Discrete Comput Geom 33:403–421 (2005)
DOI: 10.1007/s00454-004-1104-8

Discrete & Computational Geometry

The Mixed Volume of Two Finite Vector Sets*

Li Xiaoyan and Leng Gangsong

Department of Mathematics, Hunan Normal University,
Changsha 410081, People's Republic of China
and
Department of Mathematics, Shanghai University,
Shanghai 200436, People's Republic of China
lixy77@sohu.com, lenggangsong@163.com

Abstract. We introduce the concept of the mixed volume of two finite vector sets in R^n. By employing the exterior differential, we prove a new and powerful inequality and establish a series of quantity relations associated with the mixed volume of two finite vector sets. As applications, we discuss some well-known results of simplices and the Hadamard inequality.

1. Introduction

The concept of mixed volumes, which forms a central part of the Bruun–Minkowski theory of convex bodies, arises naturally if one combines the two fundamental concepts of Minkowski addition and volume (see [23]). The theory of mixed volumes provides a unified treatment of various important metric quantities in geometry, such as volume, surface area, and mean width. The mixed volumes and the relatively new dual mixed volumes introduced by Lutwak (see [18]) have provided many powerful tools for solving problems involving projections and sections of convex bodies. They have also played an important role in isoperimetric problems in geometric analysis (see [6], [7], [18], [19], and [27]).

In this paper we give the definition of the mixed volume of two finite vector sets, which is the discrete form of a mixed volume. As its special case, we introduce the concepts of the mixed Cayley–Menger determinant and the mixed vertex angles of simplices, which are extensions of the well-known Cayley–Menger determinant and Bartoš vertex angles, respectively. In this paper we establish a series of inequalities and equalities

* This research was supported by National Natural Sciences Foundation of China (10271071).

for the previously stated three new concepts, most of them are generalizations of some well-known results.

This paper, excluding the Introduction, is divided into four sections. In Section 2 we introduce the new concept of the mixed volume of two finite vector sets. By employing the exterior differential, we prove a powerful inequality which is analogous to the Minkowski inequality for mixed volumes of convex bodies, and, furthermore, study the dispersion of this inequality. In Section 3 we define the mixed Cayley–Menger determinant involving two simplices, and prove some new results associated with the two simplices. In Section 4 we introduce the concept of the mixed vertex angles of two simplices. We study the relationships between the vertex angles of simplices and the mixed volume of vector sets, and establish a number of new inequalities of the mixed vertex angles. Meanwhile, we obtain a new and simple proof of the well-known sine theorem. In Section 5 we reprove the inverse forms of the Hadamard inequality by using a new method, and show a series of new results for parallelotopes.

2. The Mixed Volume of Two Finite Vector Sets

In this section we introduce the concept of the mixed volume of two finite vector sets.

Let $X = \{x_1, x_2, \dots, x_k\}$ be a vector set from R^n, and let $\Pi_{[X]}$ be the k-parallelotope which has $\{x_1, x_2, \dots, x_k\}$ as k edge vectors, and which lies in a k-hyperplane $\pi_{[X]}$. Let $V_{[X]}$ denote the k-dimensional volume of $\Pi_{[X]}$. In particular, when $k = n$, then $\pi_{[X]} = R^n$.

Definition 1. Let $X = \{x_1, x_2, \dots, x_k\}$ and $Y = \{y_1, y_2, \dots, y_k\}$ $(k \in N)$ be two vector sets from R^n. The mixed volume $V_{[X,Y]}$ of X and Y can be defined by

$$V_{[X,Y]} = \sqrt{|\det(\langle x_\alpha, y_\beta\rangle)_{1\le\alpha,\beta\le k}|}.$$

Obviously, the mixed volume $V_{[X,Y]}$ of two finite vector sets X and Y have following properties:

(1) $V_{[X,X]} = V_{[X]}$.
(2) $V_{[\lambda_1 X,\ \lambda_2 Y]} = |\lambda_1\lambda_2|^{k/2} V_{[X,\ Y]}$, for $\lambda_1, \lambda_2 \in R$.
(3) $V^2_{[\{x_1+x_1',x_2,\dots,x_k\},\ Y]} \le V^2_{[X,Y]} + V^2_{[X',Y]}$, where $X' = \{x_1', x_2, \dots, x_k\}$.
(4) $V^2_{[\{x_1+x,x_2+x,\dots,x_k+x\},\ Y]} \le V^2_{[X,Y]} + \sum_{i=1}^n V^2_{[X_i,Y]}$, where $X_i = \{x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_k\}$.
(5) If X or Y is a linearly dependent vector set from R^n, it follows that $V_{[X,Y]} = 0$. In particular, when $k > n$, we have $V_{[X,Y]} = 0$.

According to the definition of $V_{[X,Y]}$, by applying the exterior differential, we obtain the following inequality which is analogous to the Minkowski inequality of mixed volumes of convex bodies.

Theorem 2.1. *Suppose that* $X = \{x_1, x_2, \dots, x_k\}$ *and* $Y = \{y_1, y_2, \dots, y_k\}$ $(1 \le k \le n)$ *are two linearly independent vector sets from* R^n, *respectively. Then*

$$V^2_{[X,Y]} \le V_{[X]}V_{[Y]} \tag{2.1}$$

with equality holds if and only if the hyperplane $\pi_{[X]} \parallel \pi_{[Y]}$. In particular, when $k = n$, the equality holds.

Proof. Let $e_1, e_2, \ldots, e_n$ be a standard orthogonal basis in R^n. Then

$$\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\}$$

is a standard orthogonal basis of $\bigwedge^k R^n$.

Since $x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_k \in \bigwedge^k R^n$, let

$$\langle x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_k \rangle = \det(\langle x_\alpha, y_\beta \rangle)_{1 \leq \alpha < \beta \leq k}.$$

Then $\langle\, ,\, \rangle$ is a bilinear form in the space $\bigwedge^k R^n$. It is easy to prove that $\langle\, ,\, \rangle$ is the Euclidean inner product in $\bigwedge^k R^n$.

Since

$$\|x_1 \wedge \cdots \wedge x_k\| = V_{[X]}, \qquad \|y_1 \wedge \cdots \wedge y_k\| = V_{[Y]}$$

and

$$|\langle x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_k \rangle| = V^2_{[X,Y]},$$

by applying the Cauchy inequality to the above two equalities, (2.1) follows.

When $k < n$, it is easy to see the equality of (2.1) if and only if

$$x_1 \wedge \cdots \wedge x_k = \lambda y_1 \wedge \cdots \wedge y_k,$$

where $\lambda \neq 0$ is a constant, equivalently, $\pi_{[X]} \parallel \pi_{[Y]}$.

When $k = n$, the space $\bigwedge^k R^n$ is one-dimensional, it implies that the equality of (2.1) holds. □

To estimate the difference of $V^2_{[X,Y]}$ and $V_{[X]}V_{[Y]}$, we define the *dispersion* by

$$\delta = V_{[X]}V_{[Y]} - V^2_{[X,Y]}.$$

Obviously, we have

(1) $\delta \geq 0$.
(2) If $\pi_{[X]} \parallel \pi_{[Y]}$, then $\delta = 0$.

For the dispersion of inequality (2.1), we have the following result.

