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Inequalities for the h-Vectors and Flag h-Vectors of Geometric Lattices*

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Abstract. We prove that the order complex of a geometric lattice has a convex ear decomposition. As a consequence, if $\Delta(L)$ is the order complex of a rank $(r+1)$ geometric lattice L, then for all $i \leq r/2$ the h-vector of $\Delta(L)$ satisfies $h_{i-1} \leq h_i$ and $h_i \leq h_{r-i}$.

We also obtain several inequalities for the flag h-vector of $\Delta(L)$ by analyzing the weak Bruhat order of the symmetric group. As an application, we obtain a zonotopal **cd**-analogue of the Dowling–Wilson characterization of geometric lattices which minimize Whitney numbers of the second kind. In addition, we are able to give a combinatorial flag h-vector proof of $h_{i-1} \leq h_i$ when $i \leq \frac{2}{7}(r + \frac{5}{2})$.

1. Introduction

The order complex of a geometric lattice is one of many simplicial complexes associated to matroids. For a geometric lattice L, the order complex of L, $\Delta(L)$, is the simplicial complex whose simplices consist of all chains in L, $\hat{0} \neq x_1 < x_2 < \cdots < x_k \neq \hat{1}$. The number of flats in each rank, also known as the Whitney numbers of the second kind, can be viewed as special cases of the flag f-vector of $\Delta(L)$. The Euler characteristic of $\Delta(L)$ is the Möbius invariant of L [Fo]. Surveys of these topics are [Ai], [Bj4], and [Za].

Other enumerative invariants of $\Delta(L)$ have not received as much attention. The explicit relationship between the flag h-vector of $\Delta(L)$ and the **cd**-index of oriented matroids and zonotopes discovered in [BER] suggests that it may be time to study the h-vector and flag h-vector of $\Delta(L)$.

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We begin with a review of the basic notions associated to geometric lattices, graded posets, h-vectors, and flag h-vectors. This is followed by an examination of the geometric lattices which minimize and maximize the flag h-vector of their order complex. As a consequence we find the zonotopes with a specified dimension and number of zones which minimize or maximize the **cd**-index. Then we show that $\Delta(L)$ has a convex ear decomposition. An immediate consequence of this decomposition is our main theorem concerning the h-vectors of order complexes of geometric lattices.

Theorem 1.1. *Let L be a rank $(r+1)$ geometric lattice. Then, for $i \leq r/2$, the h-vector of $\Delta(L)$ satisfies*

$$h_{i-1} \leq h_i, \tag{1}$$

$$h_i \leq h_{r-i}. \tag{2}$$

As is frequently the case with theorems of this type, the proof is not combinatorial. The remaining sections are devoted to understanding the flag h-vector of $\Delta(L)$ with an eye toward providing a combinatorial proof of (1). This leads us to an examination of the weak Bruhat order on the symmetric group.

2. Definitions

We take all posets in this work to be finite. A poset P is *graded* if all maximal chains have the same length and we call this length the *rank* of P. A graded poset has an associated rank function ρ which assigns to each element y of P a positive integer such that $\rho(y) = k$, where k is the length of the longest chain of the form $y_0 < y_1 < \cdots < y_k = y$.

A *lattice* is a poset such that each pair of elements, x and y, has a least upper bound, or *join*, denoted $x \vee y$, and a greatest lower bound, or *meet*, denoted $x \wedge y$. Consequently, a lattice has a unique minimal element $\hat{0}$ such that $x \geq \hat{0}$ for all $x \in L$, and a unique maximal element $\hat{1}$ with $x \leq \hat{1}$ for all $x \in L$. An element of L which covers $\hat{0}$ is an *atom*, and L is *atomic* if every element in L can be written as the join of atoms.

A *geometric lattice* is a graded atomic lattice whose rank function satisfies the *semimodular condition* that for any $x, y \in L$,

$$\rho(x \vee y) + \rho(x \wedge y) \leq \rho(x) + \rho(y).$$

A broad class of geometric lattices arise from the affine dependencies of a finite set of points X in Euclidean space. In this case the rank k elements of the lattice are subsets of the form $T \cap X$ where T is a $(k-1)$-dimensional subspace spanned by the elements of X. These subsets are ordered by inclusion. Points are in *general position* if every set of $k+1$ points spans a k-dimensional subspace. One particularly useful geometric lattice arises from the *near pencil* arrangement on n points in r-dimensional space which consists of $(n-r+1)$ points on a line with the remaining $(r-1)$ points in general position. For ease of reference we call this lattice the rank $r+1$ near pencil on n atoms (see Fig. 1).

A *matroid* $M = (X, \overline{})$ is a set X (for us X is always finite) with a closure operation satisfying the exchange property (see Section 1.4 of [Ox] for more details). For $A \subseteq X$

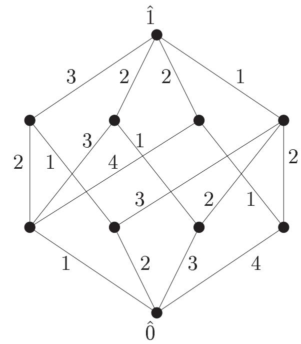

Fig. 1. An EL-labeling of the rank 3 near pencil on four atoms.

denote the *closure of* A by $\overline{A}$. A *simple matroid* is a matroid such that $\overline{\emptyset} = \emptyset$ and $\overline{a} = a$ for every element $a \in X$. The closed sets, or *flats*, of a matroid, when partially ordered by inclusion form a geometric lattice. In fact a result of Birkhoff shows there is a bijection between geometric lattices and simple matroids [Bi].

A set $S \subseteq X$ is *independent* if $x \notin \overline{S - x}$ for any $x \in S$. A *basis* of a matroid is a maximal independent set and a *circuit* is a minimal dependent set. A *loop* in a matroid $M = (X, \overline{})$ is an element $e \in X$ that is contained in no basis, while a *coloop* is an element which is contained in every basis of the matroid. Let B be a basis of a matroid M. If $e \notin B$, then $B \cup e$ contains a unique circuit, $C(e, B)$. The element e is in $C(e, B)$, and we call $C(e, B)$ the *fundamental circuit of e with respect to B*.

By a basis of a geometric lattice we mean a collection of atoms whose cardinality is the rank of P and whose join is $\hat{1}$. Particular bases which will be useful to us are the *nbc-bases*. In order to define an nbc-basis we first fix an arbitrary linear order ω on the atoms of the geometric lattice L. A *broken circuit* of (L, ω) is a circuit with its least element removed. The nbc-bases of (L, ω) are the bases of L that do not contain a broken circuit. All such bases must contain the least atom. Indeed, if B is a basis of L which does not contain the least element, then it will contain the broken circuit formed by removing the least element from the fundamental circuit contained in the union of B and the least element.

