a dilate of Q, where

$$V_p(K, Q) := \frac{p}{d} \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot Q) - V(K)}{\varepsilon}.$$

The existence of the limit is proved in [12].

In [12] it was shown that if μ is an *even* Borel measure (i.e., assumes the same values on antipodal Borel sets) that is not concentrated on a closed hemisphere of S^{d-1}, then for each $p > 1$, there exists a unique convex body K_p that is symmetric about the origin such that

$$S_p(K_p, \cdot) = \mu,$$

provided $p \neq d$. The L_p Minkowski problem as originally formulated cannot be solved for all even measures when $p = d$. The following *normalized* version of the L_p Minkowski problem was formulated in [19]: What are the necessary and sufficient conditions on a Borel measure μ to guarantee the existence of a convex body K_p^* containing the origin in its interior, such that

$$\frac{1}{V(K_p^*)} S_p(K_p^*, \cdot) = \mu?$$

For all real $p \neq d$ the two versions of the problems are equivalent in that

$$K_p = V(K_p^*)^{1/(p-d)} K_p^*$$

or equivalently

$$K_p^* = V(K_p)^{-1/p} K_p.$$

It was shown in [19] that the normalized L_p Minkowski problem has a solution for all $p > 1$ if the data measure is even (again assuming the measure is not concentrated on a subsphere of S^{d-1}).

It is the aim of this note to present two alternate approaches to the Minkowski problem which show that when the data is a discrete measure, the normalized version of the L_p Minkowski problem always has a solution (assuming, as usual, that the measure is not concentrated on a closed hemisphere of S^{d-1}). It is important to emphasize that all of our results for $p > d$ were first obtained by Guan and Lin [8] and independently by Chou and Wang [5], and all of our results for $p > 1$ were first obtained by Chou and Wang [5]. The sole aim of our work is to present polytopal approaches easily accessible to the convex, discrete, and computational geometry community.

Since the classical case $p = 1$ has been completely solved, we restrict our attention to $p > 1$. Thus, throughout we always assume that the index $p > 1$.

1. Main Results

Let $\mathcal{K}^d$ denote the space of compact convex subsets of $\mathbb{R}^d$ with nonempty interiors, and let $\mathcal{P}^d$ denote the subset of convex polytopes. The members of $\mathcal{K}^d$ are called *convex bodies*. We write $\mathcal{K}_0^d$ for the set of convex bodies which contain the origin as an interior point, and put $\mathcal{P}_0^d := \mathcal{P}^d \cap \mathcal{K}_0^d$.

For $K \in \mathcal{K}^d$, let $F(K, u)$ denote the *support set* of K with exterior unit normal vector u, i.e., $F(K, u) = \{x \in K \colon \langle x, u\rangle = h(K, u)\}$. The $(d-1)$-dimensional support sets of a polytope $P \in \mathcal{P}^d$ are called the *facets* of P. If $P \in \mathcal{P}^d$ has facets $F(P, u_i)$ with areas a_i, $i = 1, \ldots, n$, then $S(P, \cdot)$ is the discrete measure

$$S(P, \cdot) = \sum_{i=1}^{n} a_i \delta_{u_i}$$

with (finite) support $\{u_1, \ldots, u_n\}$ and $S(P, \{u_i\}) = a_i$, for each $i = 1, \ldots, n$, and where δ_{u_i} denotes the probability measure with unit point mass at u_i.

Just as the L_p surface area measure of a convex body $K \in \mathcal{K}_0^d$ satisfies

$$S_p(K, \cdot) = h(K, \cdot)^{1-p} S(K, \cdot),$$

the normalized L_p surface area measure of K is defined by

$$S_p^*(K, \cdot) := \frac{h(K, \cdot)^{1-p}}{V(K)} S(K, \cdot).$$

A convex body K is uniquely determined by its L_p surface area measure if $p > 1$ and $p \neq d$ (for $p = d$ one has uniqueness up to a dilation), uniqueness holds for the normalized L_p surface area measure and all $p > 1$.

Again for a polytope $P \in \mathcal{P}_0^d$ with outer unit facet normals $u_1, \dots, u_n$ and facet areas $a_1, \dots, a_n > 0$, $i = 1, \dots, n$, the discrete measures $S_p(P, \cdot)$ and $S_p^*(P, \cdot)$ are given by

$$S_p(P, \cdot) = \sum_{i=1}^{n} h(P, u_i)^{1-p} a_i \delta_{u_i}$$

and

$$S_p^*(P, \cdot) = \sum_{i=1}^{n} \frac{h(P, u_i)^{1-p}}{V(P)} a_i \delta_{u_i}.$$

In the case of a discrete measure $\mu = \sum_{j=1}^{n} \alpha_j \delta_{u_j}$ with unit vectors $u_1, \dots, u_n$ not contained in a closed hemisphere and $\alpha_1, \dots, \alpha_n > 0$, any solution of the L_p Minkowski problem for the data μ is necessarily a polytope with facet normals $u_1, \dots, u_n$ (see Theorem 4.6.4 of [23]). The main step in our approach to the L_p Minkowski problem for general measures and general convex bodies is to solve first the L_p Minkowski problem for discrete measures and polytopes.

