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Extremal Convex Planar Sets*

Lu Ting[1] and Joseph B. Keller[2]

[1]Courant Institute of Mathematical Sciences, New York University,
251 Mercer Street, New York, NY 10012, USA
ting@cims.nyu.edu

[2]Departments of Mathematics and Mechanical Engineering,
Stanford University, Stanford, CA 94305-2125, USA
keller@math.stanford.edu

Abstract. Each convex planar set K has a perimeter C, a minimum width E, an area A, and a diameter D. The set of points $(E, C, A^{1/2}, D)$ corresponding to all such sets is shown to occupy a cone in the non-negative orthant of R^4 with its vertex at the origin. Its three-dimensional cross section S in the plane $D = 1$ is investigated. S lies in a rectangular parallelepiped in R^3. Results of Lebesgue, Kubota, Fukasawa, Sholander, and Hemmi are used to determine some of the boundary surfaces of S, and new results are given for the other boundary surfaces. From knowledge of S, all inequalities among E, C, A, and D can be found.

1. Introduction

Every convex planar set K has a perimeter C, a minimum width E, an area A, and a diameter D. (The width $w(\theta)$ of a set K in the direction θ is the distance between two support lines orthogonal to the direction θ and containing K. E is the minimum value of $w(\theta)$. D is the maximum distance between two points of K.) We consider the set $\widehat{S}$ of points $P = (E, C, A^{1/2}, D)$ in R^4 corresponding to all convex planar sets K containing at least two points. From $\hat{S}$ all inequalities relating these four quantities can be determined.

The problem of finding $\hat{S}$ is equivalent to one of the problems stated by Santaló [10]. Following Blaschke [1], he asked for a complete set of inequalities satisfied by a given set of n geometric magnitudes corresponding to a convex planar set K. The set of inequalities is complete when n numbers satisfy them if and only if they are the n

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geometric magnitudes for some convex planar set K. Santaló [10], Hernández Cifre and Segura Gomis [5], and Hernández Cifre [4] have determined complete sets of inequalities for various collections of three geometrical magnitudes of a planar convex set, but they did not consider four magnitudes.

As a first step in determining $\widehat{S}$, we note that whenever P is a point in $\widehat{S}$ corresponding to a set K, then for any positive α the point αP corresponds to the set αK. Therefore $\widehat{S}$ is a cone in the non-negative orthant of R^4, with its vertex at the origin. Furthermore, since $D > 0$ for any K, we can set $\alpha = 1/D$ so that $\alpha P = (E/D, C/D, A^{1/2}/D, 1)$. Thus it suffices to determine the three-dimensional cross section of $\widehat{S}$ in the plane $D = 1$. We call this cross section S and use the coordinates (E, C, A) to describe its points, using A instead of $A^{1/2}$. From now on we consider only convex sets with $D = 1$.

S is contained in the rectangular parallelepiped defined by the following well-known inequalities [12]:

$$0 \le E \le 1, \tag{1.1a}$$

$$2 \le C \le \pi, \tag{1.1b}$$

$$0 \le A \le \pi/4. \tag{1.1c}$$

All three inequalities on the left become equalities when any one of them does, and then K is a straight line segment of length 1. Equality on the right in (1.1c) holds only when K is a circular disk and then the other two equalities on the right hold also. Therefore S extends from $(0, 2, 0)$ at one corner of the parallelepiped to $(1, \pi, \pi/4)$ at the diagonally opposite corner.

In (1.1a) and (1.1b), equality on the right holds simultaneously for all sets of constant width 1, i.e., with $w(\theta) \equiv 1$. Thus each such set yields a point $(1, \pi, A)$ on the boundary of S. Lebesgue [8], [9] showed that the value of A for these domains is minimized by the Reuleaux triangle, for which $A = (\pi - \sqrt{3})/2$. (A Reuleaux triangle is obtained from an equilateral triangle by replacing each side by the arc of a circle centered at the opposite vertex, with radius equal to a side of the triangle.) Furthermore, there are such sets with all values of A from the maximum $\pi/4$ down to this minimum. Therefore the sets of constant width 1 correspond to a line segment on the boundary of S, defined by

$$(E, C, A) = (1, \pi, A), \qquad (\pi - \sqrt{3})/2 \le A \le \pi/4. \tag{1.2}$$

The projection of S on the EC plane is determined by two results of Kubota [6]. For given values of E (and $D = 1$), he determined the minimum value $C_1(E)$ and the maximum value $C_2(E)$ of C (Fig. 1), and sets $K_1(E)$ and $K_2(E)$ which achieve them (Fig. 2). The values of $C_1(E)$ and $C_2(E)$, and the inequalities on C, are given by

$$\begin{aligned} 2E \sin^{-1} E + 2(1 - E^2)^{1/2} & \\ \equiv\ & C_1(E) \le C \le C_2(E) \\ \equiv\ & 2 \sin^{-1} E + 2(1 - E^2)^{1/2}, \qquad 0 \le E \le 1. \end{aligned} \tag{1.3}$$

The curves $C = C_1(E)$ and $C = C_2(E)$ meet at their endpoints $(E, C) = (0, 2)$ and $(E, C) = (1, \pi)$. These points are the diagonally opposite corners of the rectangle

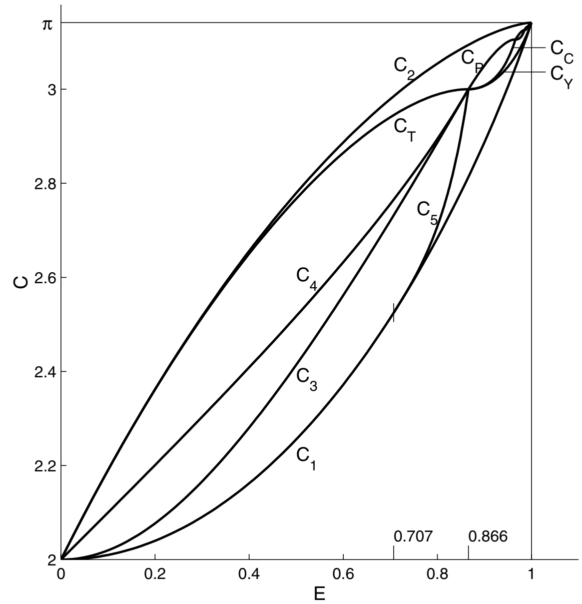

Fig. 1. The projection of S on the EC plane.

defined by (1.1a) and (1.1b). S lies within the region between the two cylindrical surfaces through these two curves, with generators parallel to the A-axis.

With the A-axis pointing vertically upward, the surface $A = A_{\max}(E, C)$ is the top boundary of S. It ends where it meets the two cylindrical surfaces $C = C_1(E)$ and $C = C_2(E)$. When $C = C_1(E)$ then $K_{\max}[E, C_1(E)] = K_1(E)$ and when $C = C_2(E)$ then $K_{\max}[E, C_2(E)] = K_2(E)$.

For each value of C in the interval (1.3), and for each E (and $D = 1$), Fukasawa [2] determined the maximum value $A_{\max}(E, C)$ of A, and a set $K_{\max}(E, C)$ which achieves it. The sets $K_{\max}(E, C)$ are shown in Fig. 2 for $E = \frac{1}{2}$ and seven values of C. Each

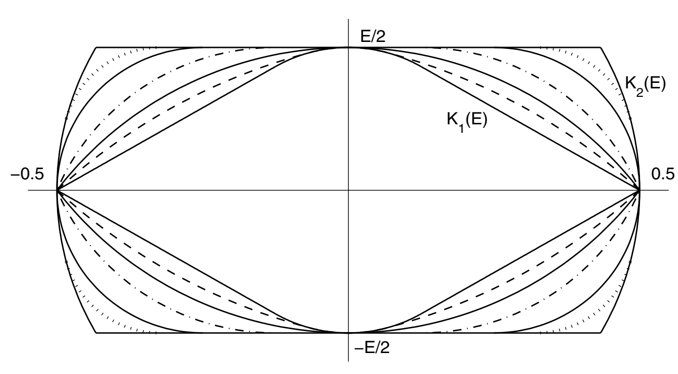

Fig. 2. The set $K_{\max}(E, C)$ maximizing A. See [2].

$K_{\max}(E, C)$ is symmetric about the x- and y-axes. The sets $K_1(E)$ and $K_2(E)$, which respectively minimize and maximize C for given E, are bounded by the inner and outer curves in this figure. In the first quadrant, K_1 is bounded by an arc of a circle of radius $E/2$ centered at the origin and by its tangent line through $(\frac{1}{2}, 0)$, and K_2 by an arc of radius $\frac{1}{2}$ centered at the origin and the line $y = E/2$, forming a corner at $(\sqrt{1-E^2}/2, E/2)$. $K_{\max}$ can be described in terms of a parameter r which ranges from $r = 0$, when $K_{\max} = K_2$, to $r = \infty$, when $K_{\max} = K_1$. We divide this range $[0, \infty)$ into three intervals with intermediate points at $r = E/2$ and $(E^2+1)/4E$. The corresponding $K_{\max}$'s are shown by the two solid lines in the figure in between those for K_2 and K_1. In those three intervals the formulas for the centers of the arcs are different [2].

In Section 2 we show that a strip of the cylindrical surface through $C_1(E)$ and a strip of that through $C_2(E)$ belong to the boundary S. We do this by exhibiting sets $K_1(r, E)$ and $K_2(r, E)$, one of which corresponds to each point of each strip. These sets are shown in Figs. 3 and 4. The strips extend respectively from $A_1(E) = A_{\max}[E, C_1(E)]$ down to $A_1^\star(E)$ and from $A_2(E) = A_{\max}[E, C_2(E)]$ down to $A_2^\star(E)$. The values of $A_1^\star(E)$ and $A_2^\star(E)$ are defined in Section 2. We call these strips the vertical boundaries of S.

The bottom boundary of S is the surface $A = A_{\min}(E, C)$ where $A_{\min}(E, C)$ is the minimum value of A for all convex sets with given E and C. Kubota [7] showed that for $0 \le E \le \sqrt{3}/2$ the minimum area of all convex sets with $D = 1$ and given E is $E/2$, achieved by a triangle with altitude E and base 1. We note that there is a one-parameter family of such triangles with different values of C. These values range from a minimum $C_3(E)$, when the vertex is above the midpoint of the base, to a maximum $C_4(E)$, when the triangle has two sides of length 1. Thus from Kubota's result we find that

$$A_{\min}(E, C) = E/2, \qquad 0 \le E \le \sqrt{3}/2, \qquad C_3(E) \le C \le C_4(E). \tag{1.4}$$

The values of C_3 and C_4 are (Fig. 1)

$$C_3(E) = 1 + \sqrt{4E^2+1}, \qquad 0 \le E \le \sqrt{3}/2, \tag{1.5}$$

$$C_4(E) = 2 + [2 - 2\sqrt{1-E^2}]^{1/2}, \qquad 0 \le E \le \sqrt{3}/2. \tag{1.6}$$

The part of the bottom boundary determined by (1.4) lies in the plane $A = E/2$.

