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Pairs of Convex Bodies with Centrally Symmetric Intersections of Translates

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Abstract. For a pair of convex bodies K and K' in E^d, the d-dimensional intersections $K \cap (x + K')$, $x \in E^d$, are centrally symmetric if and only if K and K' are represented as direct sums $K = R \oplus P$ and $K' = R' \oplus P'$ such that: (i) R is a compact convex set of some dimension m, $0 \le m \le d$, and $R' = z - R$ for a suitable vector $z \in E^d$, (ii) P and P' are isothetic parallelotopes, both of dimension $d - m$.

1. Introduction and Main Result

Several results of convex geometry characterize pairs of convex bodies whose intersections of translates satisfy given geometric properties. Thus, convex bodies K and K' in the Euclidean space E^d are homothetic ellipsoids if and only if for any translate $x + K'$, $x \in E^d$, the intersection of the boundaries of K and $x + K'$ lies in a hyperplane (see [1]). Similarly, convex bodies K and K' in E^d are homothetic simplexes if and only if the d-dimensional intersections $K \cap (x + K')$, $x \in E^d$, belong to a unique homothety class (more generally, to at most countably many homothety classes) of convex bodies (see [5]).

We study below the following problem of a similar spirit, related to centrally symmetric convex bodies.

Problem. Describe the pairs of convex bodies K and K' in E^d such that all d-dimensional intersections $K \cap (x + K')$, $x \in E^d$, are centrally symmetric.

In what follows we need some definitions. A *convex body* is a compact convex set with nonempty interior in E^d. A set $X \subset E^d$ is called *centrally symmetric* if and only if there is a point $z \in E^d$ such that $X - z = z - X$; in this case X is symmetric about z.

We say that sets X and X' in E^d are *similarly represented as direct sums*

$$X = X_1 \oplus \cdots \oplus X_k, \qquad X' = X'_1 \oplus \cdots \oplus X'_k$$

if there are subspaces $L_1, \ldots, L_k \subset E^d$ forming a direct sum such that both X_i and X'_i lie in L_i for all $i = 1, \ldots, k$.

A *parallelotope* is a compact convex set in E^d that is a direct sum of finitely many line segments. Two parallelotopes P and P' of the same dimension k $(1 \le k \le d)$ are called *isothetic* provided they can be similarly represented as direct sums

$$P = P_1 \oplus \cdots \oplus P_k, \qquad P' = P'_1 \oplus \cdots \oplus P'_k,$$

where P_i and P'_i are parallel line segments for all $i = 1, \ldots, k$.

Our main result is given in the following theorem.

Theorem 1. *For a pair of convex bodies K and K' in E^d, the following three conditions are equivalent*:

(1) *All nonempty intersections $K \cap (x + K')$, $x \in E^d$, are centrally symmetric.*
(2) *All d-dimensional intersections $K \cap (x + K')$, $x \in E^d$, are centrally symmetric.*
(3) *K and K' are similarly represented as direct sums*

$$K = R \oplus P \quad \text{and} \quad K' = R' \oplus P'$$

such that conditions (i) *and* (ii) *below are satisfied*:
 (i) *R is a compact convex set of some dimension m, $0 \le m \le d$, and $R' = z - R$ for a suitable vector $z \in E^d$,*
 (ii) *P and P' are isothetic parallelotopes, both of dimension $d - m$.*

Observation 1. *The cases $m = d$ and $m = 0$ in condition* (3) *of Theorem* 1 *are interpreted as follows*: *$m = d$ means that $K' = z - K$ for a suitable vector $z \in E^d$, while $m = 0$ means that K and K' are isothetic parallelotopes, both of dimension d.*

Corollary 1 [3]. *A convex body $K \subset E^d$ is a parallelotope if and only if there is a real number $\lambda \in]0, 1[$ such that all nonempty intersections $K \cap (x + \lambda K)$, $x \in E^d$, are centrally symmetric.*

2. Auxiliary Theorems

The proof of Theorem 1 is organized by induction on $d = \dim E^d$ and uses Theorems 2 and 3 below. Recall that a subset F of a convex body $M \subset E^d$ is called an *exposed face* of M provided there is a hyperplane H supporting M such that $F = M \cap H$. In what follows, $\mathcal{F}(M)$ denotes the family of exposed faces of M.

If an exposed face F of M consists of a single point (respectively, of a line segment), then it is called an *exposed point* (respectively, an *exposed line segment*). Throughout this paper we denote by $\mathcal{F}_0(M)$ and $\mathcal{F}_1(M)$ the family of exposed points and the family of exposed line segments of M, respectively. Generally, the endpoints of an exposed line segment of M are not exposed points themselves, but they are extreme points of the body. Recall that a point $x \in M$ is *extreme* if no open line segment $]y, z[$, $y \neq z$, with the property $x \in]y, z[\subset M$ exists. In a standard way, the set of extreme points of M is denoted ext M.