Theorem 2.2 *Suppose that $X = \{x_1, x_2, \ldots, x_k\}$ and $Y = \{y_1, y_2, \ldots, y_k\}$ $(1 \leq k \leq n)$ are two vector sets from R^n, respectively, and u, v, p, q are real non-negative numbers with $q \leq p$. If*

$$pV^2_{[X]} < u^2 \quad \textit{and} \quad pV^2_{[Y]} < v^2,$$

then

$$(p-q)\delta \geq (u^2 - pV_{[X]}^2)^{1/2}(v^2 - pV_{[Y]}^2)^{1/2} + pV_{[X]}V_{[Y]} - uv \tag{2.2}$$

with equality holds if and only if the hyperplane $\pi_{[X]} \parallel \pi_{[Y]}$ *and* $vV_{[X]} = uV_{[Y]}$.

Proof. Since

$$u^2 - (p-q)V_{[X]}^2 \geq u^2 - pV_{[X]}^2 > 0,$$

let

$$\begin{aligned} f(x) &= \left[u^2 - (p-q)V_{[X]}^2\right]x^2 - 2[uv - (p-q)V_{[X]}V_{[Y]}]x + [v^2 - (p-q)V_{[Y]}^2] \\ &= (ux - v)^2 - (p-q)(xV_{[X]} - V_{[Y]})^2. \end{aligned} \tag{2.3}$$

According to the above suppose we have $u \neq 0$, furthermore,

$$f\left(\frac{v}{u}\right) = -(p-q)\left(\frac{v}{u}V_{[X]} - V_{[Y]}\right)^2 \leq 0.$$

Since, $f(x) \to +\infty \ (x \to +\infty)$ and $f(x) \to +\infty \ (x \to -\infty)$, therefore, $f(x)$ has two different real zeros belonging to $(-\infty, v/u)$ and $(v/u, +\infty)$ respectively, or, has a zero of order 2, i.e., $x_{1,2} = v/u$. Hence, the discriminant of the above quadratic (2.3) is non-negative, i.e.,

$$[u^2 - (p-q)V_{[X]}^2][v^2 - (p-q)V_{[Y]}^2] \leq [uv - (p-q)V_{[X]}V_{[Y]}]^2.$$

Since $uv - (p-q)V_{[X]}V_{[Y]} > 0$, and using Theorem 2.1, we have that

$$[u^2 - (p-q)V_{[X]}^2]^{1/2}[v^2 - (p-q)V_{[Y]}^2]^{1/2} \leq uv - (p-q)V_{[X,Y]}^2. \tag{2.4}$$

On the other hand, implying the Cauchy inequality, we obtain

$$\begin{aligned} &[u^2 - (p-q)V_{[X]}^2]^{1/2}[v^2 - (p-q)V_{[Y]}^2]^{1/2} \\ &\qquad = [(u^2 - pV_{[X]}^2) + qV_{[X]}^2]^{1/2}[(v^2 - pV_{[Y]}^2) + qV_{[Y]}^2]^{1/2} \\ &\qquad \geq (u^2 - pV_{[X]}^2)^{1/2}(v^2 - pV_{[Y]}^2)^{1/2} + qV_{[X]}V_{[Y]}. \end{aligned} \tag{2.5}$$

Combining (2.4) and (2.5), we get that

$$uv - pV_{[X,Y]}^2 + qV_{[X,Y]}^2 \geq (u^2 - pV_{[X]}^2)^{1/2}(v^2 - pV_{[Y]}^2)^{1/2} + qV_{[X]}V_{[Y]}. \tag{2.6}$$

Rearranging (2.6), the desired (2.2) follows.

From the conditions of equalities (2.4) and (2.5), we obtain that equality (2.2) holds if and only if the hyperplane $\pi_{[X]} \parallel \pi_{[Y]}$ and $vV_{[X]} = uV_{[Y]}$. □

When $q = 0$, we get the following inequality which is analogous to the Aczél inequality.

Corollary 1. *Under the hypotheses of Theorem* 2.2, *we have that*

$$(u^2 - pV_{[X]}^2)(v^2 - pV_{[Y]}^2) \le (uv - pV_{[X,Y]}^2)^2$$

with equality holds if and only if the hyperplane $\pi_{[X]} \parallel \pi_{[Y]}$ *and* $vV_{[X]} = uV_{[Y]}$.

3. Some Results Involving Two Simplices

The quantity relations involving two simplices have been an interesting subject. The well-known Neuberg–Pedoe inequality is the first inequality involving two triangles [20]. Following Pedoe, a number of inequalities for two simplices have been established (see [15], [21], and [29]). Yang and Zhang in [28] generalized the Neuberg–Pedoe inequality to R^n. In [28] Yang and Zhang effectively applied the well-known Cayley–Menger determinant to obtain their elegant result.

Let $\mathcal{A} = \{A_0, A_1, \ldots, A_m\}$ $(m \ge n)$ be a set of points in R^n, and let $\rho_{ij} = \|A_i A_j\|$ $(0 \le i, j \le m)$, then the *Cayley–Menger determinant* (see [2]) of $\mathcal{A}$ is defined by

$$M(\mathcal{A}) = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & \rho_{01}^2 & \cdots & \rho_{0m}^2 \\ 1 & \rho_{10}^2 & 0 & \cdots & \rho_{1m}^2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \rho_{m1}^2 & \rho_{m2}^2 & \cdots & 0 \end{vmatrix}.$$

When $\mathcal{A}$ is the vertex set of an n-simplex (i.e., $m = n$) with volume V, there is a famous result which combines the Cayley–Menger determinant with the volume of simplex as follows (see [28]):

$$V^2 = \frac{(-1)^n M(\mathcal{A})}{2^{n+1} n!^2}.$$

Now we introduce the mixed Cayley–Menger determinant of two point sets as follows.

Definition 2. Suppose $\mathcal{A} = \{A_0, A_1, \ldots, A_m\}$ and $\mathcal{B} = \{B_0, B_1, \ldots, B_m\}$ $(m \ge n)$ are two point sets in R^n and $d_{ij} = \overrightarrow{A_i B_j}$ $(0 \le i, j \le m)$. Let

$$M(\mathcal{A}, \mathcal{B}) = \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & \boxed{d_{ij}^2} & \\ 1 & & & \end{vmatrix}_{0 \le i,\ j \le m}. \tag{3.1}$$

Then $M(\mathcal{A}, \mathcal{B})$ is called the mixed Cayley–Menger determinant of the point sets $\mathcal{A}$ and $\mathcal{B}$.

For the point set $\mathcal{A} = \{A_0, A_1, \dots, A_m\}$ $(m \geq n)$, let $a_i = \overrightarrow{A_0A_i}$ $(i = 1, 2, \dots, m)$. We call $X = \{a_1, a_2, \dots, a_m\}$ the *edge-vector set* of $\mathcal{A}$ at A_0. There is a relationship between the mixed Cayley–Menger determinant $M(\mathcal{A}, \mathcal{B})$ and the mixed volume of two edge-vector sets as follows.