Example 2.1. Let L be the geometric lattice of the rank 3 matroid on $\{1, 2, 3, 4, 5\}$ (with the natural order) whose bases consist of all triples except $\{1, 2, 3\}$ and $\{3, 4, 5\}$. Then the nbc-bases of L are the triples $\{1, 2, 4\}$, $\{1, 2, 5\}$, $\{1, 3, 4\}$, $\{1, 3, 5\}$. Notice that $\{1, 4, 5\}$ is not an nbc-basis because it contains the broken circuit $\{4, 5\}$.

Let Δ be a simplicial complex, i.e., Δ is a collection of subsets of a vertex set X satisfying $x \in \Delta$ for any $x \in X$ and if $F \in \Delta$ and $G \subseteq F$ then $G \in \Delta$. Maximal faces of Δ are *facets* and Δ is *pure* if all its facets have the same dimension. A pure d-dimensional simplicial complex is said to be *shellable* if there is an ordering of its facets $F_1, F_2, \ldots, F_t$ such that $F_j \cap \bigcup_{i=1}^{j-1} F_i$ is a pure $(d - 1)$-dimensional complex for $j = 2, \ldots, t$. Such an ordering is called a *shelling*. Equivalently, a linear ordering ψ

on the facets of a complex is a shelling if and only if it satisfies the following criterion (see, for instance, [Bj4]):

Property *M*: For all facets F and F' of Δ such that $F' <^{\psi} F$ there is a facet F'' with $F'' <^{\psi} F$ such that $F' \cap F \subseteq F'' \cap F$ and $|F'' \cap F| = |F| - 1$.

Given a poset P with $\hat{0}$ and $\hat{1}$, the *order complex* $\Delta(P)$ of P is the simplicial complex whose vertices are the elements of $P - \{\hat{0}, \hat{1}\}$ and whose simplices are the chains of $P - \{\hat{0}, \hat{1}\}$. Thus the facets correspond to maximal chains. P is shellable if there exists a shelling of $\Delta(P)$.

A poset P admits an *R-labeling* if there is a map from the edges of P to the positive integers (or more generally to some partially ordered set) such that in any interval $[x, y] = \{z \in P : x \leq z \leq y\}$ of P there is a unique saturated chain with increasing labels (known as a rising chain). For y_2 covering y_1 in P, denote the label on edge (y_1, y_2) by $\lambda(y_1, y_2)$. Then a rising chain in $[x, y]$ is a maximal chain $x = y_0 < y_1 < \cdots < y_k = y$ with $\lambda(y_0, y_1) \leq \lambda(y_1, y_2) \leq \cdots \leq \lambda(y_{k-1}, y_k)$.

An *EL-labeling* [Bj1] of a poset is an R-labeling in which the unique rising chain in an interval $[x, y]$ comes first lexicographically among all of the chains in $[x, y]$. An EL-labeling of a geometric lattice can be obtained by labeling the atoms $\{1, \ldots, n\}$ and labeling the edge (x, y) with the minimal atom j such that $x \vee j = y$ (see Fig. 1). We call this ordering the *minimal labeling* of the facets of $\Delta(L)$ and for a given facet F we denote its minimal label $\lambda(F)$. Notice with this labeling distinct chains have distinct labeling sequences.

A result of Björner shows that ordering the maximal chains of an EL-labeled poset lexicographically on the chain labels gives a shelling of the associated order complex [Bj1, Theorem 2.3].

3. The Flag *f*-Vector and Flag *h*-Vectors of Geometric Lattices

For P a rank $r + 1$ poset, the number of simplices in $\Delta(P)$ of cardinality k is denoted $f_k(\Delta(P))$, and $f(\Delta(P)) = (f_0, \ldots, f_r)$ is known as the *f-vector* of the order complex. The *h-vector* of $\Delta(P)$ is defined as $h(\Delta(P)) = (h_0, \ldots, h_r)$ where

$$\sum_{i=0}^{r} f_i (x-1)^{r-i} = \sum_{i=0}^{r} h_i x^{r-i}.$$

Given a rank $r + 1$ poset P, and $S \subseteq [r] = \{1, 2, \ldots, r\}$, let P_S be the rank selected subposet of P defined by $P_S = \{x \in P : \rho(x) \in S, x = \hat{0}, \text{ or } x = \hat{1}\}$. The number of maximal chains of P_S is denoted $f_S(P)$; that is, $f_S(P)$ counts the number of chains in P in which the ranks are the elements of S. The collection $\{f_S\}$, $S \subseteq [r]$, is known as the *flag f-vector* of the poset. The flag f-vector gives a natural refinement of the f-vector of a poset's associated order complex as $f_i(\Delta(P)) = \sum_{|S|=i} f_S(P)$.

Let $\mathscr{B}_{r+1,n}$ denote the rank $r + 1$ truncated Boolean algebra on n atoms. $\mathscr{B}_{r+1,n}$ is isomorphic to the rank n Boolean algebra $\mathscr{B}_n$ for rank i, $i \leq r$, and rank $r + 1$ of $\mathscr{B}_{r+1,n}$ consists of the maximal element $\hat{1}$. Every $(r + 1)$-subset of $[n]$ is a basis of $\mathscr{B}_{r+1,n}$ and every i-subset $i \leq r+1$ is independent. We shall see that the flag f-vector for rank $r+1$ geometric lattices on n atoms is maximized by $\mathscr{B}_{r+1,n}$.

Dowling and Wilson [DW1] proved that for rank $r+1$ geometric lattices with n atoms the singleton flags $f_{\{i\}}$, $1 \leq i \leq r$, are minimized by the near pencil lattice. In her Ph.D. thesis [Ny], the first author gave an explicit formula for f_S of the near pencil lattice and proved that this minimizes f_S, $S \subseteq [r]$, for all geometric lattices of rank $r+1$ with n atoms. In this section we prove the stronger result that h_S is minimized by the near pencil for all $S \subseteq [r]$.

Define the flag h-vector $\{h_S(P)\}$, for $S \subseteq [r]$, by

$$h_S(P) = \sum_{T \subseteq S} (-1)^{|S|-|T|} f_T(P). \tag{3}$$

Again, the flag h-vector refines the h-vector since $h_i(\Delta(P)) = \sum_{|S|=i} h_S(P)$. An extremely useful combinatorial interpretation of the flag h-vector is that it counts the number of chains with a specified descent set in an R-labeled poset. We describe this interpretation below.

For a maximal chain m: $\hat{0} = x_0 < x_1 < \cdots < x_{r+1} = \hat{1}$, the *descent set* of m is $D(m) = \{i : \lambda(x_{i-1}, x_i) > \lambda(x_i, x_{i+1})\}$. The following proposition can be found in [Bj1, Theorem 2.7] and [St2, Theorem 3.13.2].