For $\sqrt{3}/2 \le E \le 1$, Sholander [11] determined the minimum area of all convex domains with given E (and $D = 1$). It is achieved only by the equilateral Yamanouti triarc $K_Y(E)$ shown as a solid dark line in Fig. 5(a) and defined in Section 3. The perimeter $C_Y(E)$ and the area $A_Y(E)$ of this triarc are given by

$$C_Y(E) = 6\sqrt{1-E^2} + 6E\big(\sin^{-1} E - \pi/3\big), \tag{1.7}$$

$$A_{\min}[C_Y(E), E] = A_Y(E) = EC_Y(E)/2 - \sqrt{3}/2. \tag{1.8}$$

Thus the curve $[E, C_Y(E), A_Y(E)]$, $\sqrt{3}/2 \le E \le 1$, lies on the bottom boundary of S, and for each value of E between $\sqrt{3}/2$ and 1 it gives the unique lowest point of S. Its projection on the E, C plane is shown in Fig. 1. At $E = \sqrt{3}/2$, $C_3 = C_4 = C_Y = 3$, $A_3 = A_4 = A_Y = \sqrt{3}/4$, and $K_Y(\sqrt{3}/2)$ is an equilateral triangle with sides of length 1.

In Section 3 we determine that part of the bottom boundary for which C is bounded below by $C_1(E)$ and bounded above by $C_3(E)$, for $0 \le E \le \sqrt{3}/2$, and by $C_C(E)$ for

$\sqrt{3}/2 \le C \le \pi$. (Fig. 1.) Here $C_C(E)$ is given by (3.12). In Section 4 we consider the bottom boundary for $0 \le E \le \sqrt{3}/2$ and $C_4(E) \le C \le C_2(E)$. For given E and C we find K to minimize A *among all* K *for which a diameter of* K *is an edge*. In Section 5 we consider the bottom boundary for $\sqrt{3}/2 \le E \le 1$ and $C_C(E) \le C \le C_2(E)$. For given E and C we find K to minimize A *among all polygons* K *containing an equilateral triangle with sides of length* 1. These results of Sections 4 and 5 yield an upper bound for the bottom boundary. We conjecture that it is actually the bottom boundary.

In Section 6 we show that S includes all points from the top boundary $A_{\max}(E, C)$ down to the bottom boundary $A_{\min}(E, C)$ where we have determined it, or down to our upper bound on $A_{\min}$ where we have not determined it. In Section 7 we summarize our results and describe the projection of S on the CA plane. Thus Figs. 1 and 8 provide the top and front views of S.

2. Vertical Boundaries

Equation (1.3) shows that S lies in the region between the two vertical cylindrical surfaces $C = C_1(E)$ and $C = C_2(E)$. The top surface, $A_{\max}(E, C)$, determined by Fukasawa [2] with $K_{\max}(E, C)$ shown in Fig. 2 for $E = 0.5$, ends on these cylindrical surfaces at $A = A_1(E)$ and $A = A_2(E)$, respectively. We now show that the boundary of S contains portions of these cylindrical surfaces. They extend downward from the curve $[C_1(E), A_1(E)]$ to the curve $[C_1(E), A_1^\star(E)]$ and from $[C_2(E), A_2(E)]$ to $[C_2(E), A_2^\star(E)]$. To do this we present two families of convex domains $K_1(r, E)$ and $K_2(r, E)$ with minimum width E, diameter $D = 1$, perimeters $C_1(E)$ and $C_2(E)$, respectively, and parameter r. In fact, for each E, all the domains $K_1(r, E)$ have the same width function, and so do all $K_2(r, E)$. The area $A_1(r, E)$ of $K_1(r, E)$ decreases continuously and monotonically from $A_1(E)$ when $r = E/2$ to $A_1^\star(E)$ when $r = E$. Similarly, $A_2(r, E)$, the area of $K_2(r, E)$, decreases continuously and monotonically from $A_2(E)$ at $r = \frac{1}{2}$ to $A_2^\star(E)$ at $r = 1$. We define $K_1^\star(E)$ and $K_2^\star(E)$ by $K_1^\star(E) = K_1(E, E)$ and $K_2^\star(E) = K_2(1, E)$.

2.1. $C = C_1(E)$

To describe $K_1(r, E)$, shown in Fig. 3, we consider the three cases:

$$\begin{array}{lll} \text{(i)} & 0 \le E \le 1/\sqrt{2}, & E/2 \le r \le E, \\ \text{(ii)} & 1/\sqrt{2} \le E \le 1, & E/2 \le r \le 1/2E, \\ \text{(iii)} & 1/\sqrt{2} \le E \le 1, & 1/2E \le r \le E. \end{array} \tag{2.1}$$

We also define the angle $\alpha(E) = \sin^{-1} E$, $0 \le \alpha \le \pi/2$. In cases (i) and (ii), (see Fig. 3(a)) $K_1(r, E)$ is bounded by two circular arcs with common center $(0, y_0)$ and radii r and $E - r$, with $y_0 = \frac{1}{2}\tan\alpha - r\sec\alpha$, and four line segments tangent to these arcs from the endpoints of the diameter, $P = (\frac{1}{2}, 0)$ and $Q = (-\frac{1}{2}, 0)$. The four heavy dots

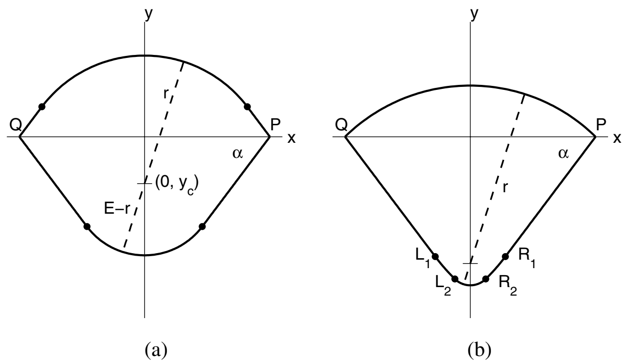

Fig. 3. The set $K_1(r, E)$ on the left vertical boundary of S on the cylinder $C = C_1(E)$. Here $E = 0.8$. (a) $r = (E + 1/E)/4$, (b) $r = (2E + 1/E)/4$.

are the points of tangency. The area of $K_1(r, E)$ is

$$A_1(r, E) = [r^2 + (E - r)^2][\alpha - \tan\alpha] + \frac{E}{2}\sec\alpha \qquad \text{(cases (i) and (ii)).} \tag{2.2}$$

When $r = E/2$, the two arcs are part of one circle with radius $E/2$ centered at the origin. In case (i) at $r = E$, the bottom arc of the boundary of K_1 is a point and $A_1(r, E)$, given by (2.2), attains its minimum value $A_1^\star(E)$:

$$A_1^\star(E) = E^2 \sin^{-1} E + \frac{E(1 - 2E^2)}{2\sqrt{1 - E^2}}, \qquad 0 \le E \le \frac{1}{\sqrt{2}}. \tag{2.3}$$

The corresponding set is $K_1^\star(E) = K_1(E, E)$ for $0 \le E \le 1/\sqrt{2}$. In case (ii) at $r = 1/2E$, the top arc passes through $(\pm\frac{1}{2}, 0)$. Then the two line segments tangent to that arc are absent and only the two line segments tangent to the lower arc remain. For larger values of r, $K_1(r, E)$ is described by case (iii).

In case (iii), $K_1(r, E)$ is bounded by four circular arcs and two line segments (see Fig. 3(b)). Two arcs have centers $(\pm\frac{1}{2}, 0)$ and common radius E, and two have the common center $(0, -\frac{1}{2}\cot\sigma)$ with radii r and $E - r$. Here $\sigma = \sin^{-1}(1/2r)$, $0 \le \sigma \le \pi/2$. One line segment starts at $P(+\frac{1}{2}, 0)$ and ends at R_1 where it is tangent to the circular arc of radius E and center $(-\frac{1}{2}, 0)$, and the other QL_1 is the mirror image of the first in the y-axis. The arcs of radii r and E are tangent to each other at R_2, and their image, at L_2. The area $A_1(r, E)$ of $K_1(r, E)$ for case (iii) is

$$A_1(r, E) = \sigma[r^2 + (E - r)^2] + E\sqrt{1 - E^2} + E^2(\alpha - \sigma) - \tfrac{1}{2}\cot\sigma. \tag{2.4}$$

When $r = E$ then $K_1(E, E) = K_1^\star(E)$, the bottom arc in Fig. 3(b) is a point, and $A_1(r, E)$ given by (2.4) reaches its minimum value $A_1^\star(E)$:

$$A_1^\star(E) = E^2 \sin^{-1} E + E\sqrt{1 - E^2} - \sqrt{E^2 - \tfrac{1}{4}}, \qquad 1/\sqrt{2} \le E \le 1. \tag{2.5}$$

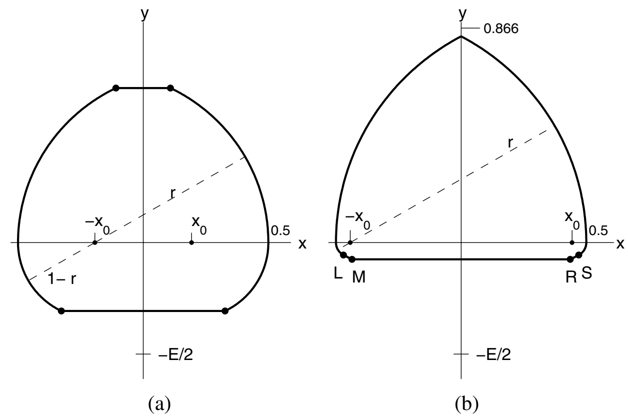

Fig. 4. The set $K_2(r, E)$ on the right vertical boundary of S, with $C = C_2(E)$. Here $E = 0.9$. (a) $r = 0.6932$, (b) $r = 0.9432$.

2.2. $C = C_2(E)$

To describe the sets $K_2(r, E)$ on the right vertical boundary where C is the maximum $C_2(E)$ while the parameter r ranges from $\frac{1}{2}$ to 1 (Fig. 4), we again consider three cases:

$$\begin{array}{lll} \text{(i)} & 0 \le E \le \sqrt{3}/2, & \frac{1}{2} \le r \le 1, \\ \text{(ii)} & \sqrt{3}/2 \le E \le 1, & \frac{1}{2} \le r \le [1+\sqrt{1-E^2}]/[2E^2], \\ \text{(iii)} & \sqrt{3}/2 \le E \le 1, & [1+\sqrt{1-E^2}]/[2E^2] \le r \le 1. \end{array} \tag{2.6}$$

In cases (i) and (ii), K_2 is symmetric about the y-axis with a diameter on the x-axis and is bounded by two horizontal line segments and four circular arcs. See Fig. 4(a). There is a horizontal top line at $y = r\sin\alpha$ of length $2r\cos\alpha - 2r + 1$ and a horizontal bottom line at $y = -(1-r)\sin\alpha$ of length $2(1-r)\cos\alpha + 2r - 1$. Two of the four circular arcs have center $(x_0, 0)$ and radii r and $1-r$ with $x_0 = r - \frac{1}{2}$. The other two arcs have center $(-x_0, 0)$ and the same radii. The lines are tangent to the arcs at the four heavy dots. The area of K_2 for cases (i) and (ii) is

$$A_2(r, E) = [r^2 + (1-r)^2](\sin^{-1} E + E\sqrt{1-E^2}) - (2r-1)^2 E. \tag{2.7}$$

When r attains its minimum value $r = \frac{1}{2}$, all the arcs belong to one circle of radius $\frac{1}{2}$ centered at the origin. Then $K_2(\frac{1}{2}, E) = K_{\max}[E, C_2(E)]$, and the area has its maximum value $A_2(E)$. In case (i) at $r = 1$, the two arcs of radius $1-r$ disappear, $K_2(1, E) = K_2^\star(E)$, and the area has its minimum value $A_2^\star(E)$:

$$A_2^\star(E) = \sin^{-1} E + E\sqrt{1-E^2} - E, \qquad 0 \le E \le \sqrt{3}/2. \tag{2.8}$$

In case (ii) when $r = (1+\sqrt{1-E^2})/2E^2$, the top line segment becomes a point. For larger r, K_2 is described by case (iii).