Theorem 3.1. *Let $M(\mathcal{A}, \mathcal{B})$ be the mixed Cayley–Menger determinant of two point sets $\mathcal{A} = \{A_0, A_1, \dots, A_m\}$ and $\mathcal{B} = \{B_0, B_1, \dots, B_m\}$ $(m \geq n)$ in R^n, let X and Y be two edge-vector sets of $\mathcal{A}$ at A_0 and $\mathcal{B}$ at B_0, respectively, and let $V_{[X,Y]}$ be the mixed volume of X and Y. Then*

$$|M(\mathcal{A}, \mathcal{B})| = 2^{m+1} V_{[X,Y]}^2. \tag{3.2}$$

Proof. For convenience, we denote $a_0 = \overrightarrow{A_0A_0}$ and $b_0 = \overrightarrow{B_0B_0}$. According to the definition of determinant $M(\mathcal{A}, \mathcal{B})$, we have

$$
\begin{aligned}
M(\mathcal{A}, \mathcal{B}) &= \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & \boxed{d_{ij}^2} & \\ 1 & & & \end{vmatrix}_{0 \leq i,\ j \leq m} = \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & \boxed{(b_j - a_i + d_{00})^2} & \\ 1 & & & \end{vmatrix} \\
&= \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & \boxed{-2\langle a_i, b_j \rangle} & \\ 1 & & & \end{vmatrix} = (-2)^{m+1} \begin{vmatrix} 0 & -1 & \cdots & -1 \\ 1 & & & \\ \vdots & & \boxed{\langle a_i, b_j \rangle} & \\ 1 & & & \end{vmatrix} \\
&= \det(\langle a_i, b_j \rangle)_{0 \leq i, j \leq m} + (-2)^{m+1} \begin{vmatrix} 0 & -1 & \cdots & -1 \\ 1 & & & \\ \vdots & & \boxed{\langle a_i, b_j \rangle} & \\ 1 & & & \end{vmatrix} \\
&= (-2)^{m+1} \begin{vmatrix} 1 & -1 & \cdots & -1 \\ 1 & & & \\ \vdots & & \boxed{\langle a_i, b_j \rangle} & \\ 1 & & & \end{vmatrix} \\
&= (-2)^{m+1} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & \boxed{\langle a_i, b_j \rangle + 1} & \\ 1 & & & \end{vmatrix}_{1 \leq i, j \leq m}
\end{aligned}
$$

$$= (-2)^{m+1} \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 1 & & & \\ \vdots & & \boxed{\langle a_i, b_j \rangle} & \\ 1 & & & \end{vmatrix}_{1 \le i, j \le m}$$

$$= (-2)^{m+1} \cdot \det(\langle a_i, b_j \rangle)_{1 \le i, j \le m}.$$

According to the definition of the mixed volume $V_{[X,Y]}$, we can get (3.2). □

Corollary 2. *Under the hypotheses of Theorem* 3.1, *if* $m \geq n+1$, *then* $M(\mathcal{A}, \mathcal{B}) = 0$.

From Theorem 3.1, and noting the volume formula of a simplex, we obtain the following theorem immediately.

Theorem 3.2. *Let* $\mathcal{A} = \langle A_0, A_1, \ldots, A_n \rangle$ *and* $\mathcal{B} = \langle B_0, B_1, \ldots, B_n \rangle$ *be two n-simplices with the volumes* $V_{\mathcal{A}}$ *and* $V_{\mathcal{B}}$ *in* R^n. *Then*

$$|M(\mathcal{A}, \mathcal{B})| = 2^{n+1} n!^2\, V_{\mathcal{A}} V_{\mathcal{B}}. \tag{3.3}$$

Let $\mathcal{A} = \langle A_0, A_1, \ldots, A_n \rangle$ and $\mathcal{B} = \langle B_0, B_1, \ldots, B_n \rangle$ be two n-simplices in R^n and let $d_{ij} = \overrightarrow{A_i B_j}$ $(0 \le i, j \le n)$. Then the matrix

$$D(\mathcal{A}, \mathcal{B}) = (d_{ij}^2)_{(n+1)\times(n+1)}$$

is called the *mixed distance matrix* of $\mathcal{A}$ and $\mathcal{B}$.

When $m = n$, let $\mathcal{A} = \langle A_0, A_1, \ldots, A_n \rangle$ be an n-simplex, and let $a_i = \overrightarrow{A_0 A_i}$ $(i = 1, \ldots, n)$. Then $X = \{a_1, a_2, \ldots, a_n\}$ is the edge-vector set of the simplex $\mathcal{A}$ at vertex A_0.

Theorem 3.3. *Let* $R_{\mathcal{A}}$ *and* $R_{\mathcal{B}}$ *be the circumradius of two n-simplices* $\mathcal{A}$ *and* $\mathcal{B}$ *in* R^n, *let X and Y be two edge-vector sets of* $\mathcal{A}$ *at* A_0 *and* $\mathcal{B}$ *at* B_0, *respectively, and let* $V_{[X,Y]}$ *be the mixed volume of X and Y. Then*

$$|\det D(\mathcal{A}, \mathcal{B})| = \frac{1}{2^{n+1}} |R_{\mathcal{A}}^2 + R_{\mathcal{B}}^2 - d^2| V_{[X,Y]}^2, \tag{3.4}$$

where $d = \overrightarrow{O_{\mathcal{A}} O_{\mathcal{B}}}$, *here,* $O_{\mathcal{A}}$ *and* $O_{\mathcal{B}}$ *are the circumcenters of* $\mathcal{A}$ *and* $\mathcal{B}$, *respectively.*

Proof. Let $\mathbb{A} = \{O_{\mathcal{B}}, A_0, \ldots, A_n\}$ and $\mathbb{B} = \{O_{\mathcal{A}}, B_0, \ldots, B_n\}$ be two $(n+2)$ points sets. According to Corollary 2, we get

$$M(\mathbb{A}, \mathbb{B}) = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & d^2 & R_{\mathcal{B}}^2 & \cdots & R_{\mathcal{B}}^2 \\ 1 & R_{\mathcal{A}}^2 & & & \\ \vdots & \vdots & & \boxed{d_{ij}^2} & \\ 1 & R_{\mathcal{A}}^2 & & & \end{vmatrix}_{0 \le i, j \le n} = 0.$$

Using the properties of the determinant, we can get

$$(d^2 - R_{\mathcal{A}}^2 - R_{\mathcal{B}}^2)M(\mathcal{A}, \mathcal{B}) - \det D(\mathcal{A}, \ \mathcal{B}) = 0.$$

Applying Theorem 3.2, (3.4) follows. □

Corollary 3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two n-simplices inscribing a unit sphere. Then*

$$\frac{|\det D(\mathcal{A}, \ \mathcal{B})|}{V_{[X,Y]}^2} = 2^{n+2}. \tag{3.5}$$

For a k-simplex $\mathcal{A} = \langle A_0, A_1, \ldots, A_k\rangle$ with k-dimensional volume V, let

$$\delta_{ij} = \tfrac{1}{2}(\|A_0A_i\|^2 + \|A_0A_j\|^2 - \|A_iA_j\|^2) \qquad (1 \le i, \ j \le k).$$

Then there is the following useful formula:

$$V = \frac{1}{k!}\det(\delta_{ij})_{1\le i,j\le k}.$$

Now we give a generalization of this formula involving two simplices.

Theorem 3.4. *Let $\mathcal{A} = \langle A_0, A_1, \ldots, A_k\rangle$ and $\mathcal{B} = \langle A_0, B_1, \ldots, B_k\rangle$ be two k-simplices which have a common vertex A_0, with k-dimensional volumes $V_{\mathcal{A}}$ and $V_{\mathcal{B}}$, respectively. Put*

$$q_{ij} = \tfrac{1}{2}(\|A_0A_i\|^2 + \|A_0B_j\|^2 - \|A_iB_j\|^2) \qquad i, \ j \in \{1, 2, \ldots, k\}.$$

Then

$$\det(q_{ij})_{k\times k} \le k!^2 \, V_{\mathcal{A}}V_{\mathcal{B}} \tag{3.6}$$

with equality holds if and only if the hyperplane $\pi_{A_k} \parallel \pi_{B_k}$, where π_{A_k} is the hyperplane containing $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ $(0 \le i_1, \ldots, i_k \le k)$. In particular, when $k = n$, then

$$\det(q_{ij})_{n\times n} = n!^2 \, V_{\mathcal{A}}V_{\mathcal{B}}. \tag{3.7}$$

Proof. Taking $x_i = \overrightarrow{A_0A_i}$, $y_i = \overrightarrow{A_0B_i}$ $(i = 1, 2, \ldots, k)$ in Theorem 2.1 and noting that

$$\langle x_i, y_j\rangle = \|x_i\|\|y_j\|\cos\alpha_{ij} = q_{ij},$$

where α_{ij} is the angle between x_i and y_j, the desired (3.6) follows.

