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An Improved Bound for Joints in Arrangements of Lines in Space*

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Abstract. Let L be a set of n lines in space. A *joint* of L is a point in $\mathbb{R}^3$ where at least three non-coplanar lines meet. We show that the number of joints of L is $O(n^{112/69}\log^{6/23} n) = O(n^{1.6232})$, improving the previous bound $O(n^{1.643})$ of Sharir.

1. Introduction

Let L be a set of n lines in space. A *joint* of L is a point in $\mathbb{R}^3$ where at least three non-coplanar lines ℓ, ℓ', ℓ'' of L meet. We denote the joint by any such triple of lines (ℓ, ℓ', ℓ'').

Let $\mathcal{J}_L$ denote the set of joints of L, and put $J(n) = \max|\mathcal{J}_L|$, taken over all sets L of n lines in space. A trivial upper bound on $J(n)$ is $O(n^2)$, as a joint is a point of intersection of (more than) two lines, but it was shown in [11], following a weaker subquadratic bound in [4], that $J(n)$ is only $O(n^{23/14}\ \mathrm{polylog}(n)) = O(n^{1.643})$. An easy construction, based on lines forming an $n^{1/2} \times n^{1/2} \times n^{1/2}$ portion of the integer grid, shows that $|\mathcal{J}_L|$ can be $\Omega(n^{3/2})$ (see Fig. 1 and [4]). The goal of this paper is to narrow the gap between these upper and lower bounds.

One of the main motivations for studying joints of a set L of lines in space is their connection to *elementary cycles* of L. An elementary cycle is a set $L' \subseteq L$ of at least three lines with the following properties: (i) The xy-projections of the lines in L' all

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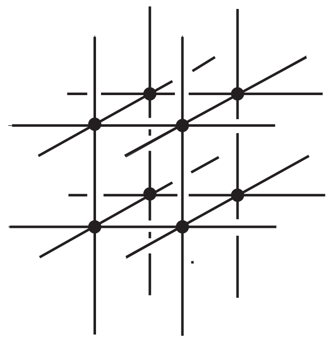

Fig. 1. The lower bound construction for joints, illustrated with $n = 12$ lines.

bound a common face in the arrangement of the xy-projections of the lines in L. (ii) As we go around the boundary of the common face, we always pass from the projection of one line ℓ to the projection of another line ℓ' such that ℓ' passes above ℓ in 3-space. See Fig. 2.

A major open problem in the study of visibility in three dimensions is to obtain a subquadratic bound on the number of elementary cycles of a set of lines in $\mathbb{R}^3$. Joints can be regarded as a degenerate case of elementary cycles. In fact, a slight random perturbation of the lines in L turns any joint incident to $O(1)$ lines into an elementary cycle with some constant probability, implying that the number of joints is strongly related to the number of elementary cycles.

Unfortunately, very little is known about the number of elementary cycles. Chazelle et al. [4] obtained a bound of $O(n^{9/5})$ for the special case of line segments (rather than lines) whose xy-projections form a (distorted) grid. Recently, Aronov et al. [2] obtained a subquadratic bound on the number of *triangular elementary cycles* (i.e., cycles formed by only three lines) for general line arrangements. Solan [13] and Har-Peled and Sharir [9] have given algorithms that eliminate all (not necessarily elementary) cycles of a set of lines in space, by cutting the lines at appropriate points. These algorithms run in subquadratic time, and cut the lines in a subquadratic number of points, provided that there exists a subquadratic bound on the number of cuts that eliminate all cycles.

The problem of joints is considerably simpler, as witnessed by the much sharper upper bound of [11], mentioned above. Still, it is a rather challenging problem, open for 10

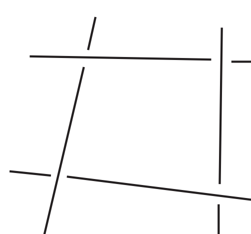

Fig. 2. An elementary cycle of lines in space.

years, to tighten the gap between the upper and lower bounds. It is our hope that better insights into the joints problem would lead to tools that could also be used to obtain subquadratic bounds for elementary cycles, and for many other problems that involve lines in space. Recently, Sharir and Welzl [12] have shown that the number of incidences between the points in $\mathcal{J}_L$ and the lines in L is $O(n^{5/3})$.

In this paper we improve the upper bound on $J(n)$ to $O(n^{112/69}\log^{6/23} n) = O(n^{1.6232})$. The proof proceeds by mapping the lines of L into points and/or hyperplanes in projective 5-space, using *Plücker coordinates* [5]. We then apply a two-stage decomposition process, which partitions the problem into subproblems, using *cuttings* of arrangements of appropriate subsets of the Plücker hyperplanes. We estimate the number of joints within each subproblem, and sum up the resulting bounds to obtain the bound asserted above. The proof adapts and applies some of the tools used by Sharir and Welzl [12] and recently enhanced by Aronov et al. [1], related mainly to the connection between joints and *reguli* spanned by the lines of L; see below for more details.