Let M and M' be a given pair of convex bodies in E^d. For any hyperplane H supporting M, denote by H' the hyperplane parallel to H and supporting M' such that M' lies on the same side from H' as M does with respect to H. In this case the exposed face $F' = M' \cap H'$ of M' will be called *associate* to the exposed face $F = M \cap H$. Generally, the relation "is associate to" is not a one-to-one correspondence between the families $\mathcal{F}(M)$ and $\mathcal{F}(M')$.

The following theorems are auxiliary for the proof of Theorem 1. (To distinguish similarly looking elements, we use θ for the zero vector of E^d.)

Theorem 2. *Let M and M' be convex bodies in E^d such that all nonempty intersections $M \cap (x + M')$, $x \in E^d$, are centrally symmetric. Then M and $-M'$ satisfy the following two conditions*:

(4) *Any exposed point a of M has an associate exposed point a' of $-M'$ such that $(-M') \cap (a' + W)$ is a translate of $M \cap (a + W)$ for a suitable neighborhood W of θ.*

(5) *Any exposed line segment $[a, b]$ of M has an associate exposed line segment $[a', b']$ of $-M'$ parallel to $[a, b]$ and such that for a suitable neighborhood W of θ the sets $(-M') \cap (a' + W)$ and $(-M') \cap (b' + W)$ are translates of $M \cap (a + W)$ and $M \cap (b + W)$, respectively, provided $a - b$ and $a' - b'$ have the same direction.*

Theorem 3. *Let M and M' be convex bodies in E^d. Then M' is a translate of M if and only if the following two conditions are satisfied*:

(6) *Any exposed point a of M has an associate exposed point a' of M' such that $M' \cap (a' + W)$ is a translate of $M \cap (a + W)$ for a suitable neighborhood W of θ.*

(7) *Any exposed line segment $[a, b]$ of M has an associate exposed line segment $[a', b']$ of M' that is a translate of $[a, b]$ and such that for a suitable neighborhood W of θ the sets $M' \cap (a' + W)$ and $M' \cap (b' + W)$ are translates of $M \cap (a + W)$ and $M \cap (b + W)$, respectively, provided $a - b$ and $a' - b'$ have the same direction.*

In a standard way, bd M and int M denote, respectively, the boundary and the interior of a convex body $M \subset E^d$. A boundary point x of M is called *regular* if there is a unique hyperplane supporting M at x. Denote by $N(M)$ the family of outward unit normals to M at its regular points. In particular, M is a polytope if and only if the set $N(M)$ is finite. Finally, $B_r(a) = \{x \in E^d : ||x - a|| \leq r\}$ stands for the closed ball with center a and radius r, and $S = \{x \in E^d : \|x\| = 1\}$ denotes the unit sphere of E^d.

3. Proof of Theorem 2

We prove condition (5) only, since the proof of (4) may be considered as a limit case of (5), by taking $a = b$ and $a' = b'$.

Let $[a, b]$ be an exposed line segment of M. Translating M, if necessary, we may assume that $a = \theta$. Denote by H a hyperplane with the property $M \cap H = [\theta, b]$, and let e be the unit vector orthogonal to H such that M and e belong to the same closed half-space P determined by H. Let G be the hyperplane through θ orthogonal to the line segment $[\theta, b]$, and let Q be the closed half-space determined by G and disjoint to b. Next, put $R = b/2 + Q$. In other words, R is the closed half-space determined by the hyperplane $b/2 + G$ and containing θ.

Choose a real number $\lambda > 0$ so small that the hyperplane $H_\lambda = \lambda e + H$ intersects $\operatorname{int} M$. Denote by D_λ the part of R that lies between H and H_λ, and let M_λ be the set of regular points of M which lie in D_λ. Now let $N(M_\lambda)$ be the set of outward unit normals to M at points from M_λ. Since $M \cap H = [\theta, b]$ and R excludes the segment $]b/2, b]$, by a compactness argument we obtain the existence of a number δ such that $0 < \delta < \min\{\lambda, \|b\|/2\}$ and the closure of $N(M_\delta)$ belongs to an open half-sphere of the unit sphere S. Indeed, otherwise we would obtain the existence of two parallel hyperplanes through θ and a point from $]\theta, b[$, respectively, both supporting M.

Translating M', if necessary, we may assume that M' is disjoint to $\operatorname{int} P$ and is supported by H at θ such that $M' \cap H$ lies in Q. By a continuity argument, we may choose a vector $x \in D_\delta$ such that $\theta \in \operatorname{int}(x + M')$ and $P \cap (x + M')$ lies in D_λ. From the above it follows that the intersection $K = M \cap (x + M')$ is a convex body situated in D_δ and that $K \cap H = [\theta, w]$ for a point $w \in]\theta, b/2]$. By the hypothesis, K is centrally symmetric.

Let F be the hyperplane supporting K and parallel to H, $F \neq H$. Obviously, F lies in the closed slab between H and $x + H$. Because K is centrally symmetric, there is an exposed line segment $[z, z - w]$ of K with the property $K \cap F = [z, z - w]$. Moreover, K is symmetric about the middle point of the line segment $[\theta, z]$.