From the condition of Theorem 2.1, we obtain (3.7). □

4. The Vertex Angles of Simplices and the Sine Theorem

For an n-simplex $\Omega = \langle A_0, A_1, \ldots, A_n\rangle$, let $\Omega_i = \langle A_0, \ldots, A_{i-1}, A_{i+1} \ldots, A_n\rangle$ be its $(n-1)$-dimensional facet whose area is S_i, with h_i the altitude of Ω from the vertex A_i $(i = 0, 1, \ldots, n)$ and φ_{ij} the internal dihedral angle between Ω_i and Ω_j. Paper [1] introduces the concept of the vertex angle of a simplex.

Bartoš in [1] gave the definition of the vertex angles of an n-dimensional simplex as follows:

Suppose that $\{e_0, e_1, \dots, e_n\}$ *is the set of the unit outer normal vector of facets of* Ω. *The vertex angle at* A_i *can be defined by*

$$\varphi_i = \arcsin \det(e_0, \dots, e_{i-1}, e_{i+1}, \dots, e_n) \qquad (i = 0, 1, \dots, n).$$

Bartoš established the following sine theorem based on his definition (see [1]).

Theorem 4.1.

$$S_0 \cdots S_{i-1} S_{i+1} \cdots S_n \sin\varphi_i = \frac{(nV)^{n-1}}{(n-1)!} \qquad (i = 0, 1, \dots, n). \tag{4.1}$$

The above sine theorem has reappeared several times in different forms and different proofs. Quite recently, it played an important role in the investigation of geometric inequalities (see [5], [14], and [24]).

In this section we give a new and simple proof of the sine theorem by applying Theorem 2.1.

Proof of Theorem 4.1. Let

$$\begin{aligned} X &= \{\overrightarrow{A_iA_0}, \dots, \overrightarrow{A_iA_{i-1}}, \overrightarrow{A_iA_{i+1}}, \dots, \overrightarrow{A_iA_n}\}, \\ Y &= \{e_0, \dots, e_{i-1}, e_{i+1}, \dots, e_n\} \qquad (i = 0, 1, \dots, n), \end{aligned}$$

noting that

$$\langle \overrightarrow{A_iA_j}, e_k \rangle = \delta_{jk}(\pm h_j) \qquad (j,\ k \neq i),$$

where

$$\delta_{jk} = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases}$$

Then

$$V^2_{[X,Y]} = |\det(\langle \overrightarrow{A_iA_j}, e_k \rangle)_{j,\ k\neq i}| = h_0 \cdots h_{i-1} h_{i+1} \cdots h_n,$$
$$V_{[X]} V_{[Y]} = n!V \sin\varphi_i.$$

Employing Theorem 2.1,

$$n!\, V \sin\varphi_i = h_0 \cdots h_{i-1} h_{i+1} \cdots h_n = \frac{(nV)^n}{S_0 \cdots S_{i-1} S_{i+1} \cdots S_n}.$$

The proof is finished. □

In [12] Leng and Zhang introduced the concept of k-order vertex angles of an n-simplex $\Omega = \langle A_0, A_1, \dots, A_n \rangle$ in R^n as follows:

Suppose that $\{e_0, e_1, \ldots, e_n\}$ *is the set of the unit outer normal vector of facets of* Ω. *Let* $e_{i_1}, e_{i_2}, \ldots, e_{i_k} (0 \le i_1 < i_2 < \cdots < i_k \le n)$ *be k vectors among this set. We call*

$$\theta_{i_1 i_2 \cdots i_k} = \arcsin \sqrt{\det(\langle e_{i_r}, e_{i_s} \rangle)_{k\times k}}, \qquad i_r, i_s \in \{i_1, i_2, \ldots, i_k\},$$

the k-order vertex angle at vertex set $\{A_{i_1}, A_{i_2}, \ldots, A_{i_k}\}$ *of* Ω.

Paper [26] extended the concept of the k-order vertex angles of a simplex to two simplices, and introduced the concept of the k-order mixed vertex angles of two simplices.

Let $\Omega = \langle A_0, A_1, \ldots, A_n \rangle$ and $\Omega' = \langle A'_0, A'_1, \ldots, A'_n \rangle$ denote two n-simplices in R^n, and let $\Omega_i = \langle A_0, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n \rangle$ and $\Omega'_i = \langle A'_0, \ldots, A'_{i-1}, A'_{i+1} \ldots, A'_n \rangle$ be their $(n-1)$-dimensional facets $(i = 0, 1, \ldots, n)$, respectively.

Definition 3. Suppose that $\{e_0, e_1, \ldots, e_n\}$ and $\{e'_0, e'_1, \ldots, e'_n\}$ are two sets of the unit outer normal vector of facets of two simplices Ω and Ω', respectively. Let

$$\tilde{\theta}_{i_1 i_2 \cdots i_k} = \arcsin \sqrt{|\det(\langle e_{i_r}, e'_{i_s} \rangle)_{k\times k}|}, \qquad i_r, i_s \in \{i_1, i_2, \ldots, i_k\}.$$

We call $\tilde{\theta}_{i_1 i_2 \cdots i_k}$ the k-order mixed vertex angle at the vertex set $\{A_{i_1}, A_{i_2}, \ldots, A_{i_k}\}$ of Ω and the vertex set $\{A'_{i_1}, A'_{i_2}, \ldots, A'_{i_k}\}$ of Ω'. In particular, $\widetilde{\varphi}_{ij}$ denotes the mixed internal dihedral angle between Ω_i and Ω'_j, and $\widetilde{\varphi}_i$ denotes the n-order mixed vertex angle at the vertices A_i and A'_i.

It is easy to see that the k-order mixed vertex angles are just k-order vertex angles when $\Omega = \Omega'$.

For k-order mixed vertex angles, we establish the following inequalities.

Theorem 4.2.

$$\sin^2 \widetilde{\theta}_{i_1 i_2 \cdots i_k} \le \sin \theta_{i_1 i_2 \cdots i_k} \sin \theta'_{i_1 i_2 \cdots i_k} \tag{4.2}$$

with equality holds if and only if the hyperplane $\pi_k \parallel \pi_{k'}$, *where* π_k *is the hyperplane containing* $\{e_{i_1}, e_{i_2}, \ldots, e_{i_k}\}$.

Proof. Taking $x_i = e_i$, $y_i = e'_i$ $(i = i_1, i_2, \ldots, i_k)$ in Theorem 2.1, we obtain the proof of the theorem. □

Using the known fact (see [8]),

$$\prod_{0 \le i_1 < \cdots < i_k \le n} \sin \theta_{i_1 i_2 \cdots i_k} \le \left[\frac{(n+1)^k}{(k+1)^n} \right]^{((n+1)/n) C_n^k}$$

with equality holds if and only if Ω is regular.