Proposition 3.1 [Bj1], [St2]. *For P a graded poset that admits an R-labeling, $h_S(P)$ is the number of maximal chains of P with labels having descent set S.*

Since (3) can be inverted to give $f_S(P) = \sum_{T \subseteq S} h_T(P)$ we see that $f_S(P)$ counts the number of maximal chains in an R-labeled poset P with descent set contained in S.

The positions of descents in maximal chains of a rank $r+1$ poset P can be encoded using the **ab**-*index* $\Psi(P)$ of the poset which we describe presently. Assign to each chain in P a word in the noncommuting variables "**a**" and "**b**" by assigning an **a** if consecutive edge labels in the chain increase and **b** if the labels decrease. Summing over all the chains in P gives the **ab**-index

$$\Psi(P) = \sum_{S \subseteq [r]} h_S(P) u_S,$$

where the word $u_S = u_1 u_2 \cdots u_r$ is given by $u_i = \mathbf{a}$ if $i \notin S$ and $u_i = \mathbf{b}$ if $i \in S$.

Through the course of this paper we use the descent set of a chain and its corresponding **ab**-monomial interchangeably where we take **a** to mean an ascent and **b** to indicate a descent in the chain label. $D(S)$ indicates the set of all permutations with descent set S and $m(S)$ refers to the **ab**-monomial with descent set S.

Next we consider the lattices which minimize and maximize the flag h-vector.

Proposition 3.2. *Let L be a rank $r+1$ geometric lattice with n atoms. Then for all $S \subseteq [r]$ we have $h_S(L) \leq h_S(\mathscr{B}_{r+1,n})$.*

Proof. Place a linear ordering on the atoms of L and $\mathscr{B}_{r+1,n}$. We construct an injection from the minimal labelings of L to the minimal labelings of $\mathscr{B}_{r+1,n}$ which preserves descent sets. This will prove the proposition since h_S is the number of minimal labelings with descent set S.

Let $\lambda(F) = (\lambda_1, \ldots, \lambda_{r+1})$ be a minimal label of a facet F in $\Delta(L)$. The uniqueness of rising chains in any interval implies that $\lambda(F)$ is completely determined by the initial segment $(\lambda_1, \ldots, \lambda_i)$, where λ_i is the label preceding the last descent of $\lambda(F)$. For instance, if $\lambda(F) = (3, 6, 9, 2, 7, 12)$, then $(3, 6, 9)$, plus the knowledge that 9 precedes the last descent of the label, determines $\lambda(F)$. Since every subset of cardinality i is a flat of $\mathscr{B}_{r+1,n}$ and $\lambda_{i+1} < \lambda_i$, there is a unique facet F' of $\Delta(\mathscr{B}_{r+1,n})$ such that $\lambda(F') = (\lambda_1, \ldots, \lambda_i, \ldots)$ and whose final descent is immediately after λ_i. The function which takes $\lambda(F)$ to $\lambda(F')$ is the required map. $\square$

Since $f_S(P) = \sum_{T \subseteq S} h_T(P)$ we immediately have the following result.

Corollary 3.3. *Among rank $r+1$ geometric lattices with n atoms, $\mathscr{B}_{r+1,n}$ maximizes $f_S(P)$ for all $S \subseteq [r]$.*

We look again to the near pencil when considering the lattice which minimize the flag h-vector.

Lemma 3.4. *Let $S \subseteq [r]$. Fix $i < r+1$. The number of orderings of $1, \ldots, r+1$ such that $r+1$ comes before i and has descent set S is independent of i.*

Proof. Suppose $r+1$ is the kth element of the permutation. There are $\binom{r-1}{r-k}$ ways to choose the elements (including i) which appear after $r+1$. Since $r+1$ effectively splits the permutation into two smaller permutations we can consider the elements to the right of $r+1$ as $1', 2', \ldots, (r+1-k)'$. Let D_r be the number of ways to arrange $1', \ldots, (r+1-k)'$ consistent with descent set S and let D_l be the number of ways to arrange the elements to the left of $r+1$ consistent with descent set S. Then

$$\sum_{k=1}^{r+1} \binom{r-1}{r-k} D_r \times D_l$$

is the number of permutations of $[r+1]$ such that $r+1$ appears before i. This number is independent of i. $\square$

Theorem 3.5. *Let L be a rank $r+1$ geometric lattice with n atoms and let P be the rank $r+1$ near pencil on n atoms. Then*

$$h_S(L) \geq h_S(P).$$

Proof. Let $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_n\}$ be the atoms of P and L, respectively, and let $\Delta(P)$ and $\Delta(L)$ be the corresponding order complexes. Order the atoms of P so that $e_1, \ldots, e_{r-1}$ are the coloops of P. What are the nbc-bases of P? Any basis must contain the coloops and e_r, the least atom in the nontrivial line. On the other hand, any basis of the form $\{e_1, \ldots, e_r, e_i\}$, $i \geq r+1$, is an nbc-basis of P. Recall that $h_S(P)$ is the number of orderings of the nbc-bases of P for which the ordering is a minimal labeling of the corresponding maximal chain of flats in $\Delta(P)$ and the descent set is S. For $i = r+1$

this is all orderings. For $i > r + 1$ an ordering is a minimal labeling if and only if e_i comes before e_r.

Now order the atoms of L so that $f_1, \dots, f_{r+1}$ is a basis of L. Clearly $B = \{f_1, \dots, f_{r+1}\}$ is an nbc-basis of L, any ordering of B is a minimal labeling of the corresponding maximal chain, and the contribution of these orderings to $h_S(L)$ is the same as the contribution of all the orderings of $e_1, \dots, e_{r+1}$ to $h_S(P)$.

For each $i > r + 1$ we form a basis B_i of L as follows. Let C_i be the fundamental circuit of f_i with respect to B. Let $B_i = B \cup \{f_i\} - \{f_j\}$, where f_j is the second highest element of C_i. For instance, if $B = \{f_1, f_2, f_3, f_4, f_5\}$ and the fundamental circuit of f_8 is $C_8 = \{f_1, f_3, f_4, f_8\}$, then $B_8 = \{f_1, f_2, f_3, f_5, f_8\}$. Now one can check that each B_i is an nbc-basis of L and that any ordering of B_i where f_i comes before the least element of C_i is a minimal labeling of the corresponding maximal chain. In combination with Lemma 3.4 this shows that there are at least as many minimal labelings with descent set S for $\Delta(L)$ as there are for $\Delta(P)$. $\square$

When the number of atoms of a rank $r + 1$ geometric lattice is not specified the standard Boolean algebra $\mathscr{B}_{r+1}$ minimizes the flag h-vector [BER, Proposition 7.4].