Theorem 1.1. *Suppose $u_1, \dots, u_n \in S^{d-1}$ are not contained in a closed hemisphere and $\alpha_1, \dots, \alpha_n$ are positive real numbers. Then, for each $p > 1$, there exists a unique polytope $P \in \mathcal{P}_0^d$ such that*

$$\sum_{j=1}^{n} \alpha_j \delta_{u_j} = \frac{h(P, \cdot)^{1-p}}{V(P)} S(P, \cdot).$$

From Theorem 1.1, we deduce the corresponding result for the L_p Minkowski problem involving discrete measures and polytopes.

Theorem 1.2. *Suppose $u_1, \dots, u_n \in S^{d-1}$ are not contained in a closed hemisphere and $\alpha_1, \dots, \alpha_n$ are positive real numbers. Then, for each $p > 1$, with $p \neq d$, there exists a unique polytope $P \in \mathcal{P}_0^d$ such that*

$$\sum_{j=1}^{n} \alpha_j \delta_{u_j} = h(P, \cdot)^{1-p} S(P, \cdot).$$

The extension of Theorem 1.1 to general measures can be obtained by approximating with discrete measures. For each approximating discrete measure, we get a polytope as the solution of the discrete L_p Minkowski problem. We then show that a subsequence of these polytopes must converge. Unfortunately, the limit body may well have the origin on its boundary. For $p \geq d$, we employ an additional argument to see that this does not occur.

Theorem 1.3. *Let μ be a Borel measure on S^{d-1} whose support is not contained in a closed hemisphere of S^{d-1}. Then, for $p > 1$, there exists a unique convex body $K \in \mathcal{K}^d$ with $0 \in K$ such that*

$$V(K) h(K, \cdot)^{p-1} \mu = S(K, \cdot);$$

moreover, $K \in \mathcal{K}_0^d$ if $p \geq d$.

In Section 4 we show that for each $p \in (1, d)$ there is a Borel measure μ_p on S^{d-1} whose support is not contained in a closed hemisphere of S^{d-1} for which the convex body $K_p \in \mathcal{K}^d$ with the property that

$$V(K_p)h(K_p, \cdot)^{p-1}\mu_p = S(K_p, \cdot) \tag{4}$$

is such that 0 is a boundary point of K_p.

The equivalence of the L_p Minkowski problem and its normalized version lets us deduce from Theorem 1.3 the following:

Theorem 1.4. *Let μ be a Borel measure on S^{d-1} whose support is not contained in a closed hemisphere of S^{d-1}. Then, for $p > 1$ with $p \neq d$, there exists a unique convex body $K \in \mathcal{K}^d$ with $0 \in K$ such that*

$$h(K, \cdot)^{p-1}\mu = S(K, \cdot);$$

moreover, $K \in \mathcal{K}_0^d$ if $p > d$.

Theorem 1.4 solves the L_p Minkowski problem for $p > d$. It would be interesting to find necessary and sufficient conditions for $1 < p < d$ which guarantee a solution to the L_p Minkowski problem.

2. Volume and Diameter Bounds

The following three lemmas will be applied in two different ways. On the one hand, we need them for our first treatment of the L_p Minkowski problem for discrete measures and polytopes which is based on Aleksandrov's mapping lemma (see [1]). Here the lemmas are applied in the very special situation where all convex bodies are polytopes containing the origin in their interiors and with the same set of outer unit facet normals and where all measures are discrete with common finite support. Except for Lemma 2.1, the proofs of the lemmas in this special case are not simpler than the ones in the general case. Therefore we present them in the general framework. Then again Lemmas 2.1–2.3 will be required for the solution of the L_p Minkowski problem in the case of general convex bodies via an approximation argument.

The next lemma provides a uniqueness result which is used to establish the injectivity of an auxiliary map (see Lemma 3.1) in our first proof of Theorem 1.1. It also yields the uniqueness assertions of Theorems 1.1 and 1.3. Moreover, an estimate established in the course of the proof of Lemma 2.1 is employed in the proof of Lemma 2.2.

Lemma 2.1. *Suppose $p > 1$, and $K, K' \in \mathcal{K}^d$ are convex bodies with $0 \in K, K'$. If μ is a Borel measure on S^{d-1} such that $V(K)h(K, \cdot)^{p-1}\mu = S(K, \cdot)$ and $V(K')h(K', \cdot)^{p-1}\mu = S(K', \cdot)$, then $K = K'$.*

Proof. Let $Q \in \mathcal{K}^d$ with $0 \in Q$. Define $\Omega := \{u \in S^{d-1}\colon h(K, u) > 0\}$ and $\Omega^c :=$

$S^{d-1}\backslash\Omega$. First note that

$$\frac{1}{d}\int_{\Omega} h(K,u)S(K,du) = \frac{1}{d}\int_{S^{d-1}} h(K,u)S(K,du) = V_1(K,K) = V(K),$$

and hence $h(K,\cdot)/(dV(K))S(K,\cdot)$ is a probability measure on Ω. Next note that

$$S(K,\Omega^c) = V(K)\int_{\Omega^c} h(K,u)^{p-1}\mu(du) = 0,$$

and therefore

$$V_1(K,Q) = \frac{1}{d}\int_{\Omega} h(Q,u)S(K,du);$$

see (2). These two facts, together with Hölder's inequality, and the assumption $p > 1$ give

$$\begin{aligned}\left(\frac{1}{d}\int_{S^{d-1}} h(Q,u)^p\mu(du)\right)^{1/p} &\geq \left(\int_{\Omega}\left(\frac{h(Q,u)}{h(K,u)}\right)^p \frac{h(K,u)S(K,du)}{dV(K)}\right)^{1/p}\\ &\geq \int_{\Omega}\frac{h(Q,u)}{h(K,u)}\frac{h(K,u)S(K,du)}{dV(K)}\\ &= \frac{V_1(K,Q)}{V(K)}.\end{aligned} \tag{5}$$