We claim that $z \in \operatorname{int} M$. Indeed, assume, for contradiction, that $z \in \operatorname{bd} M$. Since $z \in K \subset D_\delta$ and F intersects the interior of M, the boundary of K in any neighborhood of z should contain a $(d-1)$-dimensional piece of $\operatorname{bd} M$. Hence any neighborhood of z contains a regular point p of K that belongs to M_δ. Let e_p be the outward unit normal of K (also of M) at p. By the symmetry of K about $z/2$, the point $q = z - p$ is a regular point of K and the outward unit normal e_q to K at q is opposite to e_p: $e_q = -e_p$. Since $\theta \in \operatorname{int}(x + M')$, we can choose p so close to z that the respective point q belongs to $\operatorname{int}(x + M')$. As a result, q lies in the boundary of M, and whence $q \in M_\delta$. Thus we have two distinct points $p, q \in M_\delta$ with $e_p = -e_q$, which is in contradiction with the choice of δ. Hence $z \in \operatorname{int} M$.

The inclusions $z \in \operatorname{bd} K$ and $z \in \operatorname{int} M$ obviously imply that $z \in \operatorname{bd}(x + M')$, otherwise z would lie in the interior of K. Moreover, the hyperplane F should coincide with $x + H$. Indeed, assume for a moment that F is different from $x + H$. In this case, one can find a point $u \in [x, z[$, which belongs to $M \cap (x + M')$ and lies between $x + H$ and F. The last is in contradiction with the choice of F.

Next we show that $z = x$. Indeed, since $z \in \operatorname{bd}(x + M')$ and since $x + H$ supports K along the line segment $[z, z - w]$, the hyperplane $x + H$ supports $x + M'$ along a line segment $[z, s]$ that contains $[z, z - w]$. Hence $M' \cap H = [z - x, s - x]$. From the inclusion $M' \cap H \subset Q$ and the fact that G supports $M' \cap H$ at θ, we conclude that $z - x = \theta$, i. e., $z = x$. As a result, $M' \cap H = [\theta, s - x]$, whence $[\theta, x - s]$ is an exposed line segment of $-M'$ associate to $[\theta, b]$.

Since $\theta \in \operatorname{int}(x + M')$ and $x = z \in \operatorname{int} M$, there is a neighborhood $W_1 \subset E^d$ of θ such that $W_1 \subset \operatorname{int}(x + M')$ and $x - W_1$ is a neighborhood of x that lies in $\operatorname{int} M$. Because K is symmetric about $x/2$, we have

$$M \cap W_1 = K \cap W_1 = x - K \cap (x - W_1) = x - (x + M') \cap (x - W_1) = (-M') \cap W_1.$$

Repeating the consideration above for the points b and $x - s$, we obtain the existence of a neighborhood $W_2 \subset E^d$ of θ such that $(M - b) \cap W_2 = (-M' - x + s) \cap W_2$. Obviously, the set $W = W_1 \cap W_2$ is a required neighborhood of θ. □

4. Proof of Theorem 3

If a convex body M' is a translate of a convex body M, then conditions (6) and (7) are trivially satisfied.

Conversely, let M and M' be a pair of convex bodies in E^d that satisfy conditions (6) and (7). We show that M' is a translate of M. This part of the proof is organized by induction on $d = \dim E^d$.

The case $d = 1$ is trivial, and the case $d = 2$ is based on the following statement.

Claim 1. *Let M and M' be convex bodies in the plane E^2 that satisfy conditions* (6) *and* (7). *Then the relation "is associate to" gives one-to-one correspondences $\mathcal{F}_0(M) \leftrightarrow \mathcal{F}_0(M')$ and $\mathcal{F}_1(M) \leftrightarrow \mathcal{F}_1(M')$.*

Proof of Claim 1. Choose a point $x \in \mathcal{F}_0(M)$, and let $x' \in \mathcal{F}_0(M')$ be associate to x such that $M' \cap (x' + W)$ is a translate of $M \cap (x + W)$ for a suitable neighborhood W of θ. Let H be a line with $M \cap H = \{x\}$, and let H' be the line parallel to H with $M' \cap H' = \{x'\}$. Assume for a moment that x has another associate point $x'_1 \in \mathcal{F}_0(M')$, that is, assume the existence of a line H_1 distinct from H such that $M \cap H_1 = \{x\}$ and of the line H'_1 parallel to H_1 and supporting M' at x'_1 only. Since $M' \cap (x' + W) = (x' - x) + M \cap (x + W)$, the line $H'' = x' - x + H_1$ supports M' at x'. Thus H'_1 and H'' are parallel lines both supporting M' from the same side. As a result, $H'_1 = H''$ and H'_1 supports M' along the line segment $[x', x'_1]$, contradicting the condition $M' \cap H'_1 = \{x'_1\}$. Hence any exposed point x of M has a unique associate exposed point x' of M'.