From (4.2) and the above inequality, we obtain the following corollary:

Corollary 4.

$$\prod_{0 \le i_1 < \cdots < i_k \le n} \sin \widetilde{\theta}_{i_1 i_2 \cdots i_k} \le \left[\frac{(n+1)^k}{(k+1)^n} \right]^{((n+1)/n) C_n^k}$$

with equality holds if and only if all the hyperplanes $\pi_k \parallel \pi_{k'}$ *and* Ω *and* Ω' *are regular.*

Theorem 4.3.

$$\cos^2 \widetilde{\theta}_{i_1 i_2 \cdots i_k} \geq \cos \theta_{i_1 i_2 \cdots i_k} \cos \theta'_{i_1 i_2 \cdots i_k} \tag{4.3}$$

with equality holds if and only if the hyperplane $\pi_k \parallel \pi_{k'}$ *and* $\theta_{i_1 i_2 \cdots i_k} = \theta'_{i_1 i_2 \cdots i_k}$ *or* $\theta_{i_1 i_2 \cdots i_k} = \pi - \theta'_{i_1 i_2 \cdots i_k}$.

Proof. Let $u = v = p = 1, q = 0$, then taking $x_i = e_i$, $y_i = e'_i$ $(i = i_1, i_2, \ldots, i_k)$ in Theorem 2.2, the desired proof of Theorem 4.3 is finished. □

Theorem 4.4. *Let* $\widetilde{\theta}_{i_1 i_2 \cdots i_k}$ *be a k-order mixed vertex angle of two n-simplices* Ω *and* Ω', *and let* $m_0, m_1, \ldots, m_n$ *be* $n + 1$ *real positive constants. Then*

$$\sum_{0 \leq i_1 < \cdots < i_k \leq n} \left(\prod_{1 \leq j \leq k} m_{i_j} \right) \sin^2 \widetilde{\theta}_{i_1 i_2 \cdots i_k} \leq \frac{(n-1)!}{k!\,(n-k)!\,n^{k-1}} \left(\sum_{i=0}^{n} m_i \right)^k . \tag{4.4}$$

Equality (4.4) *occurs if and only if all the hyperplanes* $\pi_k \parallel \pi_{k'}$, $\sin \theta_{i_1 i_2 \cdots i_k} = c \sin \theta'_{i_1 i_2 \cdots i_k}$ *(where* $0 \leq c \leq 1$*), and all the following equalities hold*:

$$\frac{m_t}{\sum_{i=0}^{n} m_i} = \frac{\cos \varphi_{rs}}{n(\cos \varphi_{rs} + \cos \varphi_{rt} \cos \varphi_{st})} = \frac{\cos \varphi'_{rs}}{n(\cos \varphi'_{rs} + \cos \varphi'_{rt} \cos \varphi'_{st})} \tag{4.5}$$

(*where* $0 \leq r, s, t \leq n$, *and* r, s, t *are pairwise unequal*). *In particular, the equality holds when all the hyperplanes* $\pi_k \parallel \pi_{k'}$, $\sin \theta_{i_1 i_2 \cdots i_k} = c \sin \theta'_{i_1 i_2 \cdots i_k}$ *(where* $0 \leq c \leq 1$*),* Ω *and* Ω' *are regular, and* $m_0 = m_1 = \cdots = m_n$.

Proof. Let A be the metric matrix of Ω, that is,

$$A = \left(\langle e_i, e_j \rangle \right)_{i,j=0}^{n} = \begin{pmatrix} 1 & & & -\cos \varphi_{ij} \\ & 1 & & \\ & & \ddots & \\ -\cos \varphi_{ij} & & & 1 \end{pmatrix}_{(n+1)\times(n+1)} .$$

Then A is a positive semi-definite symmetric matrix of rank n [28].

Let

$$C = \begin{pmatrix} m_0 & & & -\sqrt{m_i m_j} \cos \varphi_{ij} \\ & m_1 & & \\ & & \ddots & \\ -\sqrt{m_i m_j} \cos \varphi_{ij} & & & m_n \end{pmatrix}_{(n+1)\times(n+1)} .$$

Since the matrix C has the form DAD where D is diagonal with diagonal entries $\sqrt{m_i}$, C is a positive semi-definite symmetric matrix. It is easy to see that the rank of C is n. So one of the eigenvalues of C is zero, and the remaining n eigenvalues are positive. Without loss of generality, we assume that $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ are positive eigenvalues of C, and λ_n is a zero eigenvalue of C.

Let $E_k(C)$ be the sum of all the $k \times k$ principal minors of C, and let $S_k(\lambda_0, \lambda_1, \dots, \lambda_n)$ be the kth elementary symmetric polynomial of the eigenvalues $\lambda_0, \lambda_1, \dots, \lambda_n$.

According to the fairly well-known fact

$$E_k(C) = S_k(\lambda_0, \lambda_1, \dots, \lambda_n)$$

(see (1.2, 10) of [9]), Maclaurin's inequality, and noting $\lambda_n = 0$, we have

$$\begin{aligned}\sum_{0\le i_1<\cdots<i_k\le n}\left(\prod_{1\le j\le k} m_{i_j}\right)\sin^2\theta_{i_1i_2\cdots i_k} &= E_k(C) = S_k(\lambda_0, \lambda_1, \dots, \lambda_n) = S_k(\lambda_0, \lambda_1, \dots, \lambda_{n-1})\\ &\le \left[S_1(\lambda_0, \lambda_1, \dots, \lambda_{n-1})/\binom{n}{1}\right]^k\binom{n}{k}\\ &= \left[S_1(\lambda_0, \lambda_1, \dots, \lambda_n)/\binom{n}{1}\right]^k\binom{n}{k}\\ &= \left[E_1(C)/\binom{n}{1}\right]^k\binom{n}{k} = \left(\frac{1}{n}\sum_{i=1}^n m_i\right)^k\binom{n}{k}.\end{aligned}$$

It is clear that the above equality holds if and only if

$$\lambda_0 = \lambda_1 = \cdots = \lambda_{n-1} = \frac{1}{n}\operatorname{tr} C = \frac{1}{n}\sum_{i=0}^n m_i.$$

Let

$$B = \left(\frac{1}{n}\sum_{i=0}^n m_i\right) I_{n+1} - C, \qquad \text{where} \quad I_{n+1} \text{ is an } (n+1)\times(n+1) \text{ unit matrix.}$$

This is equivalent to saying that the matrix B has a non-zero eigenvalue $(1/n)\sum_{i=0}^n m_i$ and n zero eigenvalues, namely, the rank of B is 1, this is also equivalent to saying that $n+1$ row vectors of B are collinear. Equivalently,

$$\frac{(1/n)\sum_{i=0}^n m_i - m_r}{\sqrt{m_r m_t}\cos\varphi_{rt}} = \frac{\sqrt{m_r m_t}\cos\varphi_{rt}}{(1/n)\sum_{i=0}^n m_i - m_t} = \frac{\sqrt{m_r m_s}\cos\varphi_{rs}}{\sqrt{m_s m_t}\cos\varphi_{st}}$$

(where $0 \le r, s, t \le n$, and r, s, t are pairwise unequal). Rearranging this, we obtain

$$\frac{m_t}{\sum_{i=0}^n m_i} = \frac{\cos\varphi_{rs}}{n(\cos\varphi_{rs} + \cos\varphi_{rt}\cos\varphi_{st})}.$$

Similarly,

$$\sum_{0\le i_1<\cdots<i_k\le n}\left(\prod_{1\le j\le k} m_{i_j}\right)\sin^2\theta'_{i_1i_2\cdots i_k} \le \left(\frac{1}{n}\sum_{i=1}^n m_i\right)^k\binom{n}{k}.$$