In case (iii), as shown in Fig. 4(b), $K_2(r, E)$ is again symmetric about the y-axis. Its boundary has a top point at $(0, r\sin\sigma)$ and a bottom line at $y = r\sin\sigma - E$ of length $2\sqrt{1-E^2}$, with $\sigma = \sin^{-1}(1/2r)$. It also has six circular arcs, four with centers at $(\pm x_0, 0) = (\pm r\cos\sigma, 0)$ and radii r and $1-r$, and two centered at the top with radius 1. The area of $K_2(r, E)$ in case (iii) is

$$A_2(r, E) = E\cos\alpha + \alpha - \sigma + [(1-r)^2 + r^2]\sigma - 2r^2 \sin\sigma\cos\sigma. \tag{2.9}$$

When $r = 1$ then $\sigma = \pi/3$, $x_0 = \frac{1}{2}$, the two bottom arcs with radius $1-r$ in Fig. 4(b) are absent, $K_2(1, E) = K_2^\star(E)$, and $A_2(r, E)$ given by (2.9) attains its minimum $A_2^\star(E)$:

$$A_2^\star(E) = E\sqrt{1-E^2} + \sin^{-1} E - \sqrt{3}/2, \qquad \sqrt{3}/2 \le E \le 1. \tag{2.10}$$

3. Bottom Boundary for $C_1(E) \le C \le C_3(E)$ and for $C_1(E) \le C \le C_C(E)$

The minimum value $A_{\min}(E, C)$ is given by (1.4) for $0 \le E \le \sqrt{3}/2$ and $C_3(E) \le C \le C_4(E)$, and by (1.8) for $\sqrt{3}/2 \le E \le 1$ and $C = C_Y(E)$. See Fig. 1. These values of $A_{\min}$ determine part of the bottom boundary of S. To determine the rest of the bottom, we begin with a result of Sholander [11]. He proved that a set K which minimizes A for given E, C, and D must be a proper triarc. A convex set K is a two-arc or three-arc (triarc) if ∂K contains two points O, P or three points O, P, Q, respectively, called vertices, such that at least one of each pair of parallel support lines of K passes through a vertex. K is a proper triarc if it is a two-arc or if it is a three-arc and its vertices are the alternate vertices of a circumscribed hexagon with opposite sides parallel.

To calculate the area of a triarc K, we let s be the arclength along ∂K measured from O to P to Q. Then $s = 0$ at O and also $s = C$ at O, $s = s_P$ at P, and $s = s_Q$ at Q. Let $b(s)$ be the distance from the support line which touches ∂K at s to the vertex through which the parallel support line passes. We say that that vertex supports the point s on ∂K, and $b(s)$ is the width of K corresponding to the direction θ normal to those parallel support lines, which we later call $w(\theta)$. Because K is a proper triarc, it follows that each vertex supports the opposite arc. Therefore we can write

$$A = \frac{1}{2}\int_0^{s_P} b(s)\,ds + \frac{1}{2}\int_{s_P}^{s_Q} b(s)\,ds + \frac{1}{2}\int_{s_Q}^{C} b(s)\,ds - 2A_{OPQ}. \tag{3.1}$$

The area A is expressed as the sum of the three areas covered by the radial lines from each vertex to the points s which it supports, minus twice the area A_{OPQ} of the triangle, which is covered three times. The minimum width of K is E, so $b(s) \ge E$ and (3.1) yields

$$A \ge (CE/2) - 2A_{OPQ}. \tag{3.2}$$

Equality holds in (3.2) if and only if $b(s) = E$ for all s except a set of measure zero.

A triarc with $b(s) = E$ for almost all s is either a Yamanouti triarc or a capped Yamanouti triarc, both of which we now describe. See Fig. 5. Such a triarc exists if the three altitudes h_O, h_P, and h_Q of the triangle OPQ are all less than E, the foot of each altitude lies on the opposite side, and the lengths of the three sides of the triangle all exceed E. Then the Yamanouti triarc with vertices OPQ is formed by drawing a circular arc of radius E centered at each vertex, and bounded by the two sides meeting

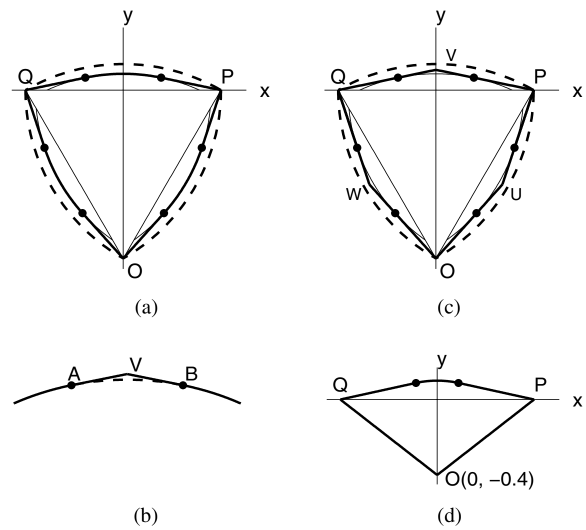

Fig. 5. Yamanouti triarcs, Reuleaux triangles, and a two-arc. (a) $K_y(E = 0.95)$, (b) capping of a convex arc, (c) $K_C(E = 0.95)$, (d) $K_B(E = 0.5)$.

at that vertex. Each arc extends beyond the opposite side. The convex hull of OPQ and these three arcs is a triarc of constant width E called a Yamanouti triarc. As shown in Fig. 5(a), for $E = 0.95$, the solid curve OPQ bounds a Yamanouti triarc symmetric about the y-axis, with vertices $O = (0, -h)$, $P = (\frac{1}{2}, 0)$, $Q = (-\frac{1}{2}, 0)$. We denote it $K_Y(h, E)$. The boundary consists of three circular arcs, each of radius E, centered at the vertices, and six tangent line segments from the vertices to the arcs. The points of tangency are shown by dots. When $h = \sqrt{3}/2$, as shown here, the vertices form an equilateral triangle, and K_Y is an equilateral Yamanouti triarc. When $E = 1$, $K_Y(\sqrt{3}/2, 1)$ is the Reuleaux triangle shown as a dashed line. Figure 5(b) demonstrates the capping of a convex arc. To cap a convex set K is to choose a capping point outside K, to draw the two segments of tangent lines from the point to K, and to replace the arc of ∂K between the points of tangency by these two segments. As shown in Fig. 5(b), the dashed arc $\overset{\frown}{AB}$ is replaced by the two tangent lines VA and VB through the capping point V. In Fig. 5(c), the solid line OPQ shows a fully capped equilateral Yamanouti triarc $K_C(E)$ for $E = 0.95$, with three caps. It has six sides tangent to the circular arcs at the dots. The dashed line is the Reuleaux triangle OPQ. Fig. 5(d) shows $K_B(h, E)$, for $h = 0.4$ and $E = 0.5$, which is a two-arc symmetric about the y-axis. It exists when (3.17) and (3.18) hold. It is bounded by four line segments and one circular arc of radius E and center $O = (0, -h)$. Two of the line segments are tangent to the arc at the dots.

If K is a Yamanouti triarc of diameter D and width E, the points of tangency must be on an arc of radius E and the chosen point must be at a distance not greater than D from the center of the arc. A capped Yamanouti triarc may have any number of nonoverlapping caps. It is fully capped if all its circular arcs are completely covered by nonoverlapping caps.

The diameter D of a Yamanouti triarc is the maximum distance between two vertices, say P and Q, so $D = 1$ when $PQ = 1$. Then $A_{OPQ} = h_O/2$. By setting the altitude $h_O = h$ we can write (3.2) for a Yamanouti triarc, or a capped Yamanouti triarc, as

$$A = (CE/2) - h. \tag{3.3}$$

If there are such triarcs with given E, C, and $D = 1$, the area is minimized by one with the maximum value of h, i.e., with the maximum value of A_{OPQ}. This occurs when $OP = OQ$, i.e., when K is an isosceles Yamanouti triarc. Furthermore, if there are both capped and uncapped Yamanouti triarcs, with given E, C, and $D = 1$, the area A is minimized by the uncapped one.

We now determine the range of values of h and E for which there is a Yamanouti triarc $K_Y(OPQ)$ with $QP = 1$ and $E < OP = OQ \le 1$. To do so we consider the possible values of h. The requirement that the foot of each altitude lies on the opposite side leads to the condition $h \ge \frac{1}{2}$. The condition $OP \ge E$ leads to $h \ge (E^2 - \frac{1}{4})^{\frac{1}{2}}$. The maximum value of h is $\sqrt{3}/2$ when $OP = OQ = 1$, and the condition $h_P = h_Q \le E$ leads to $h \le E(1 - E^2)^{-1/2}/2$. The upper and lower bounds on h are equal when $E = 1/\sqrt{2}$. Thus we have the following conditions on h and E:

$$\max[\tfrac{1}{2}, \sqrt{E^2 - \tfrac{1}{4}}] \le h \le \min\left[\frac{E}{2\sqrt{1-E^2}}, \frac{\sqrt{3}}{2}\right], \qquad \frac{1}{\sqrt{2}} \le E \le 1. \tag{3.4}$$

For each h and E satisfying (3.4) there is a unique Yamanouti triarc, $K_Y(OPQ) = K_Y(h, E)$, with altitude h, minimum width E, $QP = 1$, and $OP = OQ \le 1$. The triarc is bounded by three circular arcs of radius E, and six line segments. The centers of the arcs are at the vertices, say $(\pm\frac{1}{2}, 0)$ and $(-h, 0)$. Two line segments start at each center, and each is tangent to one of the two arcs with different centers. The perimeter C of $K_Y(h, E)$ is given by

$$C(h, E) = 4(h^2 - E^2 + \tfrac{1}{4})^{1/2} + 2(1 - E^2)^{1/2} + (2\delta + 2\nu)E. \tag{3.5}$$

Then (3.3) yields

$$A_Y(h, E) = EC(h, E)/2 - h. \tag{3.6}$$

Here δ and ν are defined by

$$\delta = \tan^{-1} 2h - \cos^{-1} E - \cos^{-1} \frac{E}{(h^2 + \frac{1}{4})^{1/2}}, \tag{3.7}$$

$$\nu = \tan^{-1} \frac{1}{2h} - \cos^{-1} \frac{E}{(h^2 + \frac{1}{4})^{1/2}}. \tag{3.8}$$