Next we prove that distinct exposed points x_1 and x_2 of M have distinct associate exposed points x'_1 and x'_2 of M'. Indeed, assume, for contradiction, that $x'_1 = x'_2$. Let H_1, H'_1 and H_2, H'_2 be the respective pairs of parallel lines with the properties

$$M \cap H_1 = \{x_1\}, \qquad M \cap H_2 = \{x_2\}, \qquad M' \cap H'_1 = M' \cap H'_2 = \{x'_1\}.$$

Let also W_1 and W_2 be some neighborhoods of θ that satisfy condition (6) for the pairs x_1, x'_1 and x_2, x'_1, respectively. Then the neighborhood $W = W_1 \cap W_2$ of θ satisfies condition (6) for each of the pairs x_1, x'_1 and x_2, x'_1. As a result, both lines $H_1 = x_1 -$

$x'_1 + H'_1$ and $H' = x_2 - x'_1 + H'_1$ support M such that $M \cap H_1 = \{x_1\}$ and $M \cap H' = \{x_2\}$. Since the lines H_1 and H' are parallel and support M from the same side, they should coincide. The last is in contradiction with $x_1 \neq x_2$.

Finally, let $x' \in \mathcal{F}_0(M')$ and let H' be a line with the property $M' \cap H' = \{x'\}$. Denote by H the line parallel to H' and supporting M such that M lies on the same side from H as M' does with respect to H'. If H supported M along a line segment $[v, w]$, then $[v, w]$ would be an exposed line segment of M with no associate in $\mathcal{F}_1(M')$. Hence the intersection $M \cap H$ is an exposed point x of M. As a result, any exposed point of M' is associate to an exposed point of M. Summing up, we obtain that the relation "is associate to" gives a one-to-one correspondence $\mathcal{F}_0(M) \leftrightarrow \mathcal{F}_0(M')$.

Let $[x, z] \in \mathcal{F}_1(M)$, and let $[x', z'] \in \mathcal{F}_1(M')$ be associate to $[x, z]$. Since the line supporting M along $[x, z]$ is uniquely defined, $[x', z']$ is a unique associate to $[x, z]$. Obviously, distinct line segments from $\mathcal{F}_1(M)$ have distinct associate line segments from $\mathcal{F}_1(M')$.

Conversely, let $[x', z'] \in \mathcal{F}_1(M')$ and let H' be the line with the property $M' \cap H' = [x', z']$. Denote by H the line parallel to H' and supporting M such that M lies on the same side from H as M' does with respect to H'. Assume for a moment that $M \cap H$ consists of a single point v, and let $v' \in \mathcal{F}_0(M')$ be associate to v. As is easily seen, v' should coincide with one of x', z'. Since any neighborhood of x' or z' contains a part of the line segment $[x', z']$, from (7) it follows that $M \cap H$ should contain a line segment parallel to $[x', z']$, contradicting the assumption $M \cap H = \{v\}$. Hence $[x', z']$ is associate to a line segment $[x, z] \in \mathcal{F}_1(M)$. Summing up, we obtain that the relation "is associate to" gives a one-to-one correspondence $\mathcal{F}_1(M) \leftrightarrow \mathcal{F}_1(M')$. □

We continue the proof of the inductive statement for $d = 2$. As is easily seen, any extreme point of a planar convex body is either an exposed point or an endpoint of an exposed line segment of the body. From Claim 1 and conditions (6) and (7) we obtain that for any extreme point x of M there is a unique extreme point x' of M' such that $M' \cap (x' + W_x)$ is a translate of $M \cap (x + W_x)$ for a suitable neighborhood W_x of θ

Denote by $\mathcal{O}(M)$ the family of open line segments $]v, z[$ such that $[v, z] \in \mathcal{F}_1(M)$. Obviously, the family

$$\mathcal{C} = \mathcal{O}(M) \cup \{\mathrm{bd}\, M \cap W_x : x \in \mathrm{ext}\, M\}$$

is an open cover for bd M. Hence bd M is the union of finitely many open arcs $V_1, \dots, V_m \in \mathcal{C}$. From Claim 1 we conclude that the respective translates $V'_1, \dots, V'_m$ of these arcs cover bd M'. The last obviously implies that M' is a translate of M.

Assume that the inductive statement ("M' is a translate of M") is true for all $d \leq n-1$, $n \geq 3$, and let M and M' be convex bodies in E^n that satisfy conditions (6) and (7). Choose a point $a \in \mathcal{F}_0(M)$, and let $a' \in \mathcal{F}_0(M')$ be associate to a. Translating, if necessary, we may assume that $a = a' = \theta$, and that $L \subset E^n$ is an $(n-1)$-dimensional subspace with the property $M \cap L = M' \cap L = \{\theta\}$ and such that both M and M' lie in the same half-space of E^n determined by L. Denote by H and H' the hyperplanes parallel to L that support M and M', respectively ($H \neq L \neq H'$). Our goal is to show that $M = M'$.

Let S_L be the unit sphere of L, and let G be the set of vectors in S_L such that each $e \in G$ is parallel to a line segment from the set $(\mathrm{bd}\, M \cup \mathrm{bd}\, M') \backslash (H \cup H')$. As follows from [4], the $(n-2)$-dimensional measure of G equals 0. Hence the complementary set $F = S_L \backslash G$ is dense in S_L.