2. The Upper Bound

2.1. *The Toolbox*

We begin by recalling and developing some of the tools we need for our proof.

Szemerédi-Trotter Point-Line Incidence Bound [15]. Let L be a set of n lines and let P be a set of m points, both in a common (two-dimensional) plane. The number $I(P, L)$ of incidences between the points of P and the lines of L satisfies

$$I(P, L) = O(n^{2/3}m^{2/3} + n + m). \tag{1}$$

This bound is tight in the worst case. See [10] for more details. A corollary of (1) is that the number of incidences between the points of P and the lines that are incident to at least k points of P is at most

$$O\left(\frac{m^2}{k^2} + m\right).$$

We use this bound to prove

Lemma 2.1. *Let L be a set of n lines in space. The number of containments between the lines of L and those planes that contain at least k lines of L is*

$$O\left(\frac{n^2}{k^2} + n\right).$$

Proof. Take a generic plane π, so that each line of L intersects π at a distinct point. A plane containing at least k lines of L intersects π at a line that contains at least k of these points. The lemma is then an immediate consequence of the corollary to the Szemerédi–Trotter theorem. $\square$

Reguli (*see* [14]). Given three pairwise skew lines ℓ_1, ℓ_2, ℓ_3, the set $\sigma = \sigma(\ell_1, \ell_2, \ell_3)$ of lines intersecting all three lines is called a *regulus*. All lines in σ are pairwise skew. If $\ell'_1, \ell'_2, \ell'_3$ are in σ, then $\sigma^\perp = \sigma(\ell'_1, \ell'_2, \ell'_3)$ constitutes another regulus, that is independent of the choice of the three lines in σ. (Note that the three generating lines ℓ_1, ℓ_2, ℓ_3 of σ *do not* belong to σ, but rather to $\sigma^\perp$.) Both σ and $\sigma^\perp$ span the same ruled surface in 3-space.

In more detail, $\bigcup_{\ell\in\sigma} \ell = \bigcup_{\ell\in\sigma^\perp} \ell$ is a *ruled surface* (which is a quadric—a hyperboloid of one sheet or a hyperbolic paraboloid) in $\mathbb{R}^3$, denoted by $\sigma^* = \sigma^*(\ell_1, \ell_2, \ell_3)$; σ and $\sigma^\perp$ are called the *generating families of* σ^* and we say that $\sigma^\perp$ is the *complementary regulus of* σ, and vice versa: $(\sigma^\perp)^\perp = \sigma$. Every point in σ^* is contained in exactly one line from σ and in exactly one line from $\sigma^\perp$. For any line ℓ in $\mathbb{R}^3$, either $\ell \in \sigma \cup \sigma^\perp$ (in particular, $\ell \subseteq \sigma^*$) or ℓ intersects σ^* in at most two points.

It follows that the number of joints in L that lie on the surface σ^* of any regulus σ is at most

$$\min\{|L \cap \sigma| \cdot |L \cap \sigma^\perp|,\ 2|L|\}.$$

This follows from the observation that at most two of the lines that form such a joint can lie in σ^*, and the third line must cross σ^*, and thus participates in at most two joints there. This allows us to apply the following pruning procedure. We fix a parameter s, whose value will be determined later. As long as there exists a regulus σ that contains more than s lines of L, we remove all these lines from L, and lose in this process at most $2n$ joints. Repeating this step at most n/s times, we eliminate all "heavy" reguli and at most $O(n^2/s)$ joints.

A similar pruning process can be applied to planes that contain more than s lines of L. Here we use the fact that any plane can contain at most n joints, because any such joint must be incident to at least one line that is not contained in the plane, and thus meets it in a single point.

To recap, we may (and will) assume in what follows that no plane or regulus contains more than s lines of L, and will add $O(n^2/s)$ to the overall bound for the number of joints.

Incidences between Lines and Reguli. Given a set L of m lines and a set R of n reguli in 3-space, the number $I(L, R)$ of incidences between the lines of L and the reguli of R (recall that we regard a regulus as a set of lines and not as the surface that they span) satisfies

$$I(L, R) = O(m^{4/7} n^{17/21} + m^{2/3} n^{2/3} + n + m). \tag{2}$$

This has recently been shown by Aronov et al. [1]. It extends and strengthens a weaker bound of $O(m^{3/5} n^{4/5} + m + n)$ proved in [12] for a more restricted situation.