For $Q = K$ or $Q = K'$ the left-hand side of (5) is equal to 1. Hence (5) and Minkowski's mixed-volume inequality (3) imply that

$$1 \geq \frac{V_1(K,K')}{V(K)} \geq \left(\frac{V(K')}{V(K)}\right)^{1/d},$$

and therefore $V(K) \geq V(K')$. By symmetry, we get $V(K) = V(K')$, and thus by the equality conditions of the Minkowski mixed-volume inequality $K = K' + t$ for some $t \in \mathbb{R}^d$. The assumption and the translation invariance of the surface area measure now yield that

$$\int_U [h(K'+t,u)^{p-1} - h(K',u)^{p-1}]\mu(du) = 0$$

for all Borel sets $U \subset S^{d-1}$. In particular, we may choose $U_t := \{u \in S^{d-1}\colon \langle t,u\rangle > 0\}$. If $t \neq 0$, then U_t is an open hemisphere. Since the support of μ is not contained in $S^{d-1}\backslash U_t$, we see that for $t \neq 0$,

$$\int_{U_t} [(h(K',u) + \langle t,u\rangle)^{p-1} - h(K',u)^{p-1}]\mu(du) > 0.$$

This shows that necessarily $t = 0$. □

In the following two lemmas we provide a priori bounds for the volume and the diameter of solutions of the L_p Minkowski problem. The constant κ_d denotes the volume of the unit ball B^d.

Lemma 2.2. *Suppose μ is a Borel measure on S^{d-1}, and the body $K \in \mathcal{K}^d$ is such that $0 \in K$ and $V(K)h(K,\cdot)^{p-1}\mu = S(K,\cdot)$. Then*

$$V(K) \geq \kappa_d \left(\frac{d}{\mu(S^{d-1})}\right)^{d/p}.$$

Proof. Apply (5) with $Q = B^d$ and use Minkowski's inequality (3) (i.e., the isoperimetric inequality in this case) to get

$$\left(\frac{1}{d}\mu(S^{d-1})\right)^{1/p} \geq \left(\frac{\kappa_d}{V(K)}\right)^{1/d},$$

which is equivalent to the assertion of the lemma. □

Subsequently, we set $\alpha_+ := \max\{0, \alpha\}$ for $\alpha \in \mathbb{R}$. Further, we write $B^d(0, r)$ for the Euclidean ball with center 0 and radius $r \geq 0$.

Lemma 2.3. *Suppose μ is a Borel measure on S^{d-1}, and the body $K \in \mathcal{K}^d$ is such that $0 \in K$ and $V(K)h(K,\cdot)^{p-1}\mu = S(K,\cdot)$. Assume that for some constant $c_0 > 0$,*

$$\int_{S^{d-1}} \langle u, v\rangle_+^p \,\mu(du) \geq \frac{d}{c_0^p} \qquad \textit{for all} \quad v \in S^{d-1}.$$

Then $K \subset B^d(0, c_0)$.

Proof. Define $R := \max\{h(K, v)\colon v \in S^{d-1}\}$ and choose $v_0 \in S^{d-1}$ so that $R = h(K, v_0)$. Then $R[0, v_0] \subset K$, and thus $R\langle u, v_0\rangle_+ \leq h(K, u)$ for $u \in S^{d-1}$. Hence

$$\begin{aligned} \frac{R^p}{c_0^p} \leq R^p \frac{1}{d}\int_{S^{d-1}} \langle u, v_0\rangle_+^p \,\mu(du) &\leq \frac{1}{d}\int_{S^{d-1}} h(K,u)^p \mu(du) \\ &= \frac{1}{d}\int_{S^{d-1}} h(K,u)h(K,u)^{p-1}\mu(du) \\ &= \frac{1}{dV(K)}\int_{S^{d-1}} h(K,u)S(K,du) = 1, \end{aligned}$$

which gives $R \leq c_0$. □

3. The L_p Minkowski Problem for Polytopes

In this section we describe two different approaches to Theorem 1.1. The first proof is based on the following auxiliary result, which is a minor modification of Aleksandrov's mapping lemma. We include the proof for the sake of completeness. Note that Aleksandrov used his mapping lemma to solve the classical Minkowski problem for polytopes.

Lemma 3.1. *Let $A, B \subset \mathbb{R}^n$ be nonempty open sets, let B be connected, and let $\varphi\colon A \to B$ be an injective, continuous map. Assume that any sequence $(x^i)_{i\in\mathbb{N}}$ in A with $\varphi(x^i) \to b \in B$ as $i \to \infty$ has a subsequence convergent in A. Then φ is surjective.*

Proof. Since $\varphi(A) \subset B$ is nonempty, it is sufficient to show that $\varphi(A)$ is open and closed in B.