Proposition 3.6 [BER]. *Let L be a geometric lattice of rank $r+1$. Then for all $S \subseteq [r]$ we have $h_S(L) \geq h_S(\mathscr{B}_{r+1})$. Hence the* **ab**-*index $\Psi(L)$ is coefficientwise greater than or equal to the* **ab**-*index of the Boolean algebra $\mathscr{B}_{r+1}$.*

Oriented matroids are signed versions of standard matroids. We refer the reader to [BLS$^+$] for more details. The elements of the oriented matroid, when partially ordered, form an Eulerian poset (see [St2]) which is known as the lattice of regions. A poset is *Eulerian* if every interval $[x, y]$, where $x \neq y$, has the same number of elements of odd rank as even rank. Of interest to our work is the fact that underlying each oriented matroid is a standard matroid along with its associated geometric lattice of flats.

A collection of hyperplanes $\mathscr{H} = \{H_e\}_{e \in E}$ is *essential* if $\bigcap_{e \in E} H_e = \{0\}$. A special class of oriented matroids, called realizable matroids, have an associated essential hyperplane arrangement. The lattice of regions of the realizable oriented matroid is isomorphic to the face lattice of the corresponding hyperplane arrangement. Every essential hyperplane arrangement has an associated *zonotope*, which is the polytope formed by taking the Minkowski sum of the normals to the hyperplanes (see [BLS$^+$]).

It was noted by Fine and proved by Bayer and Klapper [BK] that when P is an Eulerian poset the **ab**-index of P, $\Psi(P)$, can be written in terms of $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a}$. When $\Psi(P)$ is expressed in terms of **c** and **d** it is referred to as the **cd**-index. The face lattice of an r-dimensional convex polytope is an Eulerian poset and the **cd**-index of the polytope is defined to be the **cd**-index of its corresponding face lattice.

When P is the lattice of regions of an oriented matroid every occurrence of **d** in the **cd**-index appears as **2d** and so it is referred to as the **c**-**2d**-index [BER]. The lattice of flats of the underlying matroid contains all of the information necessary to determine the **c**-**2d**-index of the lattice of regions of an oriented matroid [BS, Theorem 3.4]. This connection is made explicit in [BER]. The following proposition indicates how to construct the **c**-**2d**-index of a zonotope given the geometric lattice underlying the associated hyperplane arrangement [BER, Corollary 3.2].

Proposition 3.7 [BER]. *Let L and Z be the underlying geometric lattice and zonotope associated to an essential hyperplane arrangement, respectively. Then the* $\mathbf{c}$-$\mathbf{2d}$*-index of the zonotope Z is given by*

$$\Psi(Z) = \omega(\mathbf{a} \cdot \Psi(L)),$$

where ω is a linear function which takes **ab** *words to* **cd** *words by replacing each occurrence of* **ab** *with* **2d**, *then replacing the remaining letters with* **c***'s.*

The hyperplane arrangement $\{x \in \mathbb{R}^{r+1} : x_i = 0\}$ for $i = \{1, \ldots, r+1\}$ has the Boolean algebra $\mathscr{B}_{r+1}$ as its underlying lattice of flats. The zonotope corresponding to this arrangement is the $(r+1)$-dimensional cube. Propositions 3.6 and 3.7 together imply the following [BER, Corollary 7.6]:

Corollary 3.8 [BER]. *Among all zonotopes of dimension $r+1$, the $(r+1)$-dimensional cube has the smallest* $\mathbf{c}$-$\mathbf{2d}$*-index.*

Consider the rank $r+1$ near pencil on n atoms and the truncated Boolean algebra, $\mathscr{B}_{r+1,n}$. As seen in Section 2 the near pencil is associated with the arrangement of $(n-r+1)$ points on a line with the remaining $r-1$ points in general position. Similarly, $\mathscr{B}_{r+1,n}$ can be associated to the arrangement of n points in general position in $\mathbb{R}^{r+1}$. In both of these point arrangements we can consider the set of rays from the origin to each point. Taking the hyperplanes normal to these rays gives an essential hyperplane arrangement whose underlying geometric lattice is the near pencil and $\mathscr{B}_{r+1,n}$, respectively. Combining Proposition 3.7 with Theorem 3.5 and Proposition 3.2 gives the following analogous result for $(r+1)$-dimensional zonotopes with n zones.

Corollary 3.9. *Let $\mathscr{H}_P$ and $\mathscr{H}_{\mathscr{B}_{r+1,n}}$ denote essential hyperplane arrangements whose underlying geometric lattice is the rank $r+1$ near pencil and truncated Boolean algebra on n atoms, respectively. Among all zonotopes of dimension $r+1$ with n zones, the zonotope corresponding to $\mathscr{H}_{\mathscr{B}_{r+1,n}}$ has the largest* $\mathbf{c}$-$\mathbf{2d}$*-index, and the zonotope corresponding to $\mathscr{H}_P$ has the smallest* $\mathbf{c}$-$\mathbf{2d}$*-index.*

4. Convex Ear Decompositions

Convex ear decompositions were introduced by Chari [Ch]. Our convex ear decomposition is motivated by a basis for $H_\star(\Delta(L), \mathbb{Z})$ constructed by Björner [Bj2]. A *convex ear decomposition* of a pure $(r-1)$-dimensional simplicial complex Δ is an ordered sequence $\Delta_1, \Delta_2, \ldots, \Delta_m$ of pure $(r-1)$-dimensional subcomplexes of Δ such that:

1. Δ_1 is the boundary complex of a simplicial r-polytope, while for each $j = 2, \ldots, m$, Δ_j is an $(r-1)$-ball which is a proper subcomplex of the boundary of a simplicial r-polytope.
2. For $j \geq 2$, $\Delta_j \cap (\bigcup_{k=1}^{j-1} \Delta_k) = \partial \Delta_j$.
3. $\bigcup_{k=1}^{m} \Delta_k = \Delta$.

Theorem 4.1 [Ch]. *If Δ has a convex ear decomposition, then for $i \le r/2$ the h-vector of Δ satisfies*

$$\begin{aligned} h_{i-1} &\le h_i, \\ h_i &\le h_{r-i}. \end{aligned}$$

For the rest of this section $(b_1, \dots, b_{r+1})$ is an ordered basis of atoms in L with corresponding chain of flats $\hat{0} < x_1 < \cdots < x_r < \hat{1}$, $x_i = b_1 \vee \cdots \vee b_i$.