We use this to prove:

Lemma 2.2. *Let L be a set of n lines in space. The number of incidences between the lines of L and those reguli that contain at least k lines of L is*

$$O\left(\frac{n^3}{k^{17/4}} + \frac{n^2}{k^2} + n\right).$$

Proof. Let $R_{\geq k}$ denote the set of these reguli, and put $t = |R_{\geq k}|$. The bound (2) implies that

$$tk \leq I(L, R_{\geq k}) = O(n^{4/7}t^{17/21} + n^{2/3}t^{2/3} + n + t).$$

The lemma then follows by bounding t using this inequality, and by substituting the resulting bound into (2). □

Mapping into Plücker Space. Let L be a set of n lines in $\mathbb{R}^3$. We may assume, without loss of generality, that no pair of lines in L are parallel. This can be enforced by an appropriate projective transformation that maps L to another set of lines that does not have parallel pairs, without changing the incidence structure between the lines and their joints.

We start by replicating the set of lines L into two sets, color one blue and the other red. We bound the number of points at which a red line and two blue lines, not in the same plane, meet.[1]

We map each blue line ℓ to its *Plücker hyperplane* π_ℓ, and map each red line ℓ into its *Plücker point* p_ℓ. Both points and hyperplanes lie in real projective 5-space, and the points all lie in a four-dimensional quadric surface Π known as the *Plücker surface*. Two lines $\ell, \ell' \in L$ meet if and only if p_ℓ lies on $\pi_{\ell'}$ (and $p_{\ell'}$ lies on π_ℓ). See [5] for more details on this transformation.

Cuttings. Let Γ be a set of n algebraic arcs or curves in the plane, of constant maximum degree, and let $1 \leq r \leq n$ be a parameter. A $(1/r)$-*cutting* of the arrangement $\mathcal{A}(\Gamma)$ of Γ is a partition of $\mathbb{R}^2$ into pairwise disjoint relatively open cells[2] of dimensions 0, 1, 2, such that each cell is *crossed by* (i.e., intersected by, but not contained in) at most n/r curves of Γ. The *size* of the cutting is the number of its cells. It has been shown (see [6] and [8]) that there always exists a $(1/r)$-cutting of size $O(r^2)$, which is asymptotically optimal.

The notion of cuttings can be extended in an obvious manner to arrangements of surfaces in higher dimensions. In general, however, optimal or near-optimal bounds for the size of the cuttings are harder to derive, and in most cases are not yet known. Still, in the case of hyperplanes in $\mathbb{R}^d$, there exist $(1/r)$-cuttings, whose cells are simplices, of optimal size $O(r^d)$ [6]. In our analysis, we repeatedly rely on a variant of this result, in which we need to construct $(1/r)$-cuttings for a four-dimensional cross section (within the Plücker surface) of an arrangement of hyperplanes in projective 5-space; see the following subsection for more details. See also [7] for related applications of cuttings for incidence counting problems.

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[1] In the first decomposition stage the colors play no significant role, but they will be more meaningful in the second decomposition stage, where each subproblem will involve two different subsets of L.

[2] In the standard definition of a cutting, the cells are required to have *constant descriptive complexity*, meaning that each of them is defined by a constant number of polynomial equalities and inequalities, involving polynomials of constant maximum degree. In our application, though, this additional property is not needed.

2.2. *Overview of the Proof*

We first present a rather informal and brief overview of the proof, which highlights the main ideas and ignores most of the technical details. We regard the lines in L as either red or blue, and bound the number of joints incident to two blue lines and one red line.

Using Plücker coordinates, we map each blue line to its Plücker hyperplane, and each red line to its Plücker point. We use a two-stage cutting-based decomposition scheme in the Plücker space, first in a primal setting and then in a dual setting that flips between points and hyperplanes (so in the second stage the blue lines are treated as points and the red lines as hyperplanes). Using standard machinery, we construct a $(1/r)$-cutting in the primal space, and obtain roughly r^4 cells (of dimensions between 0 and 4), each crossed by at most n/r hyperplanes and containing at most roughly n/r^4 points. After the flip to the dual space, applied to the crossing hyperplanes and the contained points of each primal cell separately, we obtain a total of about r^8 cells, each crossed by at most roughly n/r^5 hyperplanes (formerly points) and containing at most roughly n/r^5 points (formerly hyperplanes).

Joints that involve at least one crossing hyperplane and at least one point in one of these dual cells can simply be interpreted as intersection points of the corresponding pairs of lines in 3-space, and their number can therefore be bounded by roughly $r^8 \cdot (n/r^5)^2 = n^2/r^2$. The largest value of r for which this reasoning applies is roughly $r = n^{1/5}$, and we thus obtain a bound close to $n^{8/5}$.

However, the analysis is not that simple. The main difficulty lies in analyzing joints that involve a line whose corresponding Plücker point lies in some lower-dimensional cell of either the primal or the dual cuttings. The problem is that one of the other lines involved in the joint may have a Plücker hyperplane that *fully contains the cell* rather than crosses it, and the number of such hyperplanes can be arbitrarily large. Estimating the number of these "containment joints" is the hardest part of the analysis. Fortunately, though, containment joints possess a lot of structure. They generally lie on a small number of planes or reguli, and we use the various incidence bounds derived above to obtain sharp bounds on the number of such joints.