Let $b^i \in \varphi(A)$, $i \in \mathbb{N}$, with $b^i \to b \in B$ as $i \to \infty$ be given. Then there are $x^i \in A$ such that $\varphi(x^i) = b^i$ for $i \in \mathbb{N}$. By assumption, there is a subsequence $(x^{i_j})_{j\in\mathbb{N}}$ with $x^{i_j} \to x \in A$ as $j \to \infty$. Since φ is continuous, $\varphi(x^{i_j}) \to \varphi(x)$ and therefore $b = \varphi(x)$. Hence $\varphi(A)$ is closed in B.

Since A is open in $\mathbb{R}^n$ and φ is continuous and injective, $\varphi(A)$ is open in B by the theorem of the invariance of domain (see Theorem 36.5 of [21] or Theorem 4.3 of [6]). □

In the following we write $H^-_{u,t} := \{y \in \mathbb{R}^d\colon \langle y, u\rangle \le t\}$ for the halfspace with (exterior) normal vector $u \in S^{d-1}$ and distance $t \ge 0$ from the origin.

For our first proof of Theorem 1.1, we can assume that the given vectors $u_1, \dots, u_n$ are pairwise distinct and not contained in a closed hemisphere. Let $\mathbb{R}^n_+$ be the set of all $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ with positive components. For $x \in \mathbb{R}^n_+$, we define the (compact, convex) polytope

$$P(x) := \bigcap_{j=1}^{n} H^-_{u_j,x_j}.$$

The compactness of $P(x)$ is implied by the assumption that $u_1, \dots, u_n$ are not contained in a closed hemisphere. Since $x \in \mathbb{R}^n_+$, 0 is an interior point of $P(x)$. Further, we remark that $x \mapsto P(x)$, $x \in \mathbb{R}^n_+$, is continuous with respect to the Hausdorff metric (see p. 57 of [23]). We put $B := \mathbb{R}^n_+$ and define

$$A := \{x \in \mathbb{R}^n_+\colon S(P(x), \{u_j\}) > 0 \text{ for } j = 1, \dots, n\}.$$

Note that if $x \in A$, then $x_j = h(P(x), u_j)$ for $j = 1, \dots, n$. Clearly, A, B are nonempty open subsets of $\mathbb{R}^n$ and B is connected. Next we define the map $\varphi\colon A \to B$ by $\varphi(x) := b = (b_1, \dots, b_n)$ with

$$b_j := \frac{h(P(x), u_j)^{1-p}}{V(P(x))} S(P(x), \{u_j\}) = S^*_p(P(x), \{u_j\}), \qquad j = 1, \dots, n.$$

We will show that φ satisfies the assumptions of Lemma 3.1 to conclude that φ is surjective. The map φ is well-defined and continuous. The continuity of φ follows from the continuity of the volume and the support function and from the weak continuity of the surface area measure, since $x \mapsto P(x)$ is continuous as well. Next we check that φ is injective. Let $x, y \in A$ be such that $\varphi(x) = \varphi(y)$. Then Lemma 2.1 yields that $P(x) = P(y)$. Hence, by the definition of A, $x_j = h(P(x), u_j) = h(P(y), u_j) = y_j$ for $j = 1, \dots, n$, and thus $x = y$.

Now let $x^i \in A$, $i \in \mathbb{N}$, be given. Assume that $b^i := \varphi(x^i) \to b \in B$ as $i \to \infty$ and put $\mu_i := S_p^*(P(x^i), \cdot)$ for $i \in \mathbb{N}$. Since

$$\mu_i(S^{d-1}) = \sum_{j=1}^{n} \mu_i(\{u_j\}) = \sum_{j=1}^{n} b_j^i \to \sum_{j=1}^{n} b_j$$

as $i \to \infty$, we obtain $\mu_i(S^{d-1}) \leq c_1 < \infty$ for all $i \in \mathbb{N}$. Hence, by Lemma 2.2 there is a constant $c_2 > 0$ such that, for $i \in \mathbb{N}$,

$$V(P(x^i)) \geq c_2 > 0. \tag{6}$$

For the discrete measure $\mu := \sum_{j=1}^{n} b_j \delta_{u_j}$ we have $\mu_i \to \mu$ weakly as $i \to \infty$. The functions f_i, f defined by

$$f_i(v) := \int_{S^{d-1}} \langle u, v \rangle_+^p \mu_i(dv), \qquad f(v) := \int_{S^{d-1}} \langle u, v \rangle_+^p \mu(dv),$$