When $1/\sqrt{2} \le E \le \sqrt{3}/2$, the maximum value of h is $h = E/2\sqrt{1 - E^2}$. Then $h_P = h_Q = E$, the two arcs centered at P and Q do not extend outside the triangle, and $K_Y(h, E)$ is a two-arc like the domain $K_B(h, E)$ shown in Fig. 5(d). We define C_5 by using $h = E/2\sqrt{1 - E^2}$ in (3.5), so that $C_5(E) = C\left[E/2\sqrt{1 - E^2}, E\right]$. $C_5(E)$

is the maximum value of C for which there is a Yamanouti triarc with width E. The curve $C = C_5(E)$ is shown in Fig. 1. The minimum value of h is $h = \sqrt{E^2 - \frac{1}{4}}$ for $1/\sqrt{2} \le E \le 1$, and $C(\sqrt{E^2 - \frac{1}{4}}, E) = C_1(E)$. Then $OP = OQ = E$, and $K_Y(h, E)$ coincides with $K_1(E, E)$, which is the limiting case of Fig. 3(a) for $r = E$. The area is then $A_Y(\sqrt{E^2 - \frac{1}{4}}, E) = A_1^\star(E)$, so the bottom surface of S meets the lower edge of the vertical surface. Thus there is a Yamanouti triarc for each pair C and E satisfying

$$C_1(E) \le C \le C_5(E), \qquad 1/\sqrt{2} \le E \le \sqrt{3}/2. \tag{3.9}$$

For $\sqrt{3}/2 \le E \le 1$ the maximum value of h is $\sqrt{3}/2$. Then $K_Y(\sqrt{3}/2, E)$ is an equilateral Yamanouti triarc and $C(\sqrt{3}/2, E) = C_Y(E)$, which is given by (1.7). The preceding result shows that at the minimum value of h, $C = C_1(E)$. Therefore there is a Yamanouti triarc for C and E satisfying

$$C_1(E) \le C \le C_Y(E), \qquad \sqrt{3}/2 \le E \le 1. \tag{3.10}$$

For E in the range $\sqrt{3}/2 \le E \le 1$ and $C > C_Y(E)$, there is no Yamanouti triarc. However, there is a capped Yamanouti triarc provided that $C \le C_C(E)$, where $C_C(E)$ is the maximum perimeter of all capped equilateral Yamanouti triarcs with $D = 1$. Then (3.3) yields

$$A = CE/2 - \sqrt{3}/2, \qquad C_Y(E) \le C \le C_C(E), \qquad \sqrt{3}/2 \le E \le 1. \tag{3.11}$$

$C_C(E)$ is attained by a fully capped equilateral Yamanouti triarc, $K_C(E)$, which can be found in a straightforward way. See Fig. 5(c). $K_C(E)$ is found to be a polygon of $3n + 6$ vertices within the Reuleaux triangle K_R, with $n = [\pi/6 \cos^{-1} E]$, where $[x] =$ the integer part of x. Thus n increases with E. At least $3n + 3$ of its vertices lie on the boundary of K_R, with at most one vertex inside the boundary in each of the three lenses, i.e., the three regions between the sides of the triangle and the corresponding circular arcs. The value of $C_C(E)$ is found to be

$$C_C(E) = 6n\sqrt{1 - E^2} + 6 \sin \gamma. \tag{3.12}$$

Here $\gamma = (\pi/6) - n \cos^{-1} E$. The corresponding value of A is $A_C(E)$ given by (3.11) with $C = C_C(E)$. The curve $C = C_C(E)$ is shown in Fig. 1.

For those values of C and E for which there is no proper Yamanouti triarc, we reconsider (3.1). The first and last integrals in (3.1) each exceeds A_{OPQ}, and $b(s) \le E$ in the second integral, so we can write

$$A \ge \frac{1}{2} \int_{s_P}^{s_Q} b(s)\, ds \ge (s_Q - s_P)E/2. \tag{3.13}$$

Equality holds in (3.13) if and only if $b(s) = E$ for almost all s in the interval (s_P, s_Q) and both the first and last integrals in (3.1) equal A_{OPQ}. Then the domain is $K_B(h, E)$ shown in Fig. 5(c), and $PQ = 1$ in order that $D = 1$. It is a two-arc since one of every pair of parallel support lines passes through one of the two vertices O and P. Sholander

[11, Corollary 3.4] proved that every two-arc has a minimum area among bodies center-equivalent to it, i.e., having the same width function and having a center about which it is symmetric.

When E and C are such that the domain K_B in Fig. 5(d) exists, that domain minimizes A. The conditions under which this domain exists are $0 \leq h \leq E \leq h_P = h_Q$. In terms of E and h these conditions are

$$0 \leq h \leq E, \quad 0 \leq E \leq \tfrac{1}{2}, \tag{3.14a}$$

$$\sqrt{E^2 - \tfrac{1}{4}} \leq h \leq E, \quad \tfrac{1}{2} \leq E \leq 1/\sqrt{2}, \tag{3.14b}$$

$$\frac{E}{2\sqrt{1-E^2}} \leq h \leq E, \quad 1/\sqrt{2} \leq E \leq \sqrt{3}/2. \tag{3.14c}$$

Then $K_B(h, E)$ is bounded by four line segments and one circular arc of radius E and center $(0, -h)$. Two of the line segments extend from $(0, -h)$ to $(\pm\frac{1}{2}, 0)$ and the other two are tangent to the arc and end at $(\pm\frac{1}{2}, 0)$. The perimeter $C_B(h, E)$ and area $A_B(h, E)$ of $K_B(h, E)$ are given by

$$C_B(h, E) = 2E\nu + 2(h^2 + \tfrac{1}{4} - E^2)^{1/2} + 2(h^2 + \tfrac{1}{4})^{1/2}, \tag{3.15}$$

$$A_B(h, E) = [E\nu + (h^2 + \tfrac{1}{4} - E^2)^{1/2}]E. \tag{3.16}$$

By using (3.12) and (3.13), we find that a domain K_B exists for all pairs E, C satisfying

$$C_1(E) \leq C \leq C_3(E), \qquad 0 \leq E \leq 1/\sqrt{2}, \tag{3.17}$$

$$C_5(E) \leq C \leq C_3(E), \qquad 1/\sqrt{2} \leq E \leq \sqrt{3}/2. \tag{3.18}$$

The two pairs of equations (3.5), (3.6) and (3.15), (3.16) are both parametric equations for $A_{\min}(E, C)$ in terms of the parameter h. The first pair apply when h and E satisfy (3.4) while the second pair apply when h and E satisfy (3.14). The corresponding ranges of E and C are given by (3.9), (3.10) and by (3.17), (3.18), respectively. The curves bounding them are shown in Fig. 1.

This completes the determination of the bottom boundary for those values of E and C satisfying $C_1(E) \leq C \leq C_4(E)$ for $0 \leq E \leq \sqrt{3}/2$ and $C_1(E) \leq C \leq C_C(E)$ for $\sqrt{3}/2 \leq E \leq 1$. See Fig. 1.

4. Bottom Boundary for $0 \leq E \leq \sqrt{3}/2$, $C_4(E) \leq C \leq C_2(E)$

For each pair E, C satisfying $0 \leq E \leq \sqrt{3}/2$ and $C_4(E) \leq C \leq C_2(E)$, we shall find a convex domain $K(E, C)$ with these values of C and E, and $D = 1$, which minimizes A *among all convex sets K for which a diameter of K is an edge*. The corresponding value $A(E, C)$ is an upper bound on $A_{\min}(E, C)$. To find K we denote by PQ that diameter of K which is an edge, and we take PQ to be horizontal with K on its upper side. See Fig. 6. Then K is contained within the curvilinear triangle bounded below by PQ, and laterally by two circular arcs with centers P and Q and common radius 1. The support

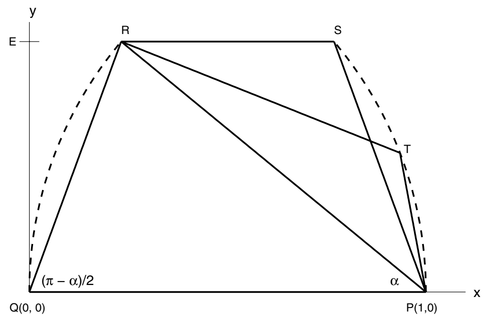

Fig. 6. The area minimizing set $K(E, C)$, on the bottom boundary, for $C_4(E) \le C \le C_T(E)$, $0 \le E \le \sqrt{3}/2$, shown for $E = 0.625$.

line of K parallel to PQ must be at least the distance E from PQ because otherwise the minimum width of K would be less than E. On the other hand, it cannot be greater than E because it contains at least one point of K, say $\overline{P}$, and the minimum width of the triangle $P\overline{P}Q$ is the distance of $\overline{P}$ from PQ. Therefore this support line is at distance E above PQ, and we denote it by RS where R and S are its intersections with the circular arcs centered at P and Q, respectively.

These conditions show that K is contained in the domain $K_2^\star(E)$ bounded above and below by the lines RS and PQ, respectively, and laterally by the circular arcs $\overset{\frown}{QR}$ and $\overset{\frown}{PS}$. See Fig. 6. For $K_2^\star(E)$ the perimeter is $C_2(E)$ and the area is $A_2^\star(E)$, which corresponds to a point on the lower edge of the vertical boundary of S. The vertices of $K_2^\star$ determine an equilateral trapezoid $K_T(E) = PQRS$ with area $A_T(E)$ and perimeter $C_T(E)$ given by

$$A_T(E) = E\cos\alpha, \qquad C_T(E) = 2\cos\alpha + 4\sin(\alpha/2). \tag{4.1}$$

K has at least one point $\overline{P}$ on RS, and it may have a segment say $\overline{PQ}$ on RS. For $C < C_2(E)$, K must have another part of its boundary in the interior of $K_2^\star(E)$. A variational argument like that of Sholander [11, pp. 172, 173] shows that ∂K cannot contain an arc of radius 1. If it did A could be reduced, with E, C, and D unchanged, by replacing a bit of the arc by its chord and simultaneously replacing a straight line segment of the boundary by two lines. Therefore ∂K cannot contain arcs on $\overset{\frown}{QR}$ or on $\overset{\frown}{PS}$. Now an area minimizing domain must be bounded by straight line segments and circular arcs of radius E or radius 1. We have just seen that ∂K cannot contain arcs of radius 1. The support lines RS and PQ are at distance E apart, so ∂K cannot contain an arc of radius E either. Therefore ∂K is a polygon.

The vertices of the polygonal boundary of K must lie on $\partial K_2^\star$, i.e., K must be circumscribed by $\partial K_2^\star$. This can be shown by considering three consecutive vertices A, B, C and moving B along the ellipse with foci A and C with the length $AB + BC$ kept constant. The area of the triangle ABC is minimized either when B lies on $\partial K_2^\star$ or when B lies on the extension of the line segment of which either A or C is an endpoint. In the latter case, either A or C is not a vertex, contradicting the assumption that A, B, C were three consecutive vertices. Therefore B must lie on $\partial K_2^\star$.

When $C = C_4(E)$, $K[E, C_4(E)]$ is an isosceles triangle, say PQR, determined by Kubota [6]. We now show that *for* $C_4(E) < C \le C_T(E)$, $K(E, C)$ *is a quadrilateral with vertices* P, Q, R, *and* T *with* T *on* RS *or on* $\overset{\frown}{PS}$. See Fig. 6. When $C = C_T(E)$ then $K[E, C_T(E)] = K_T(E)$ because only for $T = S$ is the perimeter of the quadrilateral $PQRT = C_T(E)$.

To prove the italicized statement for $C \le C_T(E)$ we denote by $\overline{QP}$ the line segment of ∂K lying on RS. First we show that aside from Q and R there is at most one vertex of K, say V, on the arc $\overset{\frown}{QR}$. Suppose there were more than one, and let V be the highest vertex. Then ∂K contains a polygonal line from V to Q inscribed in the arc $\overset{\frown}{VQ}$. A calculation like that of Hemmi [3] or that given below, shows that the area of K is reduced by shortening the polygonal line and moving either V or $\overline{Q}$ to keep constant the length of the broken line from $\overline{Q}$ to Q. Thus the minimizing K can have no vertex other than Q and V on $\overset{\frown}{QR}$ if R is not a vertex of K. Similarly, K can have at most the vertices P and U, say, on $\overset{\frown}{PS}$ if S is not a vertex of K. Therefore if neither R nor S is a vertex of K, then K has at most six vertices, P, Q, V, $\overline{Q}$, $\overline{P}$, U.