In the remainder of this section we present the technical details of the steps that we have just outlined.

2.3. *The Primal Partitioning Stage*

We construct a $(1/r)$-*cutting* Ξ of the arrangement of the set H of the Plücker hyperplanes, or, more precisely, of its cross section within the Plücker surface Π. The cutting is obtained by taking a random sample R of r hyperplanes of H, by triangulating each cell of $\mathcal{A}(R)$, and by taking the cross sections within Π of the resulting simplices. The actual construction is somewhat more involved, and follows the technique of Chazelle and Friedman [6], which uses additional samplings within some of the cells constructed above.[3] Omitting the routine details (for which see [6]), we end up with a larger sample,

[3] It might be simpler to digest the following analysis by ignoring the Chazelle–Friedman refinement. This will only affect the polylogarithmic factor appearing in the overall bound.

which we still denote by R, consisting of $O(r)$ hyperplanes, and yielding a cutting that consists of $O(r^4 \log r)$ cells of constant descriptive complexity (each cell is the intersection of some j-simplex, for $1 \le j \le 5$, with Π), so that each cell is crossed by at most n/r blue Plücker hyperplanes. (The size of the cutting is a consequence of the Zone Theorem of Aronov et al. [3], which implies that the complexity of the *zone* of Π in $\mathcal{A}(R)$, that is, the sum of the complexities of all the cells of $\mathcal{A}(R)$ crossed by Π, is $O(r^4 \log r)$, from which it follows that the cells of $\mathcal{A}(R)$ that are crossed by Π can be triangulated into $O(r^4 \log r)$ simplices.) Moreover, by splitting cells into subcells, if necessary, we may also assume that each cell contains at most $n/(r^4 \log r)$ red Plücker points. (Recall that lower-dimensional cells may be *contained* in many more blue hyperplanes, but each is *crossed* by at most n/r of them.)

We now bound the number of red–blue–blue joints by applying a case analysis on the location, within the cutting Ξ, of the Plücker point of the red line in the joint.

Vertices of Ξ. Consider a joint (ℓ_1, ℓ_2, ℓ_3), for $\ell_1, \ell_2, \ell_3 \in L$, such that p_{ℓ_1} is a vertex of Ξ. The number of such joints is at most $\sum_v d_v$, where the sum is over the vertices v of Ξ and d_v is the number of lines $\ell \in L$ such that π_ℓ passes through v. We denote the set of these lines as $L^{(v)}$. We may assume that v is a vertex formed as the transversal intersection of Π with (at least) four hyperplanes of R. Any other vertex of Ξ will not coincide with a Plücker point p_ℓ, for $\ell \in L$, provided that the triangulation is performed in a sufficiently generic manner.[4]

We fix a hyperplane π_ℓ, for $\ell \in L^{(v)}$, and intersect it with all hyperplanes of R and with Π. Since the four hyperplanes of R that form the vertex v intersect there transversally, their cross sections within $\pi_\ell \cap \Pi$ also intersect transversally at v, so this point is a vertex of the three-dimensional arrangement of these cross sections. The number of such vertices, within $\pi_\ell \cap \Pi$, is at most $O(r^3)$, for a total bound of $O(nr^3)$ on the number of joints at vertices of Ξ.

Edges of Ξ. Let γ be an intersection curve of three hyperplanes of R with Π. (As in the case of vertices, only edges of Ξ contained in such curves are of interest, if the triangulation is sufficiently generic. Note also that we consider here full intersection curves, each consisting of many edges of Ξ.) Let ℓ_1, ℓ_2, ℓ_3 be the three corresponding lines of L. Suppose first that these lines are pairwise skew and thus define a regulus σ. Let $\ell \in L$ be such that $p_\ell \in \gamma$. Then ℓ lies in σ^* (and belongs to σ). Let (ℓ, ℓ', ℓ'') be a joint that involves ℓ. It is impossible that both $\pi_{\ell'}, \pi_{\ell''}$ fully contain γ, because then ℓ', ℓ'' would belong to $\sigma^\perp$ and thus would not meet at all. Hence, say, $\pi_{\ell'}$ crosses γ, and ℓ' crosses σ^*, in at most two points. In other words, we can charge the joint under consideration to one of these crossing points of ℓ' with σ^*. The number of such crossings is at most $2n$ for each regulus σ, for a total of $O(nr^3)$ joints.