For any vector $e \in F$, denote by T_e the $(n-1)$-dimensional subspace of E^n orthogonal to e. Let M_e (respectively, M'_e) be the orthogonal projection of M (respectively, of M') on T_e. Due to the choice of F, any boundary point of M_e (respectively, of M'_e) is the orthogonal projection of a unique boundary point of M (respectively, of M').

Claim 2. *For any $e \in F$, the orthogonal projections M_e and M'_e satisfy conditions* (6) *and* (7).

Proof of Claim 2. Let z be an exposed point of M_e, and let R be an $(n-2)$-dimensional affine set in T_e with the property $M_e \cap R = \{z\}$. If $l(e)$ is the one-dimensional subspace of L containing e, then $R + l(e)$ is a hyperplane in E^n that supports M at a single point, say x. Hence, x is an exposed point of M. By condition (6), M' has an exposed point x' associate to x, and there is a neighborhood $W \subset E^n$ of θ such that $M' \cap (x' + W)$ is a translate of $M \cap (x + W)$. Denote by z' and V, respectively, the orthogonal projections of x' and W on T_e. Then V is a neighborhood of θ in T_e such that $M'_e \cap (z' + V)$ is a translate of $M_e \cap (z + V)$.

Similarly, by condition (7), for any exposed line segment $[u, z]$ of M_e, the set M'_e contains an exposed line segment $[u', z']$ that is associate to $[u, z]$ and is a translate of $[u, z]$. If $u - z$ and $u' - z'$ have the same direction, then, as above, there exists a neighborhood V of θ in T_e such that $M'_e \cap (u' + V)$ is a translate of $M_e \cap (u + V)$ and $M'_e \cap (z' + V)$ is a translate of $M_e \cap (z + V)$. □

By the inductive assumption, from Claim 2 it follows that M'_e is a translate of M_e for any $e \in F$. Since

$$M_e \cap (L \cap T_e) = M'_e \cap (L \cap T_e) = \{\theta\}$$

and both M_e and M'_e lie in the same half-space of T_e determined by its $(n-2)$-dimensional subspace $L \cap T_e$, we have that $M_e = M'_e$.

Obviously, $M \subset M_e + l(e)$ for any $e \in F$. If $x \notin M$, then, using the density of F in S_L, we can find a vector $e \in F$ such that the line $x + l(e)$ through x is disjoint to M. Then the orthogonal projection of x on T_e does not belong to M_e, whence $x \notin M_e + l(e)$. Summing up, we obtain that $M = \cap\{M_e + l(e) : e \in F\}$.

Similarly, $M' = \bigcap\{M'_e + l(e) : e \in F\}$. Since $M_e = M'_e$ for all $e \in F$, we finally have $M' = M$. □

5. Auxiliary Lemmas

This section contains some more auxiliary statements necessary for the proof of Theorem 1.

Lemma 1. *Let X be a nonempty set in E^d, and put $Y = z - X$ for some $z \in E^d$. Then any nonempty intersection $X \cap (x + Y)$, $x \in E^d$, is symmetric about* $(x + z)/2$.

Proof. Obviously, a set $T \subset E^d$ is symmetric about a point $v \in E^d$ if and only if $T - v$ is symmetric about θ. Also, the intersection $T \cap (-T)$, if nonempty, is symmetric about θ. These two observations and the equality

$$X \cap (x+Y) - \frac{x+z}{2} = X \cap (x+z-X) - \frac{x+z}{2} = \left(X - \frac{x+z}{2}\right) \cap \left(\frac{x+z}{2} - X\right)$$

imply that $X \cap (x+Y)$, if nonempty, is symmetric about $(x+z)/2$. □

Lemma 2. *Let subspaces $L_1, \ldots, L_k \subset E^d$ form a direct sum, and let $S_i, T_i \subset L_i$ and $x_i \in L_i$ be such that $S_i \cap (x_i + T_i) \neq \emptyset$ for all $i = 1, \ldots, k$. Put*

$$S = S_1 \oplus \cdots \oplus S_k, \qquad T = T_1 \oplus \cdots \oplus T_k, \qquad x = x_1 + \cdots + x_k.$$

Then the intersection $S \cap (x+T)$ is centrally symmetric if and only if all intersections $S_i \cap (x_i + T_i)$, $i = 1, \ldots, k$, are centrally symmetric.

Proof. Obviously,

$$S \cap (x+T) = [S_1 \cap (x_1 + T_1)] \oplus \cdots \oplus [S_k \cap (x_k + T_k)].$$

If each set $S_i \cap (x_i + T_i)$ is symmetric about $z_i \in L_i$, $i = 1, \ldots, k$, and $z = z_1 + \cdots + z_k$, then the equality

$$\begin{aligned} S \cap (x+T) - z &= [S_1 \cap (x_1 + T_1) - z_1] \oplus \cdots \oplus [S_k \cap (x_k + T_k) - z_k] \\ &= [z_1 - S_1 \cap (x_1 + T_1)] \oplus \cdots \oplus [z_k - S_k \cap (x_k + T_k)] \\ &= z - S \cap (x+T) \end{aligned}$$

implies that $S \cap (x+T)$ is symmetric about z.