When $\overline{Q} = R$ and $\overline{P} \ne S$, the arc can be decreased by shortening the polygonal line from R to Q and increasing the length of the polygonal line $PU\overline{P}$. This can be continued until there is no vertex V on the arc $\overset{\frown}{QR}$. Thus in this case, K has at most five vertices, P, Q, R, $\overline{P}$, U, so it is a special case of the hexagon above with $V = \overline{Q} = R$. The area of the hexagon with the vertices P, Q, V, $\overline{Q}$, $\overline{P}$, U and given C in the interval $C_4(E) < C \le C_T(E)$ is minimized when K is a quadrilateral $PQRT$ where T is on RS or on $\overset{\frown}{PS}$, or the symmetrical case $QPST$ where T is on RS or on $\overset{\frown}{QR}$. The former case corresponds to the vertices V and $\overline{Q}$ coinciding with R and either $U = P$ or $\overline{P} = R$. This result can be proved by calculating $dA/d\ell$ for the triangles $QV\overline{Q}$ and $PU\overline{P}$ due to displacement of either V or $\overline{Q}$ for $QV\overline{Q}$ and of U or $\overline{P}$ for $PU\overline{P}$. Say the minimum of the four values of $dA/d\ell$ occurs by moving V or $\overline{Q}$. Then we can reduce the area of K by shortening the perimeter of $PU\overline{P}$ and lengthening the perimeter of $QV\overline{Q}$. The inequality of the derivatives $dA/d\ell$ will be maintained, so we can continue until either $V = \overline{Q} = R$ or $U = P$. In the first case we can then continue reducing A by moving $\overline{P}$ and U until either $\overline{P} = R$ or $U = P$. In the second case we can continue until $V = \overline{Q} = R$. In both cases we end up with the quadrilateral $PQRT$ with T on RS or on $\overset{\frown}{PS}$.

For $C_4(E) \le C \le C_T(E)$, the quadrilateral K with vertices $PQRT$ is shown in Fig. 6 when T is on the arc $\overset{\frown}{PS}$. Two of its sides, PQ and QR, are those of the triangle PQR. The vertex T where the other two sides meet can be parameterized by its height h, $0 \le h \le E$. It is at $\left(\sqrt{1-h^2}, h\right)$. In terms of h, $\tau = \sin^{-1} h$, and α, we can write C and A for K as follows:

$$\begin{aligned} C(h, E) &= 1 + 2\sin(\alpha/2) + 2\sin(\tau/2) \\ &\quad + \sqrt{(E-h)^2 + (\cos\alpha + \cos\tau - 1)^2}, \end{aligned} \tag{4.2}$$

$$A(h, E) = h/2 + \sin(\alpha/2)\cos[(\alpha/2) + \tau], \qquad 0 \le h \le E. \tag{4.3}$$

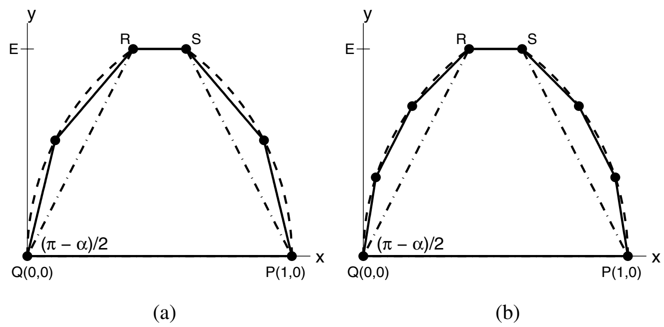

Fig. 7. The area minimizing set $K(E, C)$, on the bottom boundary, for $C_T(E) \le C \le C_2(E)$, $0 \le E \le \sqrt{3}/2$, shown for $E = 0.8$, $n_1 = n_2 = n$, and $\beta = \gamma = \alpha/[2(n+1)]$. (a) $n = 1$, (b) $n = 2$.

When $h = 0$, K is an isosceles triangle. When $h = E$, K is a trapezoid with $\tau = \alpha$, perimeter $C_T(E) = C(E, E)$ and area $A_T(E) = A(E, E)$ given by (4.1).

For $C_T(E) < C < C_2(E)$, K is a polygon formed by replacing the two sides QR and PS of the trapezoid $K_T(E)$ by polygonal lines, as in the results of Hemmi [3]. These polygons are inscribed in the two circular arcs $\widehat{QR}$ and $\widehat{PS}$ of radius 1 with centers at the endpoints of the base of the trapezoid, as is shown in Fig. 7. Each arc subtends the angle α at its center. The polygonal lines are inscribed in these circular arcs, so their vertices lie on the arcs. We denote by n_1 and n_2 the number of interior vertices of the polygonal lines on the left and right, respectively, so that they consist of $n_1 + 1$ and $n_2 + 1$ line segments, respectively.

To determine n_1, n_2 and the lengths of the segments, we begin by considering the polygonal line on the left. Let $\theta_1, \theta_2, \ldots, \theta_{n_1+1}$ denote the angles subtended by these segments at P, the center of the circle, and let L_1 be the length of the polygonal line. Then the lengths of the segments must add up to L_1 and the angles must add up to α:

$$\sum_{j=1}^{n_1+1} 2\sin\left(\frac{\theta_j}{2}\right) = L_1, \qquad \sum_{j=1}^{n_1+1} \theta_j = \alpha. \tag{4.4}$$

The area $\mathcal{A}_1$ bounded by the polygonal line and the side QR of the trapezoid is

$$\sum_{j=1}^{n_1+1} \tfrac{1}{2}\sin\theta_j - \tfrac{1}{2}\sin\alpha = \mathcal{A}_1. \tag{4.5}$$

In order to minimize $\mathcal{A}_1$ subject to the constraints (4.4) we introduce two Lagrange multipliers λ_1, λ_2 and consider $\mathcal{A}_1 - \lambda_1 L_1 - \lambda_2\alpha$. Upon using (4.4) and (4.5) in this expression and setting the derivative with respect to θ_j equal to zero, we get $\frac{1}{2}\cos\theta_j - \lambda_1\cos(\theta_j/2) - \lambda_2 = 0$. Since $\cos\theta_j = 2\cos^2(\theta_j/2) - 1$, this becomes a quadratic equation for $\cos(\theta_j/2)$. Therefore there are just two values of θ_j in the range $0 \le \theta_j \le \pi/2$. We call the larger one 2β and the smaller 2γ, so that each θ_j is either 2β or 2γ. It is easy to show that when $\mathcal{A}_1$ is minimized, at most one of the θ_j is equal to the smaller value 2γ, say $\theta_{n_1+1} = 2\gamma$. Then $\theta_1 = \theta_2 = \cdots = \theta_{n_1} = 2\beta$, so (4.4) and (4.5) become

$$\begin{gathered} 2n_1 \sin\beta + 2\sin\gamma = L_1, \qquad 2n_1\beta + 2\gamma = \alpha, \\ n_1 \sin 2\beta + \sin 2\gamma - \sin\alpha = 2\mathcal{A}_1. \end{gathered} \tag{4.6}$$

Since $\gamma \leq \beta$, the minimum value of β occurs when $\gamma = \beta$. Then the second equation in (4.6) shows that $\gamma = \beta = \alpha/2(n_1 + 1)$. The other two equations in (4.6) yield for L_1 and $\mathcal{A}_1$ the values

$$\ell_{n_1+1} = 2(n_1 + 1)\sin\frac{\alpha}{2(n_1 + 1)}, \qquad a_{n_1+1} = \frac{n_1 + 1}{2}\sin\frac{\alpha}{n_1 + 1} - \frac{1}{2}\sin\alpha. \tag{4.7}$$

From (4.7) we conclude that if $L_1 = \ell_{n_1+1}$, then the polygonal line which minimizes $\mathcal{A}_1$ subject to (4.4) consists of $n_1 + 1$ equal line segments. Each subtends the angle $\beta = \alpha/2(n_1 + 1)$ and the minimum value of $\mathcal{A}_1$ is a_{n_1+1}. If L_1 lies in the interval $\ell_{n_1} \leq L_1 < \ell_{n_1+1}$ the minimizing polygonal line still consists of $n_1 + 1$ segments, with n_1 of them equal and each subtending an angle 2β, and one shorter subtending the smaller angle 2γ. To determine the values of β and γ we first solve the second equation in (4.6) for $\beta = (\alpha - 2\gamma)/2n_1$ and then eliminate β from the first equation in (4.6) to get

$$2n_1 \sin[(\alpha - 2\gamma)/2n_1] + 2\sin\gamma = L_1. \tag{4.8}$$

From (4.8) we find that $dL_1/d\gamma = 2\cos\gamma - 2\cos\beta > 0$, so (4.8) determines a unique value of γ, and then the last equation in (4.6) gives $\mathcal{A}_1$, which we write as $\mathcal{A}_1 = \mathcal{A}(L_1)$.

By proceeding in exactly the same way on the right, we find the minimizing polygonal line and the minimal area $\mathcal{A}_2 = \mathcal{A}(L_2)$ for a given value of the length L_2. Then the sum of the two areas is $\mathcal{A}(L_1) + \mathcal{A}(L_2)$, and the sum of the lengths is $L_1 + L_2 = L_{12}$ where $L_{12} = C - 2\sqrt{1 - E^2}$. We must now find the values of L_1 and L_2, subject to this constraint on their sum, which minimizes the sum of the areas.

First we consider the case in which the derivative with respect to L_1 of the sum of the areas is zero. In view of the constraint, the derivative of the sum of the areas with respect to L_1 is $\mathcal{A}'(L_1) - \mathcal{A}'(L_2)$, which vanishes when $L_1 = L_2$. Then $L_1 = L_2 = L_{12}/2$ and the sum of the areas is $\mathcal{A}(L_1) + \mathcal{A}(L_2) = 2\mathcal{A}(L_{12}/2)$. It also follows that $n_1 = n_2 = n$, say, where n is defined by

$$\ell_n \leq L_{12}/2 < \ell_{n+1}. \tag{4.9}$$

To determine when the sum of the areas with $L_1 = L_2$ is a local minimum, we compute its second derivative with respect to L_1 at $L_1 = L_2$. In view of the constraint, we get $d^2\left[\mathcal{A}(L_1) + \mathcal{A}(L_2)\right]/dL_1^2 = 2d^2\mathcal{A}(L_1)/dL_1^2$. From (4.6) we find

$$\frac{d\mathcal{A}_1}{dL_1} = \frac{d\mathcal{A}_1/d\gamma}{dL_1/d\gamma} = \frac{\cos 2\gamma - \cos 2\beta}{\cos\gamma - \cos\beta} = \cos\gamma + \cos\beta. \tag{4.10}$$

Now differentiating (4.10) with respect to L_1 yields

$$\frac{d^2\mathcal{A}_1}{dL_1^2} = \frac{1}{dL_1/d\gamma}\left(-\sin\gamma + \frac{1}{n}\sin\beta\right). \tag{4.11}$$

Since $\beta \geq \gamma$ it follows from (4.8) that $dL_1/d\gamma > 0$. Therefore, for $n = 1$ and all γ, and also for $n > 1$ and γ sufficiently small, the expression (4.11) yields $d^2\mathcal{A}_1/dL_1^2 > 0$. Then at $L_1 = L_2$, $\mathcal{A}(L_1) + \mathcal{A}(L_2)$ has a local minimum.