Lemma 4.2. *Let $(b_1, \dots, b_{r+1})$ be a minimal labeling of a facet of $\Delta(L)$. Then $B = \{b_1, \dots, b_{r+1}\}$ is an nbc-basis of L.*

Proof. Suppose B contains a broken circuit. If x_i is the lowest ranked flat which contains a broken circuit, then $b_i \ne \lambda(x_{i-1}, x_i)$. □

Lemma 4.3 (Switching Lemma). *Let $(b_1, \dots, b_i, b_{i+1}, \dots, b_{r+1})$ be a minimal labeling of a facet of $\Delta(L)$. If $b_i < b_{i+1}$, then $(b_1, \dots, b_{i+1}, b_i, \dots, b_{r+1})$ is also a minimal labeling of a facet of $\Delta(L)$.*

Proof. For two flats $y < x$ in L let $\{x - y\} = \{\text{atoms } e : y \vee e = x\}$. Suppose $(b_1, \dots, b_{i+1}, b_i, \dots, b_{r+1})$ is not a minimal labeling. Then there exists an atom e such that either $e \in \{(x_{i-1} \vee b_{i+1}) - x_{i-1}\}$ and $e < b_{i+1}$, or $e \in \{x_{i+1} - (x_{i-1} \vee b_{i+1})\}$ and $e < b_i$. In the first case, $e \in \{x_{i+1} - x_i\}$. However, this implies that b_{i+1} is not the least atom in $x_{i+1} - x_i$. In the second case, either $e \in x_i - x_{i-1}$, which implies that b_i is not the minimal atom in $x_i - x_{i-1}$, or $e \in x_{i+1} - x_i$. However, this last is impossible since $e < b_i < b_{i+1}$ and b_{i+1} is the least atom in $x_{i+1} - x_i$. □

Let B be a basis of L. Associated to any ordering $(b_1, \dots, b_{r+1})$ of B is the facet $F = b_1 < b_1 \vee b_2 < \cdots < b_1 \vee \cdots \vee b_r$ of $\Delta(L)$. The *basis labeling* of F (with respect to B) is $(b_1, \dots, b_{r+1})$. This may or may not be the same as $\lambda(F)$.

Let $B_1, \dots, B_m$ be the nbc-bases of L in lexicographic order. For each j, $1 \le j \le m$, let Σ_j be the union of all the facets of $\Delta(L)$ associated to all possible orderings of B_j. Each Σ_j is isomorphic to the order complex of the rank $r + 1$ Boolean algebra and as a simplicial complex is the boundary of the first barycentric subdivision of the r-simplex. Now define Δ_j to be the pure subcomplex of Σ_j whose facets are the facets of Σ_j whose minimal labeling and basis labeling coincide. Except for $\Sigma_1 = \Delta_1$, each Δ_j is a proper subcomplex of Σ_j [Bj4, Lemma 7.6.2].

Proposition 4.4. *If $2 \le j \le m$, then Δ_j is a closed $(r - 1)$-ball.*

Proof. It is sufficient to show that Δ_j is nonempty and shellable. To see that Δ_j is nonempty, we note that for any basis B of L the minimal ordering and the basis ordering are the same for the maximal chain corresponding to ordering B in reverse if and only if B is an nbc-basis.

Order the facets of Δ_j in *reverse* lexicographic order with respect to the basis labeling of the corresponding maximal chains. We show that this ordering satisfies property M. Suppose F' and F are facets of Δ_j and $\lambda(F') < \lambda(F)$. Let $c = (\hat{0} = x_0 < x_1 < \cdots < x_k < x_{k+1} = \hat{1})$ be the chain which represents their intersection. We must find F'', a facet of Δ_j, lexicographically after F such that $c \subseteq (F \cap F'')$ and $|F \cap F''| = r - 1$. Let m be the least index such that the length of $[x_m, x_{m+1}]$ is greater than 1. Since $\lambda(F)$ is lexicographically before $\lambda(F')$, there are basis atoms $b < \hat{b}$ in B_j such that the chain corresponding to F contains as a short saturated chain $y < (y \vee b) < (y \vee b \vee \hat{b})$ with $x_m \leq y$ and $(y \vee b \vee \hat{b}) \leq x_{m+1}$. Let F'' be the facet corresponding to interchanging b and $\hat{b}$ in $\lambda(F)$. Then $|F \cap F''| = r - 1$ and $\lambda(F'') < \lambda(F)$. By the switching lemma, F'' is a facet of Δ_j. □

Proposition 4.5. *If $j \geq 2$, then $\Delta_j \cap (\bigcup_{k=1}^{j-1} \Delta_k) = \partial \Delta_j$.*

Proof. Let G be a face in $\Delta_j \cap (\bigcup_{k=1}^{j-1} \Delta_k)$. By definition G is not a facet. The boundary of Δ_j is equal to the boundary of $\overline{\Sigma_j - \Delta_j}$ (topological closure). So it is sufficient to show that G is contained in a facet of $\overline{\Sigma_j - \Delta_j}$.

Write $G = x_1 < \cdots < x_k$. By assumption $G \subset F$, F is a facet of Δ_j, and $G \subset F'$, F' is a facet whose corresponding minimal labeling basis B' is lexicographically before B_j. Therefore, there is some pair x_m, x_{m+1} such that $B' \cap \{x_{m+1} - x_m\}$ is lexicographically before $B_j \cap \{x_{m+1} - x_m\}$. Hence, the unique increasing minimally labeled saturated chain of $[x_m, x_{m+1}]$ is not contained in B_j. Now let $\hat{F}$ be a facet of Σ_j obtained as follows. First saturate the interval $[x_m, x_{m+1}]$ by adding in the atoms of $B_j \cap \{x_{m+1} - x_m\}$ in increasing order. Then extend this to a saturation of the chain corresponding to G in any way which results in a facet of Σ_j. Such an F'' contains G and must be in $\overline{\Sigma_j - \Delta_j}$ since its minimal label and its basis label are not equal. □

The two previous propositions show that $\Delta(L)$ has a convex ear decomposition. An immediate consequence is Theorem 1.1. Since $h_0 \leq h_1 \leq \cdots \leq h_{\lceil r/2 \rceil}$, a natural question is whether or not the g-vector of $\Delta(L)$ is an M-vector. The g-vector is $(g_0, g_1, \ldots, g_{\lceil r/2 \rceil})$, where $g_i = h_i - h_{i-1}$. A sequence of nonnegative integers is an M-vector if it is the Hilbert function of a quotient of a polynomial ring. See, for instance, Theorem 2.2 on p. 56 of [St1] for an equivalent numerical definition of M-vector.

Problem 4.6. Is the g-vector of the order complex of a geometric lattice an M-vector?

Note: After this paper was written the second author discovered a proof that the g-vector of any space with a convex ear decomposition is an M-vector.