$v \in S^{d-1}$, are continuous and positive since the support of μ_i, μ is not contained in a closed hemisphere. Since f_i converges uniformly to f as $i \to \infty$ and the sphere is compact, there is a constant $c_3 > 0$ such that $f_i(v) \geq c_3$ for all $v \in S^{d-1}$ and $i \in \mathbb{N}$. Lemma 2.3 now implies that there is a constant c_4 such that, for $i \in \mathbb{N}$,

$$P(x^i) \subset B^d(0, c_4). \tag{7}$$

By (7) there exists a convergent subsequence of $P(x^i)$, $i \in \mathbb{N}$. To simplify the notation, we assume that $P(x^i) \to P \in \mathcal{P}^d$ as $i \to \infty$. Note that by (6) P has indeed nonempty interior. Clearly, $0 \in P$ and the facets of P are among the support sets $F(P, u_1), \ldots, F(P, u_n)$ of P with normal vectors $u_1, \ldots, u_n$. We next show that $0 \in \operatorname{int}(P)$. For this, assume that 0 is a boundary point of P. Then there is a facet $F(P, u_j)$ of P with $0 \in F(P, u_j)$ and $S(P, \{u_j\}) > 0$, and therefore $h(P, u_j) = 0$. Consequently, we get $h(P(x^i), u_j) \to 0$ and $S(P(x^i), \{u_j\}) \not\to 0$, as $i \to \infty$. In view of (7) this implies that

$$b_j^i = V(P(x^i))^{-1} \frac{S(P(x^i), \{u_j\})}{h(P(x^i), u_j)^{p-1}} \to \infty$$

as $i \to \infty$, a contradiction.

Since $0 \in \operatorname{int}(P)$, we conclude that $h(P(x^i), u_j) \not\to 0$ as $i \to \infty$, for $j = 1, \ldots, n$, and therefore also $S(P(x^i), \{u_j\}) \not\to 0$; here we use (6) and $b_j^i \to b_j \neq 0$ as $i \to \infty$. This finally shows that $S(P, \{u_j\}) > 0$ for $j = 1, \ldots, n$.

Thus we get $P = P(x)$ for $x := (h(P, u_1), \ldots, h(P, u_n)) \in A$ and $x^i \to x$ as $i \to \infty$.

Now Lemma 3.1 shows that φ is surjective, which implies the existence assertion of the theorem. Uniqueness has already been established in Lemma 2.1. □

We now give a second, variational proof of Theorem 1.1. An obvious advantage of this approach is that it may be turned into a nonlinear reconstruction algorithm for retrieving a convex polytope from its L_p surface area measure. The main difficulty consists in showing that the solution of an auxiliary optimization problem is a convex polytope which contains the origin in its interior.

The following lemma will be used to verify that a convex polytope which is defined as the solution of an auxiliary optimization problem is indeed the solution of the normalized L_p Minkowski problem stated in Theorem 1.1. Lemma 3.2 can be found on p. 280 of [1].

Lemma 3.2. *Let $u_1, \dots, u_n \in S^{d-1}$ be pairwise distinct vectors which are not contained in a closed hemisphere. For $x \in \mathbb{R}^n_+$, let $P(x) := \bigcap_{i=1}^n H^-_{u_i,x_i}$ and $\tilde{V}(x) := V(P(x))$. Then $\tilde{V}$ is of class C^1 and $\partial_i \tilde{V}(x) = S(P(x), \{u_i\})$ for $i = 1, \dots, n$.*

Proof. The second assertion can be checked by a direct argument. Alternatively, it can be obtained as a very special case of Theorem 6.5.3 in [23]. Here one has to choose $\Omega = \{u_1, \dots, u_n\}$, a positive, continuous function $f\colon S^{d-1} \to \mathbb{R}$ with $f(u_j) = x_j$, and a continuous function $g_i\colon S^{d-1} \to \mathbb{R}$ with $g_i(u_j) = \delta_{ij}$, for $j = 1, \dots, n$. The first assertion then follows, since $x \mapsto S(P(x), \{u_i\})$ is continuous on $\mathbb{R}^n_+$ (see the first proof of Theorem 1.1). □

We start with the second proof of Theorem 1.1. Again we can assume that $u_1, \dots, u_n$ are pairwise distinct unit vectors not contained in a closed hemisphere. Let $\alpha_1, \dots, \alpha_n > 0$ be fixed. We denote by $\mathbb{R}^n_\star$ the set of all $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ with nonnegative components. Then we define the compact set

$$M := \{x \in \mathbb{R}^n_\star\colon \phi(x) = 1\},$$

where

$$\phi(x) := \frac{1}{d} \sum_{i=1}^n \alpha_i x_i^p.$$

For $x \in M$, we again write $P(x)$ for the convex polytope defined by

$$P(x) := \bigcap_{i=1}^n H^-_{u_i,x_i}.$$

Clearly, for any $x \in M$, $0 \in P(x)$ and $P(x)$ has at most n facets whose outer unit normals are from the set $\{u_1, \dots, u_n\}$. Moreover, $h(P(x), u_i) \le x_i$ with equality if $S(P(x), \{u_i\}) > 0$, for $i = 1, \dots, n$. Since M is compact and the function $x \mapsto V(P(x)) =: \tilde{V}(x)$, $x \in M$, is continuous, there is a point $z \in M$ such that $\tilde{V}(x) \le \tilde{V}(z)$ for all $x \in M$. We will prove that $P(z)$ is the required polytope.