For $n \geq 1$, $\mathcal{A}(L_1) + \mathcal{A}(L_2)$ will continue to have a local minimum at $L_1 = L_2$ as γ increases, until it reaches the value $\gamma = \gamma_0$ at which $d^2\mathcal{A}/dL_1^2 = 0$. To find γ_0 we equate the right side of (4.11) to zero, and eliminate β by using the second equation in (4.6). Then we get the following equation for γ_0:

$$\sin\gamma_0 = \frac{1}{n}\sin\left(\frac{\alpha - 2\gamma_0}{2n}\right), \qquad 0 \leq \gamma_0 < \beta = \frac{\alpha - 2\gamma_0}{2n} < \frac{\alpha}{2(n+1)}, \tag{4.12}$$

for $n \geq 2$. Thus $\mathcal{A}(L_1) + \mathcal{A}(L_2)$ has a local interior minimum $\mathcal{A}_I(L_{12})$ at $L_1 = L_2$ if $n = 1$, or if $n \geq 2$ and $0 < \gamma < \gamma_0$. Its value is

$$\mathcal{A}_I(L_{12}) = 2\mathcal{A}(L_{12}/2). \tag{4.13}$$

For $n \geq 2$ and $\gamma > \gamma_0$, the sum has a local maximum at $L_1 = L_2$. Then the minimum lies on the boundary, which case we consider next.

To treat the case of a minimum on the boundary with $L_1 \neq L_2$, we assume without loss of generality that $L_1 < L_2$. It can be shown that minimization requires both L_1 and L_2 to lie in the same interval $[\ell_n, \ell_{n+1}]$. Then either $\ell_n = L_1 < L_2 \leq \ell_{n+1}$ or $\ell_n < L_1 < L_2 = \ell_{n+1}$. In the first case $L_{12} \leq \ell_n + \ell_{n+1}$ and $\mathcal{A}(L_1)+\mathcal{A}(L_2) = \mathcal{A}(\ell_n)+\mathcal{A}(L_{12}-\ell_n)$. In the second case $L_{12} \geq \ell_n + \ell_{n+1}$ and $\mathcal{A}(L_1) + \mathcal{A}(L_2) = \mathcal{A}(L_{12} - \ell_{n+1}) + \mathcal{A}(\ell_{n+1})$. Thus the boundary minimum $\mathcal{A}_B(L_{12})$ is given by

$$\begin{aligned} \mathcal{A}_B(L_{12}) &= \mathcal{A}(\ell_n) + \mathcal{A}(L_{12} - \ell_n) && \text{if} \quad L_{12} \leq \ell_n + \ell_{n+1}, \\ &= \mathcal{A}(\ell_{n+1}) + \mathcal{A}(L_{12} - \ell_{n+1}) && \text{if} \quad L_{12} \geq \ell_n + \ell_{n+1}. \end{aligned} \tag{4.14}$$

Here $\ell_n = 2n\sin(\alpha/2n)$.

The interior and boundary minima become equal when

$$\mathcal{A}_I(L_{12}) = \mathcal{A}_B(L_{12}). \tag{4.15}$$

For each n and E, (4.15) determines a value $C_n(E)$ satisfying (4.9). For slightly smaller values of C, the left side of (4.15) is smaller than the right side, and for slightly larger values of C the right side is smaller than the left side.

We can express this result as follows: For given n and E, let C lie in the interval (4.9) and let $C_n(E)$ be the solution of (4.15) in that interval. For $C \leq C_n(E)$ the minimum value of the sum of the two areas is given by (4.13), and for $C \geq C_n(E)$ the minimum is given by (4.14). In the first case $L_1 = L_2$ and each polygonal line has $n + 1$ sides. In the second case $L_1 = \ell_n$ and the corresponding polygonal line has n sides, while $L_2 = L_{12} - \ell_n$ and the corresponding polygonal line has $n + 1$ sides. The total area of the convex domain $K(E, C)$ is the sum of the area $A_T(E)$ of the trapezoid plus the minimum of the sum of the areas outside it, which is $\mathcal{A}_I(L_{12})$ when $C \leq C_n(E)$ and $\mathcal{A}_B(L_{12})$ when $C \geq C_n(E)$.

These considerations complete the determination of the set $K(E, C)$ which has the minimum area $A(E, C)$, among all sets K for which a diameter is an edge, when E and C satisfy

$$C_4(E) \leq C \leq C_2(E), \qquad 0 \leq E \leq \sqrt{3}/2. \tag{4.16}$$

Since $A_{\min}(E, C) \leq A(E, C)$, this result gives an upper bound on $A_{\min}(E, C)$.

5. Bottom Boundary for $\sqrt{3}/2 \le E \le 1$ and $C_C(E) \le C \le C_2(E)$

Next we consider the bottom boundary for E and C in the intervals $\sqrt{3}/2 \le E \le 1$ and $C_C(E) \le C \le C_2(E)$. For each such E and C we seek a set $K(E, C)$ which minimizes A among all sets with those values of E and C. When $C = C_C(E)$, K is the fully capped equilateral Yamanouti triarc $K_C(E)$ (see Fig. 5(c)), and when $C = C_2(E)$ it is the set $K_2(r, E)$ shown in Fig. 4(b) with $r = 1$.

Both $K_C(E)$ and $K_2(1, E)$ contain an equilateral triangle with sides of length 1, and $K_C(E)$ is a polygon. Therefore we now *assume that for* $C < C_2(E)$, $K(E, C)$ *is a polygon containing an equilateral triangle* OPQ *with sides of length* 1. Then from the condition that $D = 1$, it follows that K is contained in the Reuleaux triangle K_R with vertices OPQ. Furthermore, from the condition that the minimum width of K is E, it follows that K contains the equilateral Yamanouti triarc $K_Y(E)$ with vertices OPQ. Thus the boundary ∂K of K passes through O, P, and Q and is divided by these vertices into three parts.

To determine $K(E, C)$ we use an analysis like that in Section 4. The argument in the second paragraph after (4.1) shows that the vertices of K must lie on K_R, unless the minimum width condition makes that impossible. In Section 4 the minimum width condition was satisfied for any positions of the vertices in the two lenses. Therefore we now seek K with all its vertices on K_R. We find that this is possible without violating the minimum width condition for all C in the range $C_P(E) \le C < C_2(E)$. Here $C_P(E)$ is the minimum value of C among all area minimizing polygons inscribed in K_R, and it is achieved by a polygon $K_P(E)$. When $E = E_n \equiv \sin(\pi/6n)$, $n = 1, 2, \ldots$, the polygon $K_P(E)$ coincides with $K_C(E)$, which then has all its vertices on the boundary of K_R. Therefore $C_C(E_n) = C_P(E_n)$ so that this construction of polygons inscribed in K_R, and that of capping sets of $K_Y(E)$ in Section 4, cover all values of C from $C_Y(E_n)$ to $C_2(E_n)$. The details are given in Appendix A.

When E does not have one of these special values E_n, there is an interval of values of C, namely, $C_C(E) < C < C_P(E)$, for which neither of the previous constructions applies. For C in this range, $K(E, C)$ does not have all its vertices on K_R. However, we can show that at most one vertex of K in each lens does not lie on K_R. The vertices on K_R can be chosen to coincide with the vertices of $K_P(E)$, so at most one more vertex must be determined in each lens. This is done in Appendix B.

6. Absence of Holes

We now show that S contains all points between the top boundary $A_{\max}(E, C)$ and the bottom boundary $A_{\min}(E, C)$, or our upper bound on it. To do so we construct a continuous family of convex sets, starting at any set $K_{\mathrm{b}}(E, C)$ corresponding to a point on the bottom boundary of S, or on our upper bound on it, and ending at a set $K_{\mathrm{t}}(E, C)$ corresponding to a point on the top boundary. All sets will have the same values of E, C, and D. Since their areas vary continuously, they will fill out the vertical segment (E, C, A) with A varying from $A_{\min}(E, C)$, or our upper bound on it, to $A_{\max}(E, C)$. We assume that each set K is placed so that it has a diameter of length $D = 1$ lying along the x-axis from $x = -\frac{1}{2}$ to $x = +\frac{1}{2}$.

First we define the support function $h(\theta)$ of a convex set K to be the distance from the origin, in K, to the support line of K with outward normal direction θ. Then the width function of K is $w(\theta) = h(\theta) + h(\theta + \pi)$. Now we form the one-parameter family of support functions

$$h_\lambda(\theta) = (1 - \lambda)h_b(\theta) + \lambda h_b(\theta + \pi), \qquad 0 \le \lambda \le \tfrac{1}{2}. \tag{6.1}$$

Here $h_b(\theta)$ is the support function of $K_b(E, C)$. Each support function h_λ defines a convex set $K_\lambda(E, C)$ with width function $w_\lambda(\theta) = h_\lambda(\theta) + h_\lambda(\theta + \pi) = w_b(\theta)$. Thus all K_λ have the same width functions as $K_b(E, C)$ so they have the same values of E, C, and $D = 1$. It follows from (6.1) that $h_{1/2}(\theta + \pi) = h_{1/2}(\theta)$, so $K_{1/2}(E, C)$ is centrally symmetric.

Next we note from Fig. 2 that the minimum width of all the sets $K_t(E, C)$ is the distance between two horizontal support lines. Therefore $w_t(\pi/2) = E$. The analysis of Sections 3 and 4 shows that for $0 \le E \le \sqrt{3}/2$, all the sets $K_b(E, C)$ on the bottom boundary, or on our upper bound on it, also have their minimum widths along the y-axis. Thus for them $w_t(\pi/2) = E$. The results of Section 5 for $\sqrt{3}/2 \le E \le 1$ and $C_1(E) \le C \le C_Y(E)$ show the same thing. However, for $\sqrt{3}/2 \le E \le 1$ and $C_Y(E) < C \le C_2(E)$ some of the sets $K_b(E, C)$ have $w_t(\pi/2) > E$. From these considerations it follows that $K_{1/2}(E, C)$ has $w_{1/2}(\pi/2)$ for all values of E and C except for those in $\sqrt{3}/2 \le E \le 1$ and $C_Y(E) < C \le C_2(E)$.

For E and C in this exceptional range, we can deform $K_{1/2}(E, C)$ continuously, keeping E, C, and D fixed and keeping the set convex, to a new set $K_v(E, C)$ with $w_v(\pi/2) = E$. Thus K_v has its minimum width in the vertical direction. We do not give the details of this deformation, because it can be done in many ways since the deformed sets do not have to satisfy any minimization condition.

Now we denote by $K_v(E, C)$ this deformed set in the exceptional range, or $K_{1/2}(E, C)$ in the rest of the range, so that for all possible E and C we have $w_v(\pi/2) = E$, where w_v is the width function of K_v. Then we form the new width function

$$w^k(\theta) = (1 - k)w_v(\theta) + kw_t(\theta), \qquad 0 \le k \le 1. \tag{6.2}$$

This width function w^k determines a continuous family of centrally symmetric convex sets, all with the same values of E, C, and D. They range from K_v at $k = 0$ to K_t at $k = 1$.

By means of these two or three deformations, we have constructed a continuous family of convex sets, all with the same E, C, and D. Their areas vary continuously from $A_{\min}(E, C)$, or our upper bound on it, to $A_{\max}(E, C)$, so all points in this interval are included in S.