The equality holds if and only if

$$\frac{m_t}{\sum_{i=0}^{n} m_i} = \frac{\cos\varphi'_{rs}}{n(\cos\varphi'_{rs} + \cos\varphi'_{rt}\cos\varphi'_{st})}$$

where $0 \le r, s, t \le n$, and r, s, t are pairwise unequal). Using Theorem 3.2 and Cauchy's inequality, we get

$$\begin{aligned}
&\sum_{0\le i_1<\cdots<i_k\le n}\left(\prod_{1\le j\le k} m_{i_j}\right)\sin^2\widetilde{\theta}_{i_1i_2\cdots i_k} \\
&\quad\le \sum_{0\le i_1<\cdots<i_k\le n}\left(\prod_{1\le j\le k} m_{i_j}\right)\sin\theta_{i_1i_2\cdots i_k}\sin\theta'_{i_1i_2\cdots i_k} \\
&\quad\le \left[\sum_{0\le i_1<\cdots<i_k\le n}\left(\prod_{1\le j\le k} m_{i_j}\right)\sin^2\theta_{i_1i_2\cdots i_k}\right]^{1/2} \\
&\qquad\times\left[\sum_{0\le i_1<\cdots<i_k\le n}\left(\prod_{1\le j\le k} m_{i_j}\right)\sin^2\theta'_{i_1i_2\cdots i_k}\right]^{1/2} \\
&\quad\le \left(\frac{1}{n}\sum_{i=1}^{n} m_i\right)^k\binom{n}{k} = \frac{(n-1)!}{k!\,(n-k)!\,n^{k-1}}\left(\sum_{i=1}^{n} m_i\right)^k.
\end{aligned}$$

Meanwhile, we obtain that equality (4.4) holds if and only if the hyperplane $\pi_k \parallel \pi_{k'}$, $\sin\theta_{i_1i_2\cdots i_k} = c\sin\theta'_{i_1i_2\cdots i_k}$ (where $0 \le c \le 1$), and all the equalities in(4.5) hold.

In particular, when $m_0 = m_1 = \cdots = m_n$ and Ω and Ω' are regular, noting $\cos\varphi_{rs} = \cos\varphi'_{rs} = (1/n)$ $(r \ne s)$, (4.5) obviously hold. □

From Theorem 4.4, we can easily derive the following interesting inequalities for the mixed vertex angles of two n-simplices.

Corollary 5. *Let $\widetilde{\theta}_{i_1i_2\cdots i_k}$ be a k-order mixed vertex angle of two n-simplices Ω and Ω', then*

$$\sum_{0\le i_1<\cdots<i_k\le n}\sin^2\widetilde{\theta}_{i_1i_2\cdots i_k} \le \frac{n!}{k!\,(n-k)!}\left(1+\frac{1}{n}\right)^k$$

with equality holds if and only if all the hyperplanes $\pi_k \parallel \pi_{k'}$, $\sin\theta_{i_1i_2\cdots i_k} = c\sin\theta'_{i_1i_2\cdots i_k}$ (where $0 \le c \le 1$), and Ω and Ω' are regular.

Corollary 6. *Let $\widetilde{\varphi}_{ij}$ be the mixed internal dihedral angle between $(n-1)$-dimensional facets Ω_i and Ω'_j. Then*

$$\sum_{0\le i<j\le n} m_i m_j \sin^2\widetilde{\varphi}_{ij} \le \frac{n-1}{2n}\left(\sum_{i=1}^{n} m_i\right)^2$$

with equality holds if and only if all the hyperplanes $\pi_{[e_i,e_j]} \parallel \pi_{[e'_i,e'_j]}$, $\sin\varphi_{i,j} = c\sin\varphi'_{i,j}$, and condition (4.5) holds.

Corollary 7. *Under the conditions of Corollary* 6, *we have*

$$\sum_{0\le i<j\le n} \sin^2 \widetilde{\varphi}_{ij} \le \frac{(n+1)(n^2-1)}{2n}$$

with equality holds if and only if all the hyperplanes $\pi_{[e_i,e_j]} \parallel \pi_{[e'_i,e'_j]}$, $\sin\varphi_{i,j} = c \sin\varphi'_{i,j}$, *and* Ω *and* Ω' *are regular*.

Corollary 8. *Let* $\widetilde{\varphi}_i$ $(i = 0, 1, \dots, n)$ *be the mixed vertex angles of* Ω *and* Ω'. *Then*

$$\sum_{i=0}^{n} \sin^2 \widetilde{\varphi}_i \le \left(1 + \frac{1}{n}\right)^n$$

with equality holds if and only if all the hyperplanes $\Omega_i \parallel \Omega'_i$ *and* Ω *and* Ω' *are regular.*

5. Inverse Forms of the Hadamard Inequality

Let $M = (m_{ij})_{n\times n}$ be an $n \times n$ positive Hermitian matrix. The classical Hadamard inequality states that

$$\det M \le \prod_{i=1}^{n} m_{ii},$$

and the equality holds if and only if M is diagonal with entries m_{ii}.

Let $\{x_1, x_2, \dots, x_n\}$ be a linearly independent vector set from R^n, and let $V(\Pi_{[X]})$ be the n-dimensional volume of the n-parallelotope $\Pi_{[X]}$ which has $\{x_1, x_2, \dots, x_n\}$ as edge-vectors, then the geometric form of the classical Hadamard inequality is as follows:

$$V(\Pi_{[X]}) \le \prod_{i=1}^{n} \|x_i\|. \tag{5.1}$$

Let $\Pi_{[\hat{x}_i]}$ be the facet of $\Pi_{[X]}$ which lies in a hyperplane π_i (i.e., an $(n-1)$-parallelotope which does not contain the vector x_i). Szasz has generalized the Hadamard inequality as follows (see [3]):

$$V^{n-1}(\Pi_{[X]}) \le \prod_{i=1}^{n} V(\Pi_{[\hat{x}_i]}). \tag{5.2}$$

The equalities in (5.1) and (5.2) occur if and only if $\{x_1, x_2, \cdots, x_n\}$ is a set of orthogonal non-zero vectors in R^n.

The other generalizations of the Hadamard inequality to the block matrices and to other types of matrices were obtained by Fisher, Johnson, Markham, Veljan, and others (see [4], [10], [11], and [13]). The estimates for the ratio $\det M / \prod_{i=1}^{n} m_{ii}$ were investigated by Johnson, Newman, Dixon, Reznikov, and others (see [3] and [22]). Some eigenvalues estimates of Wolkowitz and Styan could also be interpreted as estimates of $\det M$ (see [25]).

Let y_i be the orthogonal component of x_i with respect to π_i. Then we call y_i the altitude vector on $\Pi_{[\hat{x}_i]}$ $(i = 1, 2, \dots, n)$.

For any linearly independent vector set $\{x_1, x_2, \dots, x_n\}$, it is easy to prove that there exists an n-parallelotope $\Pi^*_{[X]}$ which has $\{x_1, x_2, \dots, x_n\}$ as n altitude vectors (see [13]).

In [13] Leng and Zhou proved the inverse forms of the Hadamard inequality and the Szasz inequality as Theorem 5.1 following. In this paper we give a new and simple proof.