5. The Weak Bruhat Order

Let $(b_1, \ldots, b_{r+1})$ be an nbc-basis of L ordered so that $b_1 < \cdots < b_{r+1}$. Then we can identify all the orderings of the basis with S_{r+1}, the symmetric group on $r + 1$ letters. For $\pi \in S_{r+1}$ we write $\pi = a_1 a_2 \cdots a_{r+1}$, where $a_i = \pi(i)$. The switching lemma tells us that if π corresponds to a minimal labeling and π' is obtained from π by interchanging a_i with a_{i+1} when $a_i < a_{i+1}$, then π' also corresponds to a minimal labeling.

Definition 5.1. Let $\pi, \pi' \in S_{r+1}$. Then $\pi \leq_{\mathrm{w}} \pi'$ if and only if π' can be obtained from π by repeated application of the above switching procedure.

Evidently $\leq_{\mathrm{w}}$ is a partial order. It is, in fact, the weak Bruhat order on S_{r+1}. An equivalent definition of $\leq_{\mathrm{w}}$ is the following. Define the *inversion* set of π to be $I(\pi) = \{(a_i, a_j) : a_i > a_j \text{ and } i < j\}$. Then $\pi \leq_{\mathrm{w}} \pi'$ if and only if $I(\pi) \subseteq I(\pi')$. See [Bj3] for more information on the weak order in general Coxeter groups.

Let $T, S \subseteq [r]$. We say *S dominates T* if there exists an injection $\varphi : D(T) \to D(S)$ such that $\pi \leq_{\mathrm{w}} \varphi(\pi)$ for all permutations $\pi \in D(T)$.

Proposition 5.2. *If S dominates T, then $h_T \leq h_S$ for all geometric lattices of rank $r + 1$ or greater.*

Proof. The switching lemma and Proposition 3.1 imply that there are at least as many facets which contribute to h_S as h_T at each step in the convex ear decomposition. $\square$

How do we find pairs T, S such that S dominates T? First some elementary facts.

Proposition 5.3. *Suppose S dominates T. Let u, v be* **ab**-*monomials, possibly equal to $\emptyset$. Then*

(1) $m(S) \cdot \mathbf{a} \cdot v$ *dominates* $m(T) \cdot \mathbf{a} \cdot v$,
(2) $u \cdot \mathbf{a} \cdot m(S)$ *dominates* $u \cdot \mathbf{a} \cdot m(T)$,
(3) $u \cdot \mathbf{a} \cdot m(S) \cdot \mathbf{a} \cdot v$ *dominates* $u \cdot \mathbf{a} \cdot m(T) \cdot \mathbf{a} \cdot v$.

Proof. We prove (3) since the other proofs are virtually identical. By definition, if $\pi \leq_{\mathrm{w}} \pi' \in S_{r+1}$, then $\pi(1) \leq \pi'(1)$ and $\pi(r) \geq \pi'(r)$. Let $\varphi : D(T) \to D(S)$ be an injection which preserves the weak Bruhat order. Let

$$\pi = s_1 \cdots s_m a_1 a_2 \cdots a_{r+2} a_{r+3} t_1 \cdots t_k$$

be a permutation such that the **ab**-monomial of the $s_1 \cdots s_m a_1$ is u, the **ab**-monomial of $a_1 \cdots a_{r+3}$ is $\mathbf{a} \cdot m(T) \cdot \mathbf{a}$, and the **ab**-monomial of $a_{r+3} \cdot t_1 \cdots t_k$ is v. Identify the ordered set $[r + 1]$ with the ordered set $[r + 3] - \{a_1, a_{r+3}\}$ in the canonical way. Define $\psi(\pi)$ to be the permutation obtained by applying φ to $a_2 \cdots a_{r+2}$ using this identification. Clearly, ψ is an injection and $\pi \leq_{\mathrm{w}} \psi(\pi)$. Since $a_1 < a_2$ and $a_{r+2} < a_{r+3}$, the descent monomial of $\psi(\pi)$ is $u \cdot \mathbf{a} \cdot m(S) \cdot \mathbf{a} \cdot v$. $\square$

Proposition 5.4. *If S dominates T, then $T \subseteq S$.*

Proof. Suppose $i \in T$. Let π be a permutation with descent set T such that $\{\pi(1), \ldots, \pi(i)\} = \{r - i + 2, \ldots, r + 1\}$. If $\pi \leq_{\mathrm{w}} \pi'$, then $\{\pi'(1), \ldots, \pi'(i)\}$ must also equal $\{r - i + 2, \ldots, r + 1\}$. Hence π' also has a descent at i. $\square$

Another place to look for $T \subseteq S$ with S dominating T is through the symmetries of the flag h-vector of the Boolean algebra. Since $h_T \leq h_S$ for all geometric lattices of rank at least $r+1$, we can begin our search by examining $\mathscr{B}_{r+1}$. This lattice has $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ symmetry. Let $\beta \in S_{r+1}$ be the permutation which reverses order, i.e., $\beta(i) = r - i + 2$. We omit the elementary proof of the following:

Proposition 5.5. *Let T be the descent set of $\pi \in S_{r+1}$.*

(1) *The descent set of $\beta \circ \pi$ is $[r] - T$.*
(2) *The descent set of $\pi \circ \beta$ is $T \circ \beta$, where $T \circ \beta = \{i \in [r] : r - i + 1 \notin T\}$.*
(3) *The descent set of $\beta \circ \pi \circ \beta$ is $r - [T]$, where $r - [T] = \{i : r - i + 1 \in T\}$.*

Combined with Proposition 3.1, the above proposition shows that in $\mathscr{B}_{r+1}$, $h_T = h_{r-T} = h_{[r]-T} = h_{T\circ\beta}$. Proposition 5.4 rules out the possibility of T being dominated by $r - T$ or $[r]-T$ except when they are equal. However, if for each i at most one of i and $r-i+1$ is in T, then $T \subseteq T \circ \beta$.

Example 5.6. Let $T = \{1\} \subseteq [3]$. Then $T \subset T \circ \beta = \{1, 2\}$. As the map which sends $\{2134\} \to \{3214\}$, $\{3124\} \to \{4312\}$, and $\{4123\} \to \{4213\}$ shows, T is dominated by $T \circ \beta$ and hence $h_T \leq h_{T\circ\beta}$ for all geometric lattices of rank 4 (or more).

Conjecture 5.7. *If $T \subseteq T \circ \beta$, then $T \circ \beta$ dominates T.*

We have verified this conjecture by computer for $r \leq 8$. For convenience, Table 1 lists all the cases that we have used in Section 6. These computations were helped by the observation that if $T \subseteq T \circ \beta$ and $T \circ \beta$ dominates T, then $r - (T \circ \beta)$ dominates $r - T$. To see this, use the fact that if $\pi \leq_{\mathrm{w}} \pi'$, then $\beta \circ \pi \circ \beta \leq_{\mathrm{w}} \beta \circ \pi' \circ \beta$. By combining Proposition 5.3 with cases where Conjecture 5.7 is known to hold many pairs T, S with S dominating T can be constructed. For instance, **abbab** dominates **abaab**, and **bbbaab** dominates **abbaaa** are known cases of Conjecture 5.7. Hence, **abbbaabaaabbab** dominates **aabbaaaaaabaab**.