Suppose next that two of the lines, say ℓ_1, ℓ_2, meet each other. Thus they define a common plane h and a common point q. If the third line ℓ_3 lies in h or passes through q, then the intersection $\pi_{\ell_1} \cap \pi_{\ell_2} \cap \pi_{\ell_3} \cap \Pi$ is two-dimensional, as is easily seen, so these three lines do not define a *one-dimensional* intersection curve that induces *edges* of Ξ.

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[4] For example, each cell can be triangulated into simplices, all emanating from some common generic point in the relative interior of the cell.

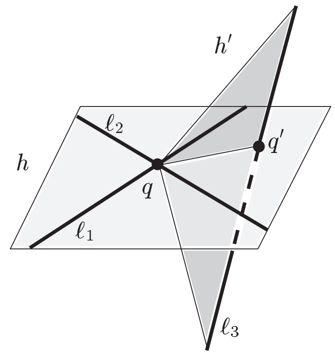

Fig. 3. The pair of planes corresponding to an edge of Ξ.

Hence ℓ_3 meets h at a single point $q' \neq q$. It follows that any line ℓ with $p_\ell \in \gamma$ either lies in h and passes through q', or passes through q and through ℓ_3, and thus lies in the plane h' spanned by q and ℓ_3. See Fig. 3. In other words, any joint on ℓ lies in $h \cup h'$, and at least one of the three lines forming the joint must cross h or h' at the joint. There are at most $2n$ such crossing points, so the number of joints in this case is at most $2n$, for a total of $O(nr^3)$ joints.

2-*Faces of* Ξ. Let φ be an intersection 2-surface of two hyperplanes of R with Π (again, only 2-faces of Ξ that lie in such 2-surfaces are of interest, and we consider full intersection 2-surfaces rather than individual 2-faces), and let ℓ_1, ℓ_2 be the two corresponding lines of L. Suppose first that ℓ_1, ℓ_2 pass through a common point q, and thus lie in a common plane h. Then any line λ with $p_\lambda \in \varphi$ either lies in h or passes through q. We can thus view φ as the union of two subsurfaces φ_q, φ_h, where φ_q (resp., φ_h) is the locus of all points representing lines passing through q (resp., lying in h).

Let (ℓ, ℓ', ℓ'') be a joint where $p_\ell \in \varphi_q$. We may assume that p_ℓ does not lie on any edge of Ξ that is contained in φ, because such points have already been accounted for. If $\pi_{\ell'}$, say, fully contains φ_q, then ℓ' must pass through q (since it touches every line that passes through q), and thus the joint in question must be the point q itself. The overall number of such joints is only $O(r^2)$. We may thus assume that both $\pi_{\ell'}$ and $\pi_{\ell''}$ cross φ_q.

Similarly, let (ℓ, ℓ', ℓ'') be a joint where $p_\ell \in \varphi_h$. If $\pi_{\ell'}$, say, fully contains φ_h, then ℓ' must lie in h. In this case the joint must lie in h. As we have already noted, h contains at most n joints, so the overall number of joints of this kind is at most $O(nr^2)$. We may thus assume that both $\pi_{\ell'}$ and $\pi_{\ell''}$ cross φ_h.

Thus, in either case, we are left with subproblems, each associated with a 2-face τ of Ξ (the surface φ is now decomposed back into its constituent 2-faces), such that τ contains at most $n/(r^4 \log r)$ red Plücker points and is crossed by at most n/r blue Plücker hyperplanes; the problem associated with τ considers red–blue–blue joints where the red point lies in τ and both blue hyperplanes cross τ. The number of subproblems is $O(r^4 \log r)$. We handle these subproblems in the second dual stage of the analysis—see below.

Finally, suppose that ℓ_1 and ℓ_2 are skew. Consider a joint (ℓ, ℓ', ℓ''), where $p_\ell \in \varphi$. Neither of the hyperplanes $\pi_{\ell'}, \pi_{\ell''}$ can fully contain φ, because then the corresponding line would have to be incident to every line that meets ℓ_1 and ℓ_2, which is clearly impossible. Hence, in this case we obtain, as above, a collection of subproblems, each associated with a 2-face τ of Ξ (a subface of φ), such that τ contains at most $n/(r^4 \log r)$ red Plücker points and is crossed by at most n/r blue Plücker hyperplanes. As above, the number of subproblems is $O(r^4 \log r)$, and they are all handled in the second dual stage of the analysis.

3-*Faces of* Ξ. Let p_ℓ be a point in the relative interior of some 3-face of Ξ, contained in the intersection of Π with some hyperplane π_{ℓ_1} in R (only such 3-faces are of interest). Any hyperplane incident to p_ℓ, with the exception of π_{ℓ_1}, crosses at least one of the two adjacent cells of Ξ. We can thus assign p_ℓ to such an adjacent cell, and count the joints on ℓ as part of the subproblem associated with that cell (losing in the reduction a total of at most n joints). Thus no special treatment is needed for points on 3-faces of Ξ. Alternatively, we can regard each 3-face τ as yielding a subproblem of its own, involving the (at most $n/(r^4 \log r)$) red points that it contains and the (at most n/r) blue hyperplanes that cross it. The number of subproblems is $O(r^4 \log r)$ and they are handled in the subsequent dual stage.