Theorem 5.1. *Let $\{x_1, x_2, \dots, x_n\}$ be a set of linearly independent vectors from R^n and let $\Pi^*_{[X]}$ be an n-parallelotope which has $\{x_1, x_2, \dots, x_n\}$ as the n altitude vectors. Furthermore, let $\Pi^*_{[\hat{x}_i]}$ be the* $(n-1)$*-parallelotope which has $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ as* $(n-1)$ *altitude vectors. Then*

$$V(\Pi^*_{[X]}) \ \geq \ \prod_{i=1}^{n} \|x_i\|, \tag{5.3}$$

$$V^{n-1}(\Pi^*_{[X]}) \ \geq \ \prod_{i=1}^{n} V(\Pi^*_{[\hat{x}_i]}). \tag{5.4}$$

The equalities in (5.3) *and* (5.4) *occur if and only if $\{x_1, x_2, \dots, x_n\}$ is a set of orthogonal non-zero vectors in R^n.*

To prove Theorem 5.1, we first prove the following lemma.

Lemma 5.2. *Let $X = \{x_1, x_2, \dots, x_k\}$ $(k \leq n)$ be a set of linearly independent vectors from R^n, and let $\Pi_{[X]}$ and $\Pi^*_{[X]}$ be as in Theorem* 5.1. *Then*

$$V(\Pi_{[X]})V(\Pi^*_{[X]}) = \left(\prod_{i=1}^{k} \|x_i\|\right)^2, \tag{5.5}$$

where $V(\Pi_{[X]})$ and $V(\Pi^*_{[X]})$ are k-dimensional volumes in R^n.

Proof. Let $Y = \{y_1, y_2, \dots, y_k\}$ be the set of k edge-vectors of $\Pi^*_{[X]}$, namely, $\Pi^*_{[X]} = \Pi_{[Y]}$. According to the definition of $\Pi^*_{[X]}$, we have

$$\langle x_i, y_j \rangle = \delta_{ij}\|x_i\|^2, \qquad \text{where} \quad \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

It is easy to see that

$$V^2_{[X,Y]} = |\det(\langle x_i, y_j \rangle)_{k\times k}| = \prod_{i=1}^{k} \|x_i\|^2,$$

$$V(\Pi_{[X]})V(\Pi^*_{[X]}) = V_{[X]}V_{[Y]}.$$

Obviously, two vector sets $X = \{x_1, x_2, \dots, x_k\}$ and $Y = \{y_1, y_2, \dots, y_k\}$ lie in a common k-dimensional hyperplane. Using the condition of equality in (2.1), the proof is completed. □

Proof of Theorem 5.1. Taking $k = n$ in Lemma 5.2 and employing inequality (5.1), we obtain

$$\left(\prod_{i=1}^{n} \|x_i\|\right)^2 = V(\Pi^*_{[X]})V(\Pi_{[X]}) \leq V(\Pi^*_{[X]}) \prod_{i=1}^{n} \|x_i\|.$$

Rearranging the above inequality, (5.3) is proved.

Taking $k = n - 1$ in Lemma 5.2, i.e., applying Lemma 5.2 to set $\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}$, we get

$$V(\Pi^*_{[\hat{x}_i]})V(\Pi_{[\hat{x}_i]}) = \prod_{\substack{j=1 \\ j \neq i}}^{n} \|x_i\|^2.$$

It follows that

$$\prod_{i=1}^{n} V(\Pi^*_{[\hat{x}_i]})V(\Pi_{[\hat{x}_i]}) = \left(\prod_{i=1}^{n} \|x_i\|\right)^{2(n-1)} = V^{n-1}(\Pi^*_{[X]})V^{n-1}(\Pi_{[X]}).$$

Applying inequality (5.2), then (5.4) follows. □

In succession, we carry on studing the estimation of the volumes of parallelotopes.

For the given n-parallelotope $\Pi_{[X]}$ which has $\{x_1, x_2, \ldots, x_n\}$ as edge vectors, let $\Pi_{[\hat{x}_i]}$ be the facet of $\Pi_{[X]}$ which lies in a hyperplane π_i, let $H = \{h_1, h_2, \ldots, h_n\}$ be n altitude vectors, and let γ_i be the angle between edge vector x_i and hyperplane π_i $(1 \leq i \leq n)$. We denote the n-parallelotope $\Pi^{\perp}_{[X]}$ which has $h_1, h_2, \ldots, h_n$ as n edge vectors, namely, $\Pi^{\perp}_{[X]} = \Pi_{[H]}$.

Theorem 5.3. *Let n-parallelotopes* $\Pi_{[X]}$ *and* $\Pi^{\perp}_{[X]}$ *be as above, then*

$$V(\Pi_{[X]})V(\Pi^{\perp}_{[X]}) = \left(\prod_{i=1}^{n} \|x_i\| \sin \gamma_i\right)^2. \tag{5.6}$$

Proof. According to the definition of $\Pi^{\perp}_{[X]}$, we know that $h_i \perp \pi_i$, $\|h_i\| = \|x_i\| \sin \gamma_i$ $(1 \leq i \leq n)$, and

$$\langle h_i, x_j \rangle = \delta_{ij} \|h_i\|^2,$$

where δ_{ij} is as previously stated.

Similar to the proof of Lemma 5.2,

$$V(\Pi_{[X]})V(\Pi^{\perp}_{[X]}) = V_{[X,H]} = \prod_{i=1}^{n} \|h_i\|^2$$

and (5.6) follows. □

From Theorems 5.1 and 5.3, it is easy to obtain the following result.

Corollary 9. *Under the above hypotheses, we have*

$$V(\Pi^{\perp}_{[X]}) \le V(\Pi_{[X]}) \le V(\Pi^{*}_{[X]}). \tag{5.7}$$

The equalities occur if and only if $\{x_1, x_2, \dots, x_n\}$ *is a set of orthogonal non-zero vectors in* R^n.

To understand the ulterior relations between $\Pi^{*}_{[X]}$, $\Pi^{\perp}_{[X]}$, and $\Pi_{[X]}$, we define the edge-vertex angle and altitude-vertex angle of an n-parallelotope as follows:

Definition 4. Let h_i be the altitude vector on $\Pi_{[\hat{x}_i]}$ $(i = 1, 2, \dots, n)$ of parallelotope $\Pi_{[X]}$ which has $\{x_1, x_2, \dots, x_n\}$ as n edge vectors. Let

$$\alpha = \arcsin \sqrt{\det\left(\left\langle \frac{x_i}{\|x_i\|}, \frac{x_j}{\|x_j\|}\right\rangle\right)_{1\le i,j\le n}};$$

we call α an edge-vertex angle of the parallelotope $\Pi_{[X].}$

Let

$$\beta = \arcsin \sqrt{\det\left(\left\langle \frac{h_i}{\|h_i\|}, \frac{h_j}{\|h_j\|}\right\rangle\right)_{1\le i,j\le n}};$$

we call β an altitude-vertex angle of the parallelotope $\Pi_{[X]}$.

Theorem 5.4.

$$\text{Sin}\alpha \sin\beta = \prod_{i=1}^{n} \sin\gamma_i. \tag{5.8}$$

Proof. Since

$$V(\Pi^{\perp}_{[X]}) = \sqrt{\det(\langle h_i, h_j\rangle)_{n\times n}} = \sin\beta \prod_{i=1}^{n} \|h_i\| = \sin\beta \prod_{i=1}^{n} \|x_i\| \sin\gamma_i,$$

$$V(\Pi_{[X]}) = \sqrt{\det(\langle x_i, x_j\rangle)_{n\times n}} = \sin\alpha \prod_{i=1}^{n} \|x_i\|.$$

According to Theorem 5.3, the proof is completed. □

Theorem 5.5. *Under the above hypotheses, then*

$$\frac{V(\Pi_{[X]})}{V(\Pi^{*}_{[X]})} = \sin^2\alpha, \tag{5.9}$$

$$\frac{V(\Pi^{\perp}_{[X]})}{V(\Pi_{[X]})} = \sin^2\beta, \tag{5.10}$$

$$\frac{V(\Pi^{\perp}_{[X]})}{V(\Pi^{*}_{[X]})} = \sin^2\alpha \sin^2\beta. \tag{5.11}$$

Proof. According to the definition of $V(\Pi_{[X]})$, we have that

$$V(\Pi_{[X]})^2 = \det(\langle x_i, x_j \rangle)_{n\times n} = \sin^2\alpha \prod_{i=1}^{n} \|x_i\|^2.$$

Combining this with equality (5.5) in Lemma 5.2, equality (5.9) follows.