Table 1. Several known examples of $T \circ \beta$ dominating T.

$T \to T \circ \beta$		$T \to T \circ \beta$
	$r = 3$	
$\{1\} \to \{1, 2\}$		$\{3\} \to \{2, 3\}$
	$r = 5$	
$\{1, 2\} \to \{1, 2, 3\}$		$\{4, 5\} \to \{3, 4, 5\}$
$\{1, 4\} \to \{1, 3, 4\}$		$\{2, 5\} \to \{2, 3, 5\}$
	$r = 7$	
$\{1, 2, 3\} \to \{1, 2, 3, 4\}$		$\{5, 6, 7\} \to \{4, 5, 6, 7\}$
$\{1, 2, 5\} \to \{1, 2, 4, 5\}$		$\{3, 6, 7\} \to \{3, 4, 6, 7\}$
$\{1, 3, 6\} \to \{1, 3, 4, 6\}$		$\{2, 5, 7\} \to \{2, 4, 5, 7\}$
$\{1, 5, 6\} \to \{1, 4, 5, 6\}$		$\{2, 3, 7\} \to \{2, 3, 4, 7\}$

If true, Conjecture 5.7 would be surprising. We are required to find φ: $D(T) \to D(T \circ \beta)$ such that $I(\pi) \subseteq I(\varphi(\pi))$. Yet, $I(\pi) \cap I(\pi \circ \beta) = \emptyset$ for all π.

6. From Flag *h*-Vectors to *h*-Vectors

Our goal is to give a combinatorial flag h-vector proof of (1). One way to do this would be to construct a matching from $(i-1)$-subsets of $[r]$ to i-subsets of r such that each $(i-1)$-subset is matched to an i-subset which dominates it. For example, here is such a matching from 2-subsets of [6] to 3-subsets of [6]:

$$\begin{array}{lll}
\{1,2\} \to \{1,2,3\} & \{2,3\} \to \{2,3,4\} & \{3,5\} \to \{1,3,5\} \\
\{1,3\} \to \{1,3,6\} & \{2,4\} \to \{2,4,6\} & \{3,6\} \to \{2,3,6\} \\
\{1,4\} \to \{1,3,4\} & \{2,5\} \to \{2,3,5\} & \{4,5\} \to \{2,4,5\} \\
\{1,5\} \to \{1,4,5\} & \{2,6\} \to \{2,5,6\} & \{4,6\} \to \{1,4,6\} \\
\{1,6\} \to \{1,5,6\} & \{3,4\} \to \{3,4,6\} & \{5,6\} \to \{4,5,6\}
\end{array}$$

This matching gives a combinatorial flag h-vector proof that $h_2 \le h_3$ for all geometric lattices whose rank is greater than or equal to 7. Table 2 gives a matching for [3]-sets of [8] to 4-sets of [8]. As we will see below, this is not possible for all r and $i \le r/2$, but it can be done for somewhat smaller i.

Lemma 6.1. *Let T be an i-subset of $[r]$. Then there exists at least $\lfloor r - \frac{5}{2}i \rfloor$ supersets of T of cardinality $i+1$ which dominate T.*

Proof. The proof is by induction on r, the case of $r = 1$ being trivial. Consider the tree in Fig. 2. Each **ab**-monomial stands for the initial descent pattern of a permutation. A

Table 2. An injection φ from 3-sets of [8] to 4-sets of [8] such that $\varphi(T)$ dominates T.

T	$\varphi(T)$	T	$\varphi(T)$	T	$\varphi(T)$
{1,2,3}	{1,2,3,4}	{1,2,5}	{1,2,4,5}	{1,2,6}	{1,2,4,6}
{1,2,8}	{1,2,3,8}	{1,3,4}	{1,3,4,8}	{1,3,6}	{1,3,6,8}
{1,3,7}	{1,3,6,7}	{1,3,8}	{1,3,5,8}	{1,4,5}	{1,4,5,8}
{1,4,6}	{1,4,6,8}	{1,4,7}	{1,4,5,7}	{1,4,8}	{1,4,7,8}
{1,5,6}	{1,2,5,6}	{1,5,7}	{1,2,5,7}	{1,5,8}	{1,2,5,8}
{1,6,7}	{1,2,6,7}	{1,6,8}	{1,2,6,8}	{1,7,8}	{1,6,7,8}
{2,3,4}	{2,3,4,5}	{2,3,5}	{2,3,5,6}	{2,3,6}	{2,3,6,8}
{2,3,7}	{2,3,4,7}	{2,3,8}	{2,3,4,8}	{2,4,5}	{2,4,5,6}
{2,4,6}	{2,4,6,7}	{2,4,7}	{2,4,5,7}	{2,4,8}	{2,4,7,8}
{2,5,6}	{2,5,6,8}	{2,5,7}	{2,3,5,7}	{2,5,8}	{2,4,5,8}
{2,6,7}	{2,3,6,7}	{2,6,8}	{2,4,6,8}	{2,7,8}	{2,6,7,8}
{3,4,5}	{1,3,4,5}	{3,4,6}	{1,3,4,6}	{3,4,7}	{1,3,4,7}
{3,4,8}	{3,4,7,8}	{3,5,6}	{1,3,5,6}	{3,5,7}	{1,3,5,7}
{3,5,8}	{2,3,5,8}	{3,6,7}	{3,4,6,7}	{3,6,8}	{3,5,6,8}
{3,7,8}	{2,3,7,8}	{4,5,6}	{1,4,5,6}	{4,5,7}	{3,4,5,7}
{4,5,8}	{3,4,5,8}	{4,6,7}	{1,4,6,7}	{4,6,8}	{3,4,6,8}
{4,7,8}	{4,5,7,8}	{5,6,7}	{4,5,6,7}	{5,6,8}	{4,5,6,8}
{5,7,8}	{1,5,7,8}	{6,7,8}	{5,6,7,8}		

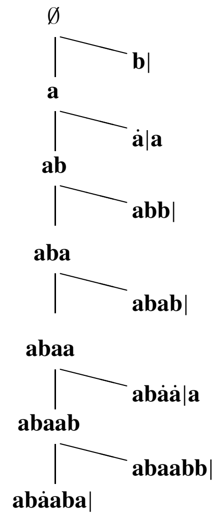

Fig. 2. Initial descent patterns.

dot above an ascent at position k means that if T has the given initial descent pattern, then T is dominated by $T \cup \{k\}$. The bottom of each branch shows how the induction hypothesis should be applied. If the monomial ends with an ascent, then the induction hypothesis is applied to everything to the right of the bar. For instance, the entry **abȧȧ|a** says to apply the induction hypothesis to the $r-4$ positions to the right of **abȧȧ**. Indeed, this ensures $\lfloor r-4-\frac{5}{2}(i-2)\rfloor + 2 \geq \lfloor r-\frac{5}{2}i\rfloor$ supersets which dominate T. When the monomial ends with ...**b**| the induction hypothesis is applied to the positions to the right of the next ascent. For instance, the monomial **ababbbbaabba** is covered by the branch endpoint **abab**|.