Cells of Ξ. As in the case of 2-faces and 3-faces, each cell τ of Ξ generates a subproblem involving the at most $n/(r^4 \log r)$ red Plücker points in τ and the at most n/r blue Plücker hyperplanes that cross τ. There are $O(r^4 \log r)$ subproblems of this type.

2.4. *The Dual Partitioning Stage*

Let τ be a cell of Ξ; we include here also the cases where τ is a 2-face or a 3-face of Ξ, and only hyperplanes that cross τ are considered. Let L_τ be the set of all lines $\ell \in L$ such that $p_\ell \in \tau$, and let L'_τ be the set of all lines $\ell \in L$ such that π_ℓ crosses τ; we have $|L_\tau| \le n/(r^4 \log r)$ and $|L'_\tau| \le n/r$. We "dualize" the problem, by mapping the lines of L_τ to (red) Plücker hyperplanes and lines of L'_τ to (blue) Plücker points in projective 5-space. Recall that we consider here joints (ℓ_1, ℓ_2, ℓ_3) where $\ell_1 \in L_\tau$, $\ell_2, \ell_3 \in L'_\tau$. Since ℓ_2, ℓ_3 are mapped to distinct points, the triple interaction of ℓ_1, ℓ_2, ℓ_3 is not localized at any point of this dual parametric 5-space. We therefore do not consider at all any triple interaction at this stage. Instead, we charge the joint in question simply to the incidence between p_{ℓ_2} and π_{ℓ_1}, or to the incidence between p_{ℓ_3} and π_{ℓ_1}. Clearly, this count is a (probably gross) overestimate of the number of joints under consideration.[5]

We construct a $(1/r)$-cutting Ξ'_τ of the cross section within Π of the hyperplanes π_ℓ, for $\ell \in L_\tau$, using, as above, a generic triangulation of the arrangement $\mathcal{A}(R_\tau)$, for an appropriate sample R_τ of $O(r)$ of these hyperplanes. As above, the size of Ξ'_τ is $O(r^4 \log r)$, and we may assume that each of its cells τ' contains at most $(n/r)/(r^4 \log r) = n/(r^5 \log r)$ blue Plücker points p_ℓ, for $\ell \in L'_\tau$, and is crossed by at most $(n/(r^4 \log r))/r = n/(r^5 \log r)$ red Plücker hyperplanes π_ℓ, for $\ell \in L_\tau$.

[5] Arguably, this is one of the weak spots of our analysis. Any method of "preserving" the triple interactions at joints would likely lead to an improved bound on $J(n)$.

We first note that the number of incident pairs $(p_\ell, \pi_{\ell'})$, involving a blue Plücker point p_ℓ of L'_τ and a red Plücker hyperplane $\pi_{\ell'}$ of L_τ that *crosses* the cell of Ξ'_τ that contains p_ℓ, is

$$O\left(r^4 \log r \cdot \left(\frac{n}{r^5 \log r}\right)^2\right) = O\left(\frac{n^2}{r^6 \log r}\right).$$

Summing this bound over all cells τ of Ξ, we obtain an overall number of

$$O\left(r^4 \log r \cdot \frac{n^2}{r^6 \log r}\right) = O\left(\frac{n^2}{r^2}\right)$$

joints of this type. Hence, in what follows we may restrict the analysis to incidences between the blue points and the red hyperplanes that *fully contain* the corresponding cell of Ξ'_τ.

We thus proceed to bound the number of "containment" incident pairs $(p_{\ell_2}, \pi_{\ell_1})$, for $\ell_2 \in L'_\tau$, $\ell_1 \in L_\tau$, applying a case analysis on the location of p_{ℓ_2} in Ξ'_τ.

Vertices of Ξ'_τ. Consider a joint (ℓ_1, ℓ_2, ℓ_3) where $\ell_1 \in L_\tau$, $\ell_2, \ell_3 \in L'_\tau$, such that p_{ℓ_2}, say, is a vertex of Ξ'_τ. As in the primal stage, the number of such joints is at most the sum $\sum_v d_v$, taken over the vertices v of Ξ'_τ, where d_v is the number of red lines $\ell \in L_\tau$ such that π_ℓ passes through v. We denote the set of these lines as $L_\tau^{(v)}$. As in the primal stage, only vertices v incident to four hyperplanes of R_τ that meet there transversally need to be considered.