Using Theorems 5.3 and 5.4, we get that

$$V(\Pi_{[X]}^{\perp})V(\Pi_{[X]}) = \sin^2\alpha \sin^2\beta \prod_{i=1}^{n} \|x_i\|^2 = \sin^2\beta V(\Pi_{[X]})^2.$$

Rearranging it, equality (5.10) follows.

Combining (5.9) with (5.10), we get (5.11). □

At last, we study the relationship of $V(\Pi_{[X]} \cup \Pi_{[Y]})$ and $V_{[X,Y]}$. In [16] Leng et al. introduce the concept of a parallelotope-stack.

Let $\Pi_{[X]}$ and $\Pi_{[Y]}$ be two k-parallelotopes in R^n. The union $\Pi_{[X]} \cup \Pi_{[Y]}$ is called the parallelotope-stack of $\Pi_{[X]}$ and $\Pi_{[Y]}$ if there is a hyperplane H such that the two parallelotopes lie in opposite half-spaces bounded by H and H contains a facet of each.

Theorem 5.6. *Let $X = \{x_1, x_2, \ldots, x_k\}$ and $Y = \{y_1, y_2, \ldots, y_k\}$ be two sets of linearly independent vectors, let $\Pi_{[X]}$ and $\Pi_{[Y]}$ be the k-parallelotopes which takes X and Y as k edge vectors, respectively. Suppose that $\Pi_{[X]} \cup \Pi_{[Y]}$ is the parallelotope-stack. Then*

$$V_{[X,Y]} \le \tfrac{1}{2} V(\Pi_{[X]} \cup \Pi_{[Y]}). \tag{5.12}$$

Proof. According to the definition of parallelotope-stack $\Pi_{[X]} \cup \Pi_{[Y]}$ of $\Pi_{[X]}$ and $\Pi_{[Y]}$ in R^n, we infer

$$V(\Pi_{[X]} \cup \Pi_{[Y]}) = V(\Pi_{[X]}) + V(\Pi_{[Y]}).$$

On the other hand, it is easy to see that

$$V(\Pi_{[X]}) = V_{[X]}, \qquad V(\Pi_{[Y]}) = V_{[Y]}.$$

Applying Theorem 1, we obtain that

$$V_{[X]} + V_{[Y]} \ge 2\sqrt{V_{[X]}V_{[Y]}} \ge 2V_{[X,Y]},$$

and this implies (5.12). □

Acknowledgements

The authors are greatly indebted to Professor Hanfang Zhang for his help.

References

1. P. Bartoš, Sinusova veta o simplexoch vE_n, *Časopis Pěst. Mat.*, **93** (1968), 273–277.
2. L. M. Blumenthal, *Theory and Applications of Distance Geometry*, Oxford University Press, London, 1953.
3. J. D. Dixon, How good is Hadamard's inequality for determinants?, *Canad. Math. Bull.*, **27** (1984), 260–264.
4. G. M. Engle and H. Schneider, The Hadamard–Fischer inequality for a class of matrices defined by eigenvalue monotonicity, *Linear and Multilinear Algebra*, **4** (1976), 155–176.
5. F. Eriksson, The law of sines for tetrahedra and n-simplices, *Geom. Dedicata*, **7** (1978), 71–80.
6. R. J. Gardner, Intersection bodies and the Busemann-Petty problem, *Trans. Amer. Math. Soc.*, **342** (1994), 435–445.
7. R. J. Gardner, *Geometric Tomography*, Cambridge University Press, Cambridge, 1995.
8. S. G. Guo, Generalization of Bartoš volume formulas for high-dimensional simplex, *J. Math. Res. Exposition*, **18** (1998), 597–600.
9. R. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, New York, 1985.
10. C. R. Johnson and T. Markham, Compression and Hadamard power inequalities for matrices, *Linear and Multilinear Algebra*, **18** (1985), 23–34.
11. C. R. Johnson and M. Newman, How bad is the Hadamard determinantal bound?, *J. Res. Nat. Bur. Standards*, **78** (1974), 167–169.
12. G. S. Leng and Y. Zhang, The generalized sine theorem and inequalities for simplices, *Linear Algebra and its Applications*, **278** (1998), 237–247.
13. G. S. Leng, Inverse forms of Hadamard inequality, *SIAM J. Matrix Anal. Appl.*, **23**(4) (2002), 990–997.
14. G. S. Leng, Z. Shen, and L. H. Tang, Inequalities for two simplices, *J. Math. Anal. Appl.*, **248** (2000), 429–437.
15. G. S. Leng and L. H. Tang, Some generalizations to several dimensional of Pedoe inequality with applications, *Acta. Math. Sinica*, **40** (1997), 14–21 (in Chinese).
16. G. S. Leng, Y. Zhang, and B. L. Ma, Largest parallelotopes contained in simplices, *Discrete Math.*, **211** (2000), 111–123.
17. E. Lutwak, Dual mixed volumes, *Pacific J. Math.*, **58** (1975), 531–538.
18. E. Lutwak, Volume of mixed bodies, *Trans. Amer. Math. Soc.*, **294** (1986), 487–500.
19. E. Lutwak, Centroid bodies and dual mixed volumes, *Proc. London Math. Soc.*, **60** (1990), 365–391.
20. D. Pedoe, An inequality for two triangles, *Math. Proc. Cambridge Philos. Soc.*, **38** (1942), 397–398.
21. D. Pedoe, Thinking geometrically, *Amer. Math. Monthly*, **77** (1970), 711–721.
22. A. G. Reznikov, Determinant inequalities with applications to isoperimetrical inequalities, *Oper. Theory Adv. Appl.*, **77** (1995), 139–244.
23. R. Schneider, *Convex Bodies: The Brunn-Minkoeski Theory*, Cambridge University Press, Cambridge, 1993.
24. D. Veljan, The sine theorem and inequalities for volumes of simplices and determinants, *Linear Algebra Appl.*, **219** (1995), 79–91.
25. H. Wolkowitz and G. Styan, Bounds for eigenvalues using traces, *Linear Algebra Appl.*, **29** (1980), 471–506.
26. D. H. Yang, The Geometric relation involving k-dimensional vertex angle of two simplicies and its applications (to appear).
27. G. Y. Zhang, Centered bodies and dual mixed volumes, *Trans. Amer. Math. Soc.*, **345** (1994), 777–801.
28. L. Yang and J. Z. Zhang, A generalization to several dimensions of the Neuberg–Pedoe inequality with applications, *Bull. Austral. Math. Soc.*, **27** (1983), 203–214.
29. Y. Zhang, A conjection for the volume of simplex with feet of perpendicular as vertices, *J. Systems Sci. Math. Sci.*, **12** (1992), 371–375.

Received July 5, 2003, *and in revised form December* 7, 2003. *Online publication June* 7, 2004.