7. Summary

The set $\widehat{S}$ of points $(E, C, A^{1/2}, D)$ in R^4 corresponding to all convex planar sets K occupies a cone. Its cross section S in the plane $D = 1$, described by the coordinates (E, C, A), lies in the rectangular parallelepiped (1.1). Figure 1 shows the top view of S, i.e., the projection of S on the EC plane. The front view of S, i.e., the projection of S

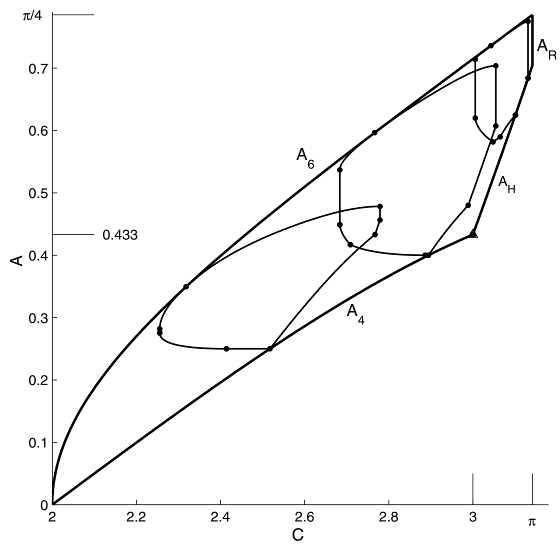

Fig. 8. The projection of S on the CA plane.

on the CA plane is shown in Fig. 8 by the heavy outermost line. It reduces to the point $(2, 0)$ at the left end, corresponding to $E = 0$. At the right end, $C = \pi$, it becomes the vertical segment A_R corresponding to sets of constant width $E = 1$. The upper boundary A_6 represents symmetric lenses of width $E \in [0, 1]$ and diameter 1. The arc A_4 of the lower boundary represents isosceles triangles of base 1, and height E in $[0, \sqrt{3}/2]$. Its end $E = \sqrt{3}/2$ corresponds to the equilateral triangle with $C = 3$. The other arc of the lower boundary, A_H, from the equilateral triangle to the Reuleaux triangle with $C = \pi$, represents polygons with all but at most three of their vertices on the Reuleaux triangle, as found by Hemmi [3]. Note that along the arc A_H, E has countably infinitely many discontinuities and is not monotonically increasing from $\sqrt{3}/2$ to 1, as C increases from 3 to π. The other three lighter lines are cross sections of S at $E = 0.5$, 0.8, and 0.95, respectively, from left to right. For $E = 0.5$ the horizontal line at the bottom connects dots at $C_3(E)$ and $C_4(E)$, and represents triangles of base 1 and height E. Proceeding counterclockwise from $C_4(E)$, the next dot is for the trapezoid $K_T(E)$ with area $A_T(E)$, then at the maximum perimeter $C_2(E)$ one dot is the minimum area $A_2^\star(E)$ and the next at the maximum area $A_2(E)$. The dot on A_6 is for the symmetric lens. Finally at the minimum perimeter $C_1(E)$ one dot is for the maximum area $A_1(E)$ and the next for the minimum area $A_1^*(E)$. Some of the dots in the other two lines correspond to these, and the others correspond to points on the curves in Fig. 1.

Fukasawa [2] determined $A_{\max}(E, C)$, which is the top boundary of S. From the result of Kubota [7] we found part of the bottom boundary $A_{\min}(E, C)$. We have also found the vertical boundaries of S, and certain additional parts of the bottom boundary $A_{\min}(E, C)$. For the remaining parts of the bottom boundary, we have obtained upper bounds which we conjecture to be the actual bottom boundary. In addition, we have proven that there are no holes in S, by showing that it extends from the bottom boundary, or from our upper bound on it, to the top boundary.

We hope that some reader will prove our conjecture, or, if it is false, find the actual bottom boundary.

Appendix A. Bottom Boundary for $\sqrt{3}/2 \le E \le 1$, $C_P(E) \le C \le C_2(E)$

For $\sqrt{3}/2 \le E \le 1$ and $C_C(E) \le C < C_2(E)$ our assumption implies that $K(E, C)$ is a polygon contained in the Reuleaux triangle K_R with vertices OPQ. It is inscribed in K_R if the condition that its minimum width is E allows that. This condition implies that no side of K can be longer than the chord at distance E from the opposite vertex. The length of this chord is $s_{\max} = 2\sin\rho$ where $\rho = \cos^{-1} E = (\pi/2) - \alpha$. Furthermore, K must have at least one chord of length $s_{\max}$ in order that the minimum width equals E.

Let the lens containing this chord be PQ, and let this lens contain m successive vertices $V_1, V_2, \ldots, V_m$. The chord of length $s_{\max}$ can be chosen to be $V_m Q$. Then the chord from P to V_m bounds a smaller lens which subtends the angle $\Phi = (\pi/3) - 2\rho$. Each of the arcs in the other two lenses subtend the angle $\pi/3$. The analysis of Section 4 shows that in each of these two arcs K will have n long equal chords and one short chord, while in the arc of angle Φ it will have m equal long chords and one short chord. Then $K(E, C)$ has $n + 1$ chords in each of the arcs of angle $\pi/3$, $m + 1$ chords in the arc of angle Φ, and one chord of length $s_{\max}$. Thus K is a convex $(2n + m + 4)$-gon inscribed in K_R. The values of m and n will be determined in the process of minimizing the area of K.[1]

Let L_1, L_2, and L_3 be the total lengths of the chords in the arc of angle $\pi/3$, $\pi/3$, and Φ, respectively, and let $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ be the corresponding areas of K in the three lenses. The analysis of Section 4, starting at (4.4) with α replaced by $\pi/3$, shows that each of the first two lenses will have n equal long chords, each subtending the angle 2β, and one short chord subtending the angle γ. For a given value of $L_{12} = L_1 + L_2$, that analysis determines the values of n, L_1, and L_2 which minimize $A_{12} = A_1 + A_2$. Similarly for the third arc, that analysis with α replaced by Φ determines the number m of long chords, each subtending an angle $2\bar{\beta}$, and the short chord subtending an angle $2\bar{\gamma}$, in terms of L_3.

The smallest values of n and m are achieved when all the long chords are equal to $s_{\max}$. Then $n = m + 1 = n^*$, $\beta = \bar{\beta}$, and $\gamma = \bar{\gamma} = \gamma^*$ and K is a $3(n^* + 1)$-gon. The common values n^* and γ^* are determined by the following equation and inequality

$$n^*\rho + \gamma^* = \pi/6, \qquad 0 < \gamma^* \le \pi/[6(n^* + 1)] \le \rho. \tag{A.1}$$

We must determine L_{12} and L_3 to minimize the total area $\mathcal{A}$ of K. The condition that the perimeter of K is C, and the relations between L_1, L_2, L_3, n, and m are

$$L_{12} + L_3 + s_{\max} = C, \qquad \ell_n < L_2 \le L_1 \le \ell_{n+1}, \qquad \bar{\ell}_m < L_3 \le \bar{\ell}_{m+1}. \tag{A.2}$$

[1] Here the convex domain $K(E, C)$ differs from Hemmi's [3] result K_H in two respects: (i) the number of sides in each lens of K_R is greater than or equal to $n^* + 1$ where $n^\star$ is defined by (A.1), and (ii) to have minimum width E, the polygon K must have at least one chord of length $s_{\max}$ subtending the angle 2ρ, say the last chord $V_{n^*}Q$ in the third lens.

Here $\ell_n = 2n \sin(\pi/3 \cdot 1/2n)$ and $\bar{\ell}_m = 2n \sin(\Phi/2m)$. The area $\mathcal{A}$ of K is

$$\mathcal{A} = \mathcal{A}_{12} + \mathcal{A}_3 + \tfrac{1}{2}\,(Es_{\max} + \sin\Phi). \tag{A.3}$$

In the limiting case $E = \sqrt{3}/2$, we have $\alpha = 2\rho = \pi/3$. The solution $K(E, C)$ is given by that in Section 4 with the smaller lens empty, i.e., $m = 0$ and $L_3 = \Phi = \mathcal{A}_3 = 0$.

For $n > n^*$, the interval $\mathcal{I}_n$ containing L_{12} is $(2\ell_n, 2\ell_{n+1})$. Then the optimum combination of L_1 and L_2 and the minimum value of $\mathcal{A}_{12}(L_{12})$ are (see Section 4)

$$\mathcal{A}_{12} = \min(A_I, A_B), \tag{A.4}$$

where

$$\begin{aligned} A_B &= \mathcal{A}(L_{12} - \ell_n) + \mathcal{A}(\ell_n) \qquad && (L_{12} \le \ell_n + \ell_{n+1}) \\ &= \mathcal{A}(L_{12} - \ell_{n+1}) + \mathcal{A}(\ell_{n+1}) \qquad && (L_{12} \ge \ell_n + \ell_{n+1}). \end{aligned} \tag{A.5}$$

$$A_I = 2\mathcal{A}(L_{12}/2). \tag{A.6}$$

These formulas hold for $n = n^*$, when $2\ell_{n^*} = 2C_C/3$. In $\mathcal{I}_n$ there can be at most two subintervals, smaller than $\mathcal{I}_n$ by $O(1/n)$, in which the slope dA_{12}/dL_{12} is increasing. If they exist, the first subinterval begins at that end of $\mathcal{I}_n$ where $\mathcal{A}_{12} = A_I$, and the second begins at the mid-point of $\mathcal{I}_n$, where $\mathcal{A}_{12} = A_B$. We find the range of the slope dA_{12}/dL_{12} in $\mathcal{I}_n$ and denote it by $R_n = [\min_n, \max_n]$.

For the smaller lens PV_{n^*}, the lower and upper bounds of L_3 are $\bar{\ell}_m$ and $\bar{\ell}_{m+1}$ for $m \ge m^*$, with the lower bound $\bar{\ell}_{m^*}$ defined as $C_C/3 - s_{\max}$. The range of the slope, $\bar{\mathcal{A}}'(\bar{\ell})$, is denoted by $\bar{R}_m = [\,\overline{\min}_m, \overline{\max}_m]$.

We consider a combination (m, n), for which (L_3, L_{12}) lies in the rectangle $[\bar{\ell}_m, \bar{\ell}_{m+1}] \times [2\ell_n, 2\ell_{n+1}]$. To find the optimum path of (L_3, L_{12}) beginning from the lower left corner of the rectangle, we first compare the ranges of the slopes $\bar{R}_m$ and R_n. If $\bar{R}_m$ and R_n do not intersect, we have either (1) $\overline{\min}_m \ge \max_n$ or (2) $\overline{\max}_m \le \min_n$.

(1) $\overline{\min}_m \ge \max_n$. The optimum path is along the left side of the rectangle, $L_3 = \bar{\ell}_m$, with the smaller lens unchanged. We increase L_{12} and $\mathcal{A}_{12}$ accordingly until L_{12} reaches its upper bound $2\ell_{n+1}$, arriving at the lower right corner of the combination. For further increment in C, we begin with the next combination $(m, n + 1)$.

(2) $\overline{\max}_m \le \min_n$. The optimum path is along the lower edge of the rectangle, where $L_{12} = 2\ell_n$, with no changes in the first two lenses. We increase L_3 and $\bar{\mathcal{A}}$ of the smaller lens until L_3 reaches its upper bound $\bar{\ell}_{m+1}$, arriving at the lower left corner of the next combination $(m + 1, n)$.