First, we show that

$$0 \in \operatorname{int}(P(z)). \tag{8}$$

This will be proved by contradiction. Let $h_i := h(P(z), u_i)$ for $i = 1, \dots, n$. Without loss of generality, assume that $h_1 = \ldots = h_m = 0$ and $h_{m+1}, \dots, h_n > 0$ for some $1 \le m < n$. Note that $m < n$ is implied by $\tilde{V}(z) > 0$. We will show that under this assumption there is some $z_t \in M$ such that $\tilde{V}(z_t) > \tilde{V}(z)$, which contradicts the definition of z. Pick a small $t > 0$ and consider

$$z_t := \left(\left(z_1^p + t^p\right)^{1/p}, \dots, \left(z_m^p + t^p\right)^{1/p}, \left(z_{m+1}^p - \alpha t^p\right)^{1/p}, \dots, \left(z_n^p - \alpha t^p\right)^{1/p}\right),$$

where

$$\alpha := \frac{\sum_{i=1}^{m} \alpha_i}{\sum_{i=m+1}^{n} \alpha_i}.$$

Since $0 < h_i \le z_i$ for $m+1 \le i \le n$, we have $z_t \in M$ if $t > 0$ is sufficiently small.

Define

$$P_t := \bigcap_{i=1}^{m} H^-_{u_i,t} \cap \bigcap_{i=m+1}^{n} H^-_{u_i,(h_i^p - \alpha t^p)^{1/p}},$$

hence $P_0 = P(z)$, $P_t \subset P(z_t)$ and $0 \in \operatorname{int}(P_t)$, if $t > 0$ is sufficiently small. We put

$$f_i := S(P(z), \{u_i\}) \quad \text{and} \quad \Delta_i(t) := S(P_t, \{u_i\}) - f_i,$$

and thus

$$dV(P_t) = t \sum_{i=1}^{m} (f_i + \Delta_i(t)) + \sum_{i=m+1}^{n} \left(h_i^p - \alpha t^p\right)^{1/p} (f_i + \Delta_i(t))$$

and

$$dV_1(P_t, P(z)) = 0 \sum_{i=1}^{m} (f_i + \Delta_i(t)) + \sum_{i=m+1}^{n} h_i (f_i + \Delta_i(t)),$$

where (2) is used.

Since an interior point of $P(z)$ is also an interior point of P_t, if $t > 0$ is sufficiently small, it follows that $P_t \to P(z)$ as $t \to 0^+$ (see p. 57 of [23]), and therefore $\Delta_i(t) \to 0$ as $t \to 0^+$. From this and since at least one facet is supposed to contain the origin, we deduce that

$$\begin{aligned}
&\lim_{t \to 0^+} \frac{V(P_t) - V_1(P_t, P(z))}{t} \\
&\qquad = \frac{1}{d} \lim_{t \to 0^+} \left(\sum_{i=1}^{m} \frac{t-0}{t} (f_i + \Delta_i(t)) + \sum_{i=m+1}^{n} \frac{\left(h_i^p - \alpha t^p\right)^{1/p} - h_i}{t} (f_i + \Delta_i(t)) \right) \\
&\qquad = \frac{1}{d} \sum_{i=1}^{m} f_i > 0.
\end{aligned}$$

Here the assumption $p > 1$ enters in a crucial way. By Minkowski's inequality (3) and since $P_t \to P(z)$ as $t \to 0^+$, we get

$$\begin{aligned}
0 < \lim_{t \to 0^+} \frac{V(P_t) - V_1(P_t, P(z))}{t} &\le \liminf_{t \to 0^+} \frac{V(P_t) - V(P_t)^{1-1/d} V(P(z))^{1/d}}{t} \\
&= V(P(z))^{1-1/d} \liminf_{t \to 0^+} \frac{V(P_t)^{1/d} - V(P(z))^{1/d}}{t}.
\end{aligned}$$

This shows that $V(P_t) > V(P(z))$ if $t > 0$ is sufficiently small. Since $P_t \subset P(z_t)$, the required contradiction follows.

From (8) it follows that

$$z \in M_+ := \{x \in \mathbb{R}^n_+ \colon \phi(x) = 1\},$$

and $\tilde{V}(x) \le \tilde{V}(z)$ for all $x \in M_+$. Hence, by the Lagrange multiplier rule there is some $\lambda \in \mathbb{R}$ such that

$$\nabla \tilde{V}(z) = \lambda \nabla \phi(z).$$

The required differentiability of $\tilde{V}$ is ensured by Lemma 3.2, and $\nabla \phi(z) \neq 0$ since $z \in \mathbb{R}^n_+$ and $\alpha_1, \dots, \alpha_n > 0$; moreover,

$$f_i = \lambda \frac{1}{d} \alpha_i p z_i^{p-1}, \qquad i = 1, \dots, n,$$

and thus $\lambda > 0$, since $f_i > 0$ for some $i \in \{1, \dots, n\}$. We deduce that $f_i > 0$ and therefore $h(P(z), u_i) = z_i$ for all $i = 1, \dots, n$. Since $\phi(z) = 1$, we obtain

$$dV(P(z)) = \sum_{i=1}^{n} f_i z_i = \lambda p \frac{1}{d} \sum_{i=1}^{n} \alpha_i z_i^p = \lambda p.$$

This shows that, for $i = 1, \dots, n$,

$$S(P(z), \{u_i\}) = f_i = \frac{d}{p} V(P(z)) \frac{p}{d} \alpha_i z_i^{p-1} = V(P(z)) h(P(z), u_i)^{p-1} \alpha_i > 0,$$

which completes our second proof of Theorem 1.1.