Let $u = m(T)$. Starting from the top of the tree we look for either an interior node which matches u exactly or a branch endpoint which matches an initial segment of u. All of the interior nodes satisfy the theorem and all of the branch endpoints demonstrate how to use the induction hypothesis to prove the theorem for u. □

Theorem 6.2. *If $i \leq \frac{2}{7}(r+\frac{5}{2})$, then there exists a matching φ from $(i-1)$-subsets of $[r]$ to i-subsets of $[r]$ such that $\varphi(T)$ dominates T for each $|T| = i-1$.*

Proof. The condition on i ensures that $r-\frac{5}{2}(i-1) \geq i$. Hence, by the above lemma, each $(i-1)$-subset of $[r]$ has at least i supersets which dominate it. Obviously any i-subset of $[r]$ has at most i subsets of cardinality $i-1$ which it dominates. The theorem is now an elementary application of Hall's marriage theorem. □

Theorem 6.2 is not optimal. We have already seen that there are suitable matchings for 2-sets to 3-sets in [6] and 3-sets to 4-sets in [8]. However, it is not always possible to obtain suitable matchings for all $i \leq r/2$.

Example 6.3. Let $T = \{2, 5, 6, 9\} \subseteq [10]$. There are no 5-supersets of T in [10] which dominate T. This can be seen by directly computing h_T and h_S in $\mathscr{B}_{11}$ where S runs over all potential supersets. In each case $h_T > h_S$.

Problem 6.4. Asymptotically, Theorem 6.2 covers a little over 57% of the inequalities in (1). How much can this be improved?

Instead of insisting on a one-to-one matching we can consider grouping subsets together. Of course, this must be done in moderation. Indeed, $h_{i-1} \leq h_i$ is just a reflection of grouping all subsets of the same cardinality together.

Example 6.5. Let $T = \{3\}$, $S = \{2\}$, $U = \{2, 3\}$, and $V = \{1, 3\}$. As the following table shows, there is a bijection which respects the weak Bruhat order from the permutations in S_4 whose descent set is T or S to those whose descent set is U or V. Hence $h_T + h_S \leq h_U + h_V$ in rank 4 (or more) geometric lattices.

$$\begin{array}{ccc}
1243 & \to & 2143 \\
1342 & \to & 1432 \\
2341 & \to & 2431 \\
1324 & \to & 3142 \\
1423 & \to & 4132 \\
2314 & \to & 3241 \\
2413 & \to & 4231 \\
3412 & \to & 3421
\end{array}$$

Combined with the previously shown $h_{\{1\}} \leq h_{\{1,2\}}$, the above example provides a combinatorial proof of $h_1 \leq h_2$ for geometric lattices of rank 4 or greater.

As noted earlier, the **cd**-index of an oriented matroid is a nonnegative linear combination of the **ab**-index of the associated geometric lattice.

Problem 6.6. Are there groupings of subsets such that the corresponding flag h-vector inequality translates to a **cd**-inequality for oriented matroids?

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References

[Ai] M. Aigner. Whitney numbers. In N. L. White, editor, *Combinatorial Geometries*, pages 139–160. Cambridge University Press, Cambridge, 1987.

[BER] L. J. Billera, R. Ehrenborg, and M. Readdy. The **c-2d**-index of oriented matroids. *J. Combin. Theory Ser. A*, 80:79–105, 1997.

[Bi] G. Birkhoff. Abstract linear dependence in lattices. *Amer. J. Math.*, 57:800–804, 1935.

[Bj1] A. Björner. Shellable and Cohen–Macaulay partially ordered sets. *Trans. Amer. Math. Soc.*, 260:159–183, 1980.

[Bj2] A. Björner. On the homology of the geometric lattice. *Algebra Universalis*, 14:107–128, 1982.

[Bj3] A. Björner. Orderings of Coxeter groups. In C. Greene, editor, *Combinatorics and Algebra*, pages 175–195. Contemporary Mathematics Series, Vol. 34. American Mathematical Society, Providence, RI, 1985.

[Bj4] A. Björner. The homology and shellability of matroids and geometric lattices. In N. L. White, editor, *Matroid Applications*, pages 226–283. Cambridge University Press, Cambridge, 1992.

[BK] M. Bayer, and A. Klapper. A new index for polytopes. *Discrete Comput. Geom.*, 6:33–47, 1991.

[BLS$^+$] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. Ziegler. *Oriented Matroids*. Cambridge University Press, Cambridge, 1993.

[BS] M. Bayer, and B. Sturmfels. Lawrence polytopes. *Canad. J. Math*, 42: 62–79, 1990.

[Ch] M.K. Chari. Two decompositions in topological combinatorics with applications to matroid complexes. *Trans. Amer. Math. Soc.*, 349:3925–3943, 1997.

[DW1] T.A. Dowling and R.M. Wilson. The slimmest geometric lattices. *Trans. Amer. Math. Soc.*, 196:203–215, 1974.

[DW2] T.A. Dowling and R.M. Wilson. Whitney number inequalities for geometric lattices. *Proc. Amer. Math. Soc.*, 47:504–512, 1975.

[Fo] J. Folkman. The homology groups of a lattice. *J. Math. Mech.*, 15:631–636, 1966.

[Ny] K.L. Nyman. Enumeration in geometric lattices and the symmetric group. Ph.D. thesis, Cornell University, Ithaca, NY, 2001.

[Ox] J. Oxley *Matroid Theory*. Oxford University Press, Oxford, 1992.

[St1] R.P. Stanley. *Combinatorics and Commutative Algebra*. Progress in Mathematics, Vol. 41. Birkhäuser, Boston, MA, 1996.

[St2] R.P. Stanley. *Enumerative Combinatorics*, Vol. 1. Cambridge University Press, Cambridge, 1997.

[Za] T. Zaslavsky. The Möbius function and the characteristic polynomial. In N. L. White, editor, *Combinatorial Geometries*, pages 114–138. Cambridge University Press, Cambridge, 1987.

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