For example, for $\pi/12 < \rho < \pi/6$, i.e., $0.866 < E < 0.966$, we have $n^* = 1$ and $m^* = 0$, $0 < \gamma^* < \pi/12$. The range $\bar{R}_0$ will not intersect R_n for $n \le 3$. For $0 < \rho < \pi/12$, we have $n^* \ge 2$ and $m^* \ge 1$.

If the ranges of the slopes of the combination (m, n) do intersect, then it is possible that the optimum path could end at the upper right corner $(\bar{\ell}_{m+1}, 2\ell_{n+1})$ and then continue with the next combination $(m + 1, n + 1)$. We need to compare the area A along the left edges of the rectangle with $L_{12} \in (2\ell_n, 2\ell_{n+1}]$ and that along the bottom edge with $L_3 \in (\bar{\ell}_m, \bar{\ell}_{m+1}]$. If their intersection $\bar{R}_m \cap R_n$ lies in a subinterval of $\bar{R}_m$ where the slope is increasing and also in that of R_n, we then have to compute a third solution

$A_{\mathrm{III}}(C) = \mathcal{A}_{12} + \bar{\mathcal{A}}$ with $C = L_{12} + L_3 + s_{\max}$ in the intersection with increments of L_{12} and L_3 adjusted to match the slopes of $\mathcal{A}_{12}$ and $\bar{\mathcal{A}}$. We then have an additional curve to be compared with in defining the bottom boundary. (Since $\bar{\Phi} < \Phi < \pi/3$, and the lower bounds of the slopes $\min_n$ and $\overline{\min}_m$ are less than 2 by $O([\Phi/2n]^2)$ and $O([\bar{\Phi}/2m]^2)$, the differences between those curves become graphically indistinguishable for n and m greater than 2.)

Appendix B. Bottom Boundary from C_C to C_P for $\sqrt{3}/2 \le E \le 1$

The convex domain $K_P(E)$ at the endpoint (C_P, A_P) was described in Appendix A. The corresponding convex domain $K_C(E)$ at the endpoint (C_C, A_C), described in Section 3, is the largest polygon circumscribing K_Y. The subdomains of K_C in the three lenses of K_R are equivalent. Therefore, it suffices to study one of the lenses. In the lens PQ, all the $n^* + 1$ sides of $K_C(E)$ are tangent to the circular arc $\overset{\frown}{T_1 T_2}$ of K_Y, with $n^* - 1$ vertices on $\overset{\frown}{PQ}$ and one vertex, say V_1', lying inside the lens. Thus we can identify those $n^* - 1$ vertices on $\overset{\frown}{PQ}$ with the vertices $V_2, \ldots, V_n$ of K_P. The two sides of vertex V_1', $V_1'P$, and $V_1'V_2$, are tangent to $\overset{\frown}{T_1 T_2}$ at T_1 and T_1', respectively, and they are shorter than $s_{\max}$, with $|V_1'V_2| = |V_1'P| = \sin\beta^* + E\tan\gamma^* = \Lambda < s_{\max}$. The side V_2V_1' of K_C is a segment of the side V_2V_1 of K_P.

The difference between the domains K_P and K_C can be seen by their difference in lens PQ. Let ℓ_*, $\mathcal{A}_*$, and ℓ^*, $\mathcal{A}^*$ denote respectively the perimeters and areas of K_C and K_P in this lens. We have

$$\begin{aligned} \ell_* &= C_C/3, \quad \mathcal{A}_* = (A_C - \sqrt{3}/4)/3, \quad \text{and} \\ \ell^* &= C_P/3, \quad \mathcal{A}^* = (A_P - \sqrt{3}/4)/3.11 \end{aligned} \tag{B.1}$$

$\mathcal{A}^*$ and $\mathcal{A}_*$ are identical in the lens QV_2 while their difference $A^* - A_* = \Delta PV_1V_1'$ appears only in the lens V_2P.

To find the domain $K(C, E)$ of minimum area, as C increases from C_* to C^*, we consider how $\mathcal{A}(\ell)$ increases from $\mathcal{A}_*$ in lens PQ, where ℓ denotes the perimeter of K in this lens. It is not hard to show that

$$E + \tfrac{1}{2} \ge \mathcal{A}'(\ell) > E/2 \qquad \text{and } \mathcal{A}' \text{ decreases as } \ell \text{ increases from } \ell_* \text{ to } \ell^*. \tag{B.2}$$

Then $K(C, E)$ is obtained by increasing the perimeter in the three lenses one by one as C exceeds C_*, $C_* + (C^* - C_*)/3$, and $C_* + 2(C^* - C_*)/3$, respectively.

To find $\mathcal{A}(\ell)$ in lens PQ, we observe that the part of K in the lens QV_2, common to K_* and K^*, remains unchanged so that the change of $\mathcal{A}$ comes from that of K_* in the lens V_2P. This follows because a negative perimeter increment $-\bar{\ell}$ in the lens V_2Q would change the area by at least $-E\bar{\ell}/2$, see (B.2). To keep ℓ the same, there has to be an increment $\bar{\ell}$ in the lens PV_2 with an area increment greater than $E\bar{\ell}/2$. Thus the resultant area increment in the lens PQ is greater than that with $\bar{\ell} = 0$. On the other hand, if we increase the perimeter in the lens V_2Q by $\bar{\ell}$ we increase the area by an amount greater than $2E\bar{\ell}$ (see Section 4, or [3]) while the compensation of $-\bar{\ell}$ in the lens PV_2 would decrease the area by an amount less than $(\frac{1}{2}+E)\bar{\ell} < 2E\bar{\ell}$, from (B.2) for $E \geq \sqrt{3}/2$. Again there is a net increment of area. Thus the boundary of K in the lens V_2Q does not change from that common to K_* and K^*.

Because of (B.2) and the symmetry of A_* in the lens PV_2 with respect to the radial bisector OV_1' of the lens, we consider the difference $\mathcal{A}-\mathcal{A}_*$ to remain in the part of the lens which is the curvilinear triangle ΔPV_2V_1 bounded by the sides V_2P and V_2V_1 and $\overset{\frown}{V_1P}$. so that $\mathcal{A}-\mathcal{A}_*$ becomes $\mathcal{A}^*-\mathcal{A}_* = \Delta PV_1'V_1$ as $\ell \to \ell^*$.

We study the boundary of $\mathcal{A}$ in the curvilinear ΔV_2V_1P for $\ell \in (\ell_*, \ell^*)$ via the displacement of the vertex V_1' to a point U outside of $\mathcal{A}_*$. To achieve the minimum area, point U must either be (i) on the boundary $V_1'V_1$ or (ii) on $\overset{\frown}{PV_1}$ of the curvilinear triangle.

Case (i). If point U lies in $V_1'V_1$, the new boundary of $\mathcal{A}$ is the sides UV_1' and UP replacing the side PV_1' of $\mathcal{A}_*$. The additional area is $\Delta PUV_1'$. With $\angle UV_1'P = 2\gamma^*$ and $|PV_1'| = \Lambda$, $\Delta PUV_1'$ is specified by the parameter $\omega = \angle V_1'PU$. As ω increases from zero to $\rho-\gamma^*$, vertex U moves from V_1' to V_1 while ℓ increases from ℓ_* to ℓ^*. For this case, we have

$$\ell_i(\omega) = \ell_* + \Lambda\left[\frac{\sin 2\gamma^* + \sin\omega}{\sin(\omega+2\gamma^*)} - 1\right] \quad \text{and} \quad \mathcal{A}_i(\omega) = \mathcal{A}_* + \frac{\Lambda^2 \sin\omega \sin 2\gamma^*}{2\sin(\omega+2\gamma^*)}. \tag{B.3}$$

Case (ii). As ℓ increases from ℓ_* to ℓ^*, U moves along $\overset{\frown}{PV_1}$ from P to V_1. Since the new side PU is at a distance greater than $\cos\gamma^* > E$ from O, we keep the other side UU' at the minimum distance E from O. The side UU' is tangent to $\overset{\frown}{T_1T_2}$ at point T_U and intersects the line V_2V_1' at U'. Point U' caps $\overset{\frown}{T_1T_2}$ at point T_1' and T_U. When $\ell \to \ell^*$, the cap vanishes as $T_U \to T_1'$ along $\overset{\frown}{T_1T_2}$ while $U' \to T_1'$ along the line $V_1'T_1'$.

Let the polar angle of U, $\angle POU$, equal $2\bar{\omega}$ and the capping angle of U', $\angle T_1'OT_U$, equal 2μ. Since $\angle UOT_U = \rho = \angle V_1OT_1'$, and $\angle POT_1' = \rho + 2\gamma^* = 2\bar{\omega} + \rho + 2\mu$, we have $\mu = \gamma^* - \bar{\omega}$. With $\bar{\omega} \in (0, \gamma^*]$, as the parameter, we have

$$\begin{aligned} \ell_{ii}(\bar{\omega}) - \ell_* &= |PU| + |UU'| - |PV_1'| - |V_1'U'| \\ &= 2\{\sin\bar{\omega} + E[\tan\mu - \tan\gamma^*]\}, \end{aligned} \tag{B.4}$$

$$\begin{aligned} \mathcal{A}_{ii}(\bar{\omega}) - \mathcal{A}_* &= \Delta PUU' - \Delta PV_1'U' \\ &= \tfrac{1}{2}\sin 2\bar{\omega} + E^2[\tan\mu - \tan\gamma^*]. \end{aligned} \tag{B.5}$$

For the minimum area increment in a lens, we have

$$\mathcal{A}(\ell) = \min[A_i(\ell), A_{ii}(\ell)], \qquad \ell \in [C_*/3, C^*/3]. \tag{B.6}$$

By analyzing $\mathcal{A}(\ell)$ for cases (i) and (ii), we find that

$$\mathcal{A}(\ell) = \mathcal{A}_{ii}(\ell) \qquad \text{for} \qquad 0 < \gamma^* \leq \arcsin(\tfrac{1}{2}\ \sin\rho), \tag{B.7}$$

$$\mathcal{A}(\ell) = \mathcal{A}_{i}(\ell) \qquad \text{for} \qquad \frac{\rho}{2} \leq \gamma^* < \rho. \tag{B.8}$$

For the intermediate range of γ^* there is a transition point ℓ_c in the interval (ℓ_*, ℓ^*), where $\mathcal{A}_{ii}(\ell)$ crosses over $\mathcal{A}_i(\ell)$, i.e., $\mathcal{A}_{ii} = \mathcal{A}_i$ and $\mathcal{A}_{ii}' > \mathcal{A}_i'$ at $\ell = \ell_c$. Thus we have for $\arcsin\left(\frac{1}{2}\ \sin\rho\right) < \gamma^* < \rho/2$,

$$\mathcal{A}(\ell) = \mathcal{A}_i(\ell), \quad \ell_* \le \ell \le \ell_c, \qquad \text{and} \qquad \mathcal{A}(\ell) = \mathcal{A}_{ii}(\ell), \quad \ell_c \le \ell \le \ell^*. \quad \text{(B.9)}$$

We can confirm (B.2) and use the statement following (B.2) to obtain the bottom boundary

$$A(C) = m\mathcal{A}^* + (2 - m)\mathcal{A}_* + \mathcal{A}(\ell) + \sqrt{3}/4, \qquad \text{(B.10)}$$

where $m = 0, 1, 2$ with $C - C_* = m(\ell^* - \ell_*) + (\ell - \ell_*)$ and $\ell_* \le \ell < \ell^*$.

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