Conversely, let the intersection $S \cap (x+T)$ be symmetric about a point $z \in E^d$. Clearly, $z \in L_1 \oplus \cdots \oplus L_k$. Denote by φ_i the parallel projection of $L_1 \oplus \cdots \oplus L_k$ onto L_i along $L_1 \oplus \cdots \oplus L_{i-1} \oplus L_{i+1} \oplus \cdots \oplus L_k$, and let $z_i = \varphi_i(z)$, $i = 1, \ldots, k$. Then

$$z_i - S_i \cap (x_i + T_i) = \varphi_i(z - S \cap (x+T)) = \varphi_i(S \cap (x+T) - z) = S_i \cap (x_i + T_i) - z_i.$$

Hence each set $S_i \cap (x_i + T_i)$ is symmetric about z_i, $i = 1, \ldots, k$. □

Lemma 3. *Let $X_1, X_2, \ldots$ be a sequence of centrally symmetric compact sets in E^d convergent in the Hausdorff metric to a bounded set X. Then the limit set X is also centrally symmetric.*

Proof. Let X_i be symmetric about a point z_i, $i = 1, 2, \ldots$ Since $X_i \to X$, all the sets X_i are situated in a neighborhood of X, and, as a result, the sequence $z_1, z_2, \ldots$ is bounded. If $z_{i_1}, z_{i_2}, \ldots$ is a subsequence of $z_1, z_2, \ldots$ that converges to a point z, then

$$X - z = \lim_{j \to \infty} (X_{i_j} - z_{i_j}) = \lim_{j \to \infty} (z_{i_j} - X_{i_j}) = z - X,$$

i.e., X is symmetric about z. □

6. Proof of Theorem 1

First we prove the equivalence of conditions (1) and (2).

Since (1) obviously implies (2), it is sufficient to show that (2) $\Rightarrow$ (1). Let $X = K \cap (x + K')$, $x \in E^d$, be nonempty, and choose a point $y \in X$. Then there is a sequence $y_1, y_2, \ldots$ of points from int K that converges to y. Consider the intersections $X_i = K \cap (x + y_i - y + K')$, $i = 1, 2, \ldots$ Since $y_i \in (\text{int}\, K) \cap (x + y_i - y + K)$, each set X_i has dimension d. By (2), all X_i are centrally symmetric. Since $X_i \to X$ in the Hausdorff metric when $i \to \infty$, X is centrally symmetric itself (see Lemma 3).

The remaining part of the proof is devoted to the equivalence of conditions (1) and (3). Lemmas 1 and 2 above obviously imply that (3) $\Rightarrow$ (1). Hence, it remains to show that (1) $\Rightarrow$ (3). Since the case $d = 1$ is trivial, we assume that $d \geq 2$. If $K' = z - K$ for a suitable vector $z \in E^d$, we have finished the proof. Assume that $K' \neq z - K$ for any $z \in E^d$. Then Theorems 2 and 3 imply the existence of an exposed line segment $[a, b]$ of K that has an associate exposed line segment $[a', b']$ of $-K'$ such that $[a', b']$ is not a translate of $[a, b]$. By a symmetry argument, we may assume that $\|a' - b'\| < \|a - b\|$.

Translating K and $-K'$, if necessary, we may assume that $a = a' = \theta$. Condition (1) implies the existence of a hyperplane H supporting both K and $-K'$ such that K and $-K'$ lie in the same closed half-space P determined by H, with $K \cap H = [\theta, b]$ and $(-K') \cap H = [\theta, b']$, where $b' \in]\theta, b[$. Moreover, there is a neighborhood W of the origin θ such that $K \cap W = (-K') \cap W$. Hence for any point $x \in \text{int}\, K \cap W$, the intersection $K \cap (x + K')$ is a convex body, centrally symmetric about $x/2$ (see Lemma 2). In particular, H supports $K \cap (x + K')$ along a line segment $[\theta, c]$, $c \in]\theta, b]$, and the hyperplane $x + H$ supports $K \cap (x + K')$ along the line segment $[x, x - c]$. Moreover, $\theta \in \text{int}(x + K')$, as shown in the proof of Theorem 2.

Denote by l the line containing the segment $[\theta, b]$, and let l_x be the line through x parallel to l. By a continuity argument, the point x above can be chosen so close to θ that the line segment $l_x \cap K$ becomes arbitrarily close to $[\theta, b]$; in particular, $l_x \cap K$ becomes longer than $[\theta, b']$. Thus we can translate $x + K'$ along the line l_x into a position $x + w + K'$, $w \in l$, such that the exposed line segment $[x + w, x + w - b']$ of the body $x + w + K'$ lies in int K.