4. The General Case

We now provide a proof of Theorem 1.3. Theorem 1.4 is an immediate consequence by the equivalence between the Minkowski problem and its normalized version, as outlined in the Introduction. Let μ be a Borel measure on S^{d-1} whose support is not contained in a closed hemisphere. As on pp. 392–393 of [23], one can construct a sequence of discrete measures μ_i, $i \in \mathbb{N}$, such that the support of μ_i is not contained in a closed hemisphere and $\mu_i \to \mu$ weakly as $i \to \infty$. By Theorem 1.1, for each $i \in \mathbb{N}$ there exists a polytope $P_i \in \mathcal{P}_0^d$ with

$$\mu_i = \frac{h(P_i, \cdot)^{1-p}}{V(P_i)} S(P_i, \cdot).$$

As in the proof of (7), we see that the sequence P_i, $i \in \mathbb{N}$, is uniformly bounded. Hence we can assume that $P_i \to K \in \mathcal{K}^d$ as $i \to \infty$ and $0 \in K$. In fact, since $\mu_i(S^{d-1}) \to \mu(S^{d-1})$ as $i \to \infty$, we get as in the proof of (6) that $V(K) > 0$, and thus $K \in \mathcal{K}^d$.

For a continuous function $f \in C(S^{d-1})$ and $i \in \mathbb{N}$ we have

$$\int_{S^{d-1}} f(u) V(P_i) h(P_i, u)^{p-1} \mu_i(du) = \int_{S^{d-1}} f(u) S(P_i, du). \tag{9}$$

Since $V(P_i) h(P_i, \cdot)^{p-1} \to V(K) h(K, \cdot)^{p-1}$ uniformly on S^{d-1} (note that $p - 1 > 0$), and since $\mu_i \to \mu$ and $S(P_i, \cdot) \to S(K, \cdot)$ weakly, as $i \to \infty$, we obtain from (9) that

$$\int_{S^{d-1}} f(u) V(K) h(K, u)^{p-1} \mu(du) = \int_{S^{d-1}} f(u) S(K, du). \tag{10}$$

The existence assertion now follows, since (10) holds for any $f \in C(S^{d-1})$.

Uniqueness had been proved in Lemma 2.1.

Now we consider the case $p \geq d$. Assume that $K \in \mathcal{K}^d$ with $0 \in K$ satisfies $V(K)h(K,\cdot)^{p-1}\mu = S(K,\cdot)$, but $0 \in \partial K$. We derive a contradiction by adapting an argument from [5].

Let $e \in S^{d-1}$ be such that ∂K can locally be represented as the graph of a convex function over (a neighborhood of) $B_r := H_{e,0} \cap B^d(0,r)$, $r > 0$, and $K \subset H^-_{-e,0}$ (see Theorem 1.12 of [2]), where $H_{e,0} := \{x \in \mathbb{R}^d : \langle x, e\rangle = 0\}$. Let μ_i and $P_i \in \mathcal{P}^d_0$ be constructed for μ as in the first part of the proof. In particular, $\mu_i(S^{d-1}) \leq c_5 < \infty$ and $0 \in \operatorname{int}(P_i)$, for all $i \in \mathbb{N}$, and $P_i \to K$ as $i \to \infty$ with respect to the Hausdorff metric. Then, for $i \geq i_0$, ∂P_i can locally be represented as the graph of a convex function g_i over B_r, and the Lipschitz constants of these functions are uniformly bounded by some constant L. We define $G_i(y) := y + g_i(y)e$ for $y \in B_r$, put $\alpha := p-1$ and write c_6, c_7 for constants independent of i and r. Then, for $i \geq i_0$,

$$\begin{aligned} c_5 \geq \mu_i(S^{d-1}) &= \frac{1}{V(P_i)} \int_{S^{d-1}} h(P_i,u)^{-\alpha} S(P_i,du) \\ &\geq c_6 \int_{G_i(B_r)} \langle x, \sigma(P_i,x)\rangle^{-\alpha} \mathcal{H}^{d-1}(dx), \end{aligned}$$

where $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure and $\sigma(P_i,x)$ is an exterior unit normal vector of P_i at $x \in \partial P_i$, which is uniquely determined for $\mathcal{H}^{d-1}$-almost all $x \in \partial P_i$. Since g_i is Lipschitz on B_r, the differential $(dg_i)_y$ exists for $\mathcal{H}^{d-1}$-almost all $y \in B_r$. Let $(e_1,\ldots,e_{d-1},e)$ be an orthonormal basis of $\mathbb{R}^d$. Then we put $\nabla g_i(y) := \sum_{j=1}^{d-1}(dg_i)_y(e_j)e_j$, whenever $(dg_i)_y$ exists. Using the area formula and the fact that

$$\sigma(P_i, G_i(y)) = \left(1 + |\nabla g_i(y)|^2\right)^{-1/2} (\nabla g_i(y) - e),$$

for $\mathcal{H}^{d-1}$-almost all $y \in B_r$, we obtain

$$\begin{aligned} c_5 &\geq c_6 \int_{B_r} \langle G_i(y), \sigma(P_i, G_i(y))\rangle^{-\alpha} \sqrt{1+|\nabla g_i(y)|^2}\,\mathcal{H}^{d-1}(dy) \\ &= c_6 \int_{B_r} (\langle y, \nabla g_i(y)\rangle - g_i(y))^{-\alpha} \sqrt{1+|\nabla g_i(y)|^2}^{\,1+\alpha} \mathcal{H}^{d-1}(dy) \\ &\geq c_6 \int_{B_r} (\langle y, \nabla g_i(y)\rangle - g_i(y))^{-\alpha}\, \mathcal{H}^{d-1}(dy). \end{aligned}$$