We fix a hyperplane π_ℓ for $\ell \in L_\tau^{(v)}$ and intersect it with all hyperplanes of R_τ and with Π. Since the four hyperplanes of R_τ that form the vertex v intersect there transversally, their cross sections within $\pi_\ell \cap \Pi$ also intersect transversally at v, so that this point is a vertex of the three-dimensional arrangement of these cross sections. The number of such vertices, within $\pi_\ell \cap \Pi$, is at most $O(r^3)$, for a total of $O(r^3 \cdot (n/(r^4 \log r)))$, which, multiplied by the number of cells τ, yields a bound of $O(nr^3)$ on the number of joints at vertices of the cuttings Ξ'_τ.

Regulus Edges of Ξ'_τ. This is the most intricate part of our analysis. Let γ be an intersection curve of three hyperplanes of R_τ with Π, representing three respective lines ℓ_1, ℓ_2, ℓ_3 (again, only such curves are of interest). Suppose first that these lines are pairwise skew, so that they form a regulus σ. Let k_σ (resp., k'_σ) denote the number of lines ℓ of L_τ (resp., of L'_τ) that are contained in $\sigma^\perp$ (resp., in σ); in 5-space these are lines for which π_ℓ contains γ (resp., p_ℓ lies in γ). We need to bound the number of incident pairs of lines $(\ell, \ell') \in L_\tau \times L'_\tau$, such that $p_{\ell'} \in \gamma$. We do not include in this count lines $\ell' \in L'_\tau$ whose points $p_{\ell'}$ are vertices of Ξ'_τ, since they have already been accounted for.

Recall that we only need to consider the case where π_ℓ contains γ; that is, $\ell \in \sigma^\perp$. A trivial upper bound on the number of joints under consideration (or, rather, the number of incident pairs (ℓ, ℓ'), as above) is $k_\sigma \cdot k'_\sigma$. Our next steps proceed by case analysis on the values of k_σ and k'_σ, which uses two threshold values s, t that we specify later, where s is the parameter used in the process of pruning away heavy reguli and planes, applied at the beginning of the analysis.

(a) $k_\sigma \le t$: In this case we bound the number of joints by $t\sum_\sigma k'_\sigma$, where the sum extends over all reguli σ with this property. Since, in 5-space, k'_σ counts points that lie on the curves representing the reguli and each point is counted only once (since we exclude vertices of the cutting), the above sum is at most tn/r. Summed over all cells τ, this yields an overall bound of $O(nr^3 t\log r)$ joints (which already dominates the bounds $O(nr^3)$ obtained for the vertices of the dual cuttings, as well as for the vertices and edges of the primal cutting).[6]

(b) $k_\sigma > t$: By the initial pruning process, we may assume that $k'_\sigma \le s$. In this case we use Lemma 2.2 to conclude that the sum $\sum_\sigma k_\sigma$, over those reguli σ for which $k_\sigma > t$ (for the fixed cell τ), is at most

$$O\left(\frac{(n/(r^4\log r))^3}{t^{17/4}}+\frac{(n/(r^4\log r))^2}{t^2}+\frac{n}{r^4\log r}\right)$$
$$=O\left(\frac{n^3}{r^{12}t^{17/4}\log^3 r}+\frac{n^2}{r^8t^2\log^2 r}+\frac{n}{r^4\log r}\right).$$

Multiplying by s and by the number of cells τ, we obtain the bound

$$O\left(\frac{n^3 s}{r^8 t^{17/4}\log^2 r}+\frac{n^2 s}{r^4t^2\log r}+ns\right)$$

on the number of joints under consideration.

Non-Regulus Edges of Ξ'_τ. Suppose next that two of the lines that define the intersection curve, say ℓ_1,ℓ_2, meet (recall that we have assumed that no pair of lines in L are parallel). Thus they define a common plane h and a common point q. If the third line $\ell_3\in L_\tau$ lies in h or passes through q, then the intersection $\pi_{\ell_1}\cap\pi_{\ell_2}\cap\pi_{\ell_3}\cap\Pi$ is two-dimensional, so these three lines do not define an *edge* of Ξ'_τ. Hence ℓ_3 meets h at a single point $q'\ne q$. It follows (see Fig. 3) that any line ℓ with $p_\ell\in\gamma$ either lies in h and passes through q', or it passes through q and through ℓ_3, and thus lies in the plane h' spanned by q and ℓ_3. In other words, any joint on ℓ lies in $h\cup h'$. We can decompose γ into two subcurves $\gamma_h,\gamma_{h'}$, where γ_h (resp., $\gamma_{h'}$) consists of all points p_ℓ for which ℓ lies in h and passes through q' (resp., lies in h' and passes through q).