Claim 3. *For any points $z \in]\theta, b[$ and $z' \in]\theta, b'[$, there is a neighborhood W of θ such that $(-K') \cap (z' + W)$ is a translate of $K \cap (z + W)$.*

Proof of Claim 3. First we choose z' to be the middle point of $[x + w, x + w - b']$. Since $K \cap (x + w + K')$ is centrally symmetric, we obtain that the point $z \in [\theta, b]$ symmetric to z' satisfies the conclusion of Claim 3. Shifting the body $x + w + K'$ both ways along the line l_x such that $[x + w, x + w - b']$ remains in K, and using the symmetry of intersections $K \cap (x + w + K')$, we obtain that any point $u \in [\theta, b]$ from a small neighborhood of z satisfies, together with z', the conclusion of Claim 3. Coming back to z', we obtain that any point $u \in [x + w, x + w - b']$ from a small neighborhood of z', satisfies, together with z, the conclusion of Claim 3. Continuing along this way, we get the proof of Claim 3. □

Claim 3 implies the following corollary.

Corollary 2. *For any points $z \in]\theta, b[$ and $z' \in]\theta, b'[$, the generated cones*

$$C_z(K) = \{z+\lambda(x-z) : x \in K, \lambda \geq 0\}, \qquad C_{z'}(K') = \{z'+\lambda(x-z') : x \in K', \lambda \geq 0\}$$

satisfy the relation $C_{z'}(K') + z' = z - C_z(K)$, and each of these cones contains the line l.

Claim 4. *The line segment $[\theta, -b']$ is an affine diameter of K', i.e., there are distinct parallel hyperplanes through θ and $-b'$, respectively, both supporting K'.*

Proof of Claim 4. Equivalently, $[x + w, x + w - b']$ is stated to be an affine diameter of $x + w + K'$. It is known (see, e.g., [2]) that a chord $[r, s]$ of a convex body $C \subset E^d$ is an affine diameter of C if and only if $[r, s]$ is a longest chord of C in the direction parallel to $[r, s]$.

Assume, for contradiction, that $[x + w, x + w - b']$ is not an affine diameter of $x + w + K'$. Then there exists a line segment $[p, q] \subset (x + w + K')$ parallel to l and longer than $[x + w, x + w - b']$. By a continuity argument, we may consider that $]p, q[\subset \operatorname{int}(x + w + K')$. Then the relative interior of the trapezoid A with bases $[x+w, x+w-b']$ and $[p, q]$ lies in the interior of $x+w+K'$. Due to Corollary 2, we may choose the point $x \in \operatorname{int} K \cap W$ and the respective point $w \in l$ such that A intersects l along a line segment $[p_1, q_1]$ that lies inside $[\theta, b]$. Obviously, the hyperplanes $x + H$ and H support the symmetric convex body $K \cap (x + w + K')$ along the line segments $[x + w, x + w - b']$ and $[p_1, q_1]$, respectively, a contradiction with the fact that $[p_1, q_1]$ is longer than $[x + w, x + w - b']$. Thus $[x + w, x + w - b']$ is a longest chord of $x + w + K'$ in the direction l, whence it is an affine diameter of $x + w + K'$. □

Claim 5. *There is a hyperplane T through θ and not containing l such that K' has a pair of $(d - 1)$-dimensional exposed faces parallel to T and containing the points θ and $-b'$, respectively.*

Proof of Claim 5. Since $[\theta, -b']$ is an affine diameter of K', there is a hyperplane T supporting K' that passes through θ and does not contain l such that the hyperplane $T - b'$ also supports K'. We prove that the sets $K' \cap T$ and $K' \cap (T - b')$ are the required $(d - 1)$-dimensional exposed faces of K'.

First we show the existence of a neighborhood V of the point $t = -b'/2$ such that the line segment $(z + l) \cap K'$ is of length at least $\|b'\|$ for any point $z \in K' \cap V$. An obvious modification of the considerations preceding Claim 3 implies the existence of a point $x \in K$ close to $-t$ and of a point $w \in l$ such that $[w + x, w + x - b']$ lies in K. Moreover, Claim 3 implies the existence of a neighborhood U of $-t$ such that $[w + x, w + x - b'] \subset K$ for all $x \in K \cap U$. Furthermore, U can be chosen such that $(-K') \cap U = K \cap U$. Then each intersection $K \cap (x + w + K')$, $x \in K \cap U$, is centrally symmetric and is supported by the hyperplane $x + H$ along the line segment $[x + w, x + w - b']$. Hence each $K \cap (x + w + K')$, $x \in K \cap U$, is supported by H along a line segment $[r, r - b']$ that lies in l and is a translate of $[x + w, x + w - b']$. Obviously, the line segment $[r - x - w, r - x - w - b']$ is of length $\|b'\|$ and lies in

$K' \cap (l - x - w) = K' \cap (l - x)$. Therefore, the segment $K' \cap (l - x)$ is of length at least $\|b'\|$. Finally, put $V = -U$.