Since

$$0 < \langle y, \nabla g_i(y)\rangle - g_i(y) \leq 2\,dL|y| + |g_i(0)|,$$

we further deduce that

$$c_5 \geq c_6 \int_{B_r} (2\,dL|y| + |g_i(0)|)^{-\alpha}\, \mathcal{H}^{d-1}(dy) = c_7 \int_0^r (2\,dLt + |g_i(0)|)^{-\alpha}\, t^{d-2}\,dt.$$

Since $|g_i(0)| \to 0$ as $i \to \infty$, we can extract a decreasing subsequence of $(|g_i(0)|)_{i\in\mathbb{N}}$. Hence the monotone convergence theorem yields that

$$c_5 \geq c_7 \int_0^r (2\,dLt)^{-\alpha} t^{d-2}\,dt,$$

which implies that $\alpha < d-1$; a contradiction. □

The following example demonstrates that the assumption $p \geq d$ in the second part of the assertion of Theorem 1.3 cannot be omitted.

Example 4.1. We now give an example of a Borel measure μ on S^{d-1} whose support is not contained in a hemisphere and such that 0 is a boundary point of the uniquely determined convex body $K \in \mathcal{K}^d$ for which $V(K)h(K,\cdot)^{p-1}\mu = S(K,\cdot)$. Let $(e_1,\dots,e_d)$ denote an orthonormal basis of $\mathbb{R}^d$ such that $\operatorname{span}\{e_1,\dots,e_{d-1}\} = \mathbb{R}^{d-1}\times\{0\}$.

For $q > 1$ we define $g(x) := |x|^q$ for $x \in \mathbb{R}^{d-1}$ and

$$K := \{(x,t) \in \mathbb{R}^{d-1}\times\mathbb{R}\colon t \geq g(x)\} \cap H^-_{e_d,1}.$$

Clearly, $K \in \mathcal{K}^d$, $0 \in \partial K$, and ∂K is C^2 in a neighborhood of 0 excluding 0. The given convex body satisfies $V(K)h(K,\cdot)^{p-1}\mu = S(K,\cdot)$ if

$$\mu := \frac{h(K,\cdot)^{1-p}}{V(K)} S(K,\cdot)$$

defines a finite measure on S^{d-1} and $S(K,\{-e_d\}) = 0$. Since indeed $S(K,\{-e_d\}) = 0$ and $h(K,u) > 0$ for $u \in S^{d-1}\backslash\{-e_d\}$, and since $S(K,\cdot)$ is absolutely continuous with respect to the spherical Lebesgue measure (with density function f_K) in a spherical neighborhood of $-e_d$, it remains to show that $h(K,\cdot)^{1-p} f_K$ is integrable in a spherical neighborhood of $-e_d$. For $r \in (0,1)$ we put $B_r := B^d(0,r) \cap H_{e_d,0}$. Then we define

$$a(x) := (1+|\nabla g(x)|^2)^{1/2}, \qquad x \in B_r\backslash\{0\},$$

where $\nabla g(x) := \sum_{i=1}^{d-1} dg_x(e_i)e_i = q|x|^{q-2}x$. For $x \in B_r\backslash\{0\}$ and

$$u := \sigma(K,(x,g(x))) = a(x)^{-1}(\nabla g(x) - e_d),$$

we get

$$h(K,u) = \langle (x,g(x)),u\rangle = a(x)^{-1}(q-1)|x|^q,$$

$$f_K(u)^{-1} = a(x)^{-(d+1)} \det\left(d^2 g(x)\right),$$

and hence

$$h(K,u)^{1-p} f_K(u) = (q-1)^{1-p} a(x)^{d+p} |x|^{q(1-p)} \left[\det\left(d^2 g(x)\right)\right]^{-1}.$$

A direct computation shows that

$$\det\left(d^2 g(x)\right) = q^{d-1}(q-1)|x|^{(q-2)(d-1)},$$

and therefore

$$h(K,u)^{1-p} f_K(u) = q^{1-d}(q-1)^{-p}|x|^{-[(q-2)(d-1)+q(p-1)]} a(x)^{d+p},$$

for $x \in B_r\backslash\{0\}$ and $u = \sigma(K,(x,g(x)))$. For a given $p \in (1,d)$, we now choose

$$q := \frac{2(d-1)}{d+p-2} \in (1,2),$$

and hence, for $x \in B_r\backslash\{0\}$ and $u = \sigma(K,(x,g(x)))$,

$$h(K,u)^{1-p} f_K(u) = q^{1-d}(q-1)^{-p} a(x)^{d+p}.$$

Since $x \mapsto a(x)$ is bounded on $B_r\backslash\{0\}$ and K is strictly convex in a neighborhood of the origin, the required integrability follows.

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