On the other hand, $[\theta, -b']$ is a longest chord of K' in the direction l. Hence the line segment $(z + l) \cap K'$ is exactly of length $\|b'\|$ for any point $z \in K' \cap V$. Since each such segment lies between the parallel hyperplanes T and $T - b'$, its endpoints lie on T and $T - b'$, respectively. Obviously, these endpoints fill some $(d - 1)$-dimensional sets in T and $T - b'$, respectively. Thus both sets $K' \cap T$ and $K' \cap (T - b')$ are $(d - 1)$-dimensional. □

Claim 6. *K and K' are similarly represented as direct sums $K = Q \oplus [\theta, b]$ and $K' = Q' \oplus [\theta, b']$, where Q and Q' are $(d - 1)$-dimensional compact convex sets in T.*

Proof of Claim 6. In view of Corollary 2 and Claim 5, it is sufficient to prove that for any two-dimensional plane L through l, both intersections $P = K \cap L$ and $P' = K' \cap L$, if two-dimensional, are isothetic parallelograms.

According to the consideration above, P is supported by the line l along the segment $[\theta, b]$, and P' is supported by l along the line segment $[\theta, -b']$. Moreover, P' is supported by the lines $R = L \cap T$ and $R - b'$ along some segments $[\theta, v']$ and $[-b', w']$, respectively, where T is the hyperplane defined in Claim 5.

Since any nonempty intersection $K \cap (x + K')$ is supported by the hyperplanes H and $x + H$ along the line segments $[\theta, w] \subset l$ and $[x, x - w]$, respectively, and since both these segments lie in L, we conclude that for any point $x \in L$ the set $P \cap (x + P')$ equals $L \cap K \cap (x + K')$. Hence $P \cap (x + P')$, $x \in L$, is centrally symmetric if and only if $K \cap (x + K')$ is centrally symmetric.

To show that P and P' are isothetic parallelograms, we consider only those intersections $P \cap (x + P')$, $x \in L$, which are parallelograms, and, as a consequence, derive the respective properties of the boundaries of P and P'. For simplicity, our considerations are performed in the plane L, such that both P and P' have nonempty interior.

Choose a point $x \in \operatorname{int} P$ such that $[x, x - v']$ intersects $[\theta, b]$ and $[x, x - b']$ intersects the boundary of P. From the central symmetry of $P \cap (x + P')$ we conclude that $P \cap (x + P')$ has to be a parallelogram. Then bd P contains a line segment $[\theta, v]$ that lies in R. By a similar argument, bd P contains a line segment $[b, w] \subset b + R$.

Considering the possible cases $\|v\| < \|v'\|$, $\|v\| > \|v'\|$, $\|v\| = \|v'\|$, we first assume that $\|v\| < \|v'\|$. Then there is a scalar $\lambda > 0$ such that $\lambda v + x + P'$ entirely contains $[\theta, v]$ and $[\lambda v + x, \lambda v + x - v']$ still intersects $[\theta, b]$. Since $P \cap (\lambda v + x + P')$ is a parallelogram, bd P contains a line segment $[v, u]$ parallel to l. Moving $\lambda v + x + P'$ further along the ray $\{\lambda v : \lambda > 0\}$ and looking for the intersection of P and $\lambda v + x + P'$, we obtain that bd P' contains a line segment $[v', u']$ parallel to l. Now moving $v + P'$ along the ray $\{\lambda b : \lambda > 0\}$ and, if necessary, again along the ray $\{\lambda v : \lambda > 0\}$, we obtain that $\|u - v\|$ is at least $\|b'\|$, and P' is a parallelogram. Further movement of $v + P'$ along the ray $\{\lambda v : \lambda > 0\}$ gives us that P is also a parallelogram isothetic to P'.

In a similar way, any of the cases $\|v\| > \|v'\|$, $\|w\| > \|w'\|$, $\|w\| < \|w'\|$ gives us that P and P' are isothetic parallelograms. It remains to assume that $\|v\| = \|v'\|$ and $\|w\| = \|w'\|$. Then moving $b' + P'$ along the ray $\{\lambda v : \lambda > 0\}$ we get that $w = v$ and $w' = v'$, i.e., that P and P' are isothetic parallelograms. □

We finalize the proof of Theorem 1 by induction on d. The case $d = 2$ is confirmed in the proof of Claim 6. Assume that (1) $\Rightarrow$ (3) is true for all $d \le n - 1$, $n \ge 2$, and let convex bodies $K, K' \subset E^n$ satisfy condition (1). By Claim 6, K and K' are similarly represented as direct sums $K = Q \oplus [\theta, b]$ and $K' = Q' \oplus [\theta, b']$, where Q and Q' are $(n-1)$-dimensional compact convex sets in the $(n-1)$-dimensional subspace T of E^n. Obviously, $Q \cap (x + Q')$, $x \in T$, is nonempty if and only if $K \cap (x + K')$ is nonempty. By condition (1), every nonempty intersection $Q \cap (x + Q')$, $x \in T$, is centrally symmetric, and, by the inductive assumption, Q and Q' satisfy condition (3), with $n - 1$ instead of d. From $K = Q \oplus [\theta, b]$ and $K' = Q' \oplus [\theta, b']$ we obviously conclude that K and K' satisfy condition (3), with n instead of d. □

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