We next repeat the preceding analysis, handling planes instead of reguli, which makes it somewhat simpler.[7] Let then $\gamma=\gamma_h\cup\gamma_{h'}$ be an intersection curve of three hyperplanes of R_τ, representing lines ℓ_1,ℓ_2,ℓ_3 that form a pair of planes h,h', as above. We focus on one of the subcurves, say γ_h. Let k_h (resp., k'_h) denote the number of lines ℓ of L_τ (resp., of L'_τ) that are contained in h; in 5-space these are lines for which π_ℓ contains γ_h (resp., p_ℓ lies in γ_h). We need to bound the number of incident pairs of lines $(\ell,\ell')\in L_\tau\times L'_\tau$, for which $p_{\ell'}\in\gamma_h$. We do not include in this count lines $\ell'\in L'_\tau$ whose points $p_{\ell'}$ are vertices of Ξ'_τ, since they have already been accounted for.

As in the case of reguli, we only consider the case where π_ℓ contains γ_h. A trivial upper bound on the number of joints under consideration is $k_h\cdot k'_h$. Our next steps proceed

[6] This is another weak spot in our analysis—see the discussion at the end of the paper.

[7] It also yields smaller bounds, as we shall see, so this part of the analysis does not really affect the final overall bound.

by case analysis on the values of k_h and k'_h, which uses the same two threshold values s, t as for the case of reguli.

(a) $k_h \le t$: In this case we bound the number of joints by $t \sum_h k'_h$, where the sum extends over all planes h with this property. Since, in 5-space, k'_h counts blue points (representing lines in L'_τ) that lie on the corresponding curves γ_h, and each point is counted only once (since we exclude vertices of the cutting), the above sum is at most tn/r. Summed over all cells τ, this yields an overall bound of $O(nr^3 t \log r)$.

(b) $k_h > t$: The pruning process allows us to assume that $k'_h \le s$. In this case we use Lemma 2.1 to conclude that the sum $\sum_h k_h$, over those planes h for which $k_h > t$ (for the fixed cell τ), is at most

$$O\left(\frac{(n/(r^4 \log r))^2}{t^2} + \frac{n}{r^4 \log r}\right) = O\left(\frac{n^2}{r^8 t^2 \log^2 r} + \frac{n}{r^4 \log r}\right).$$

Multiplying by s and by the number of cells τ, we obtain the bound

$$O\left(\frac{n^2 s}{r^4 t^2 \log r} + ns\right) \tag{3}$$

on the number of joints under consideration.

2-Faces of Ξ'_τ. The analysis follows closely that for the 2-faces of the primal cutting Ξ. Specifically, let φ be an intersection 2-surface of two hyperplanes of R_τ with Π, and let ℓ_1, ℓ_2 be the two corresponding lines of L_τ. Suppose first that ℓ_1, ℓ_2 pass through a common point q, and thus lie in a common plane h. Then any line ℓ with $p_\ell \in \varphi$ either lies in h or passes through q. We can thus view φ as the union of two surfaces φ_q, φ_h, where φ_q (resp., φ_h) is the locus of all (points representing) lines passing through q (resp., lying in h).

Let (ℓ, ℓ', ℓ'') be a joint where $\ell \in L_\tau$, $\ell', \ell'' \in L'_\tau$, and, say, $p_{\ell'} \in \varphi_q$. We may assume that $p_{\ell'}$ does not lie on any edge of Ξ'_τ that is contained in φ, because such points have already been taken care of. As above, we assume that π_ℓ fully contains φ_q. Then ℓ must pass through q, and thus the joint in question must be the point q itself. The overall number of such joints is only $O(r^2)$, for an overall bound of $O(r^6 \log r)$.

Similarly, let (ℓ, ℓ', ℓ'') be a joint as above, where $p_{\ell'} \in \varphi_h$. Assume now that π_ℓ fully contains φ_h. Then ℓ must lie in h. In this case the joint must lie in h. We then proceed exactly as in the analysis of non-regulus edges of Ξ'_τ. (In case (a) of the analysis, the sum $\sum_h k'_h$ is at most n/r, since it counts lines of L'_τ without multiplicity, as we ignore the corresponding Plücker points that lie on edges of Ξ'_τ.) This yields the same bounds as in cases (a) and (b) of the non-regulus edges, i.e., a total bound of $O(n^2 s/(r^4 t^2 \log r) + ns + nr^3 t \log r)$ for the number of joints of this kind.

Finally, the case where ℓ_1 and ℓ_2 are skew can be ignored. Indeed, consider a joint (ℓ, ℓ', ℓ''), where, say, $p_{\ell'} \in \varphi$. Then the hyperplane π_ℓ cannot fully contain φ, because then the line ℓ would have to be incident to every line that meets ℓ_1 and ℓ_2, which is clearly impossible.

Cells of Ξ'_τ. Since cells are full-dimensional, no "containment" incidence can arise in this case.

3-Faces of Ξ'_τ. Let $p_{\ell'}$, where $\ell' \in L'_\tau$, be a blue point in the relative interior of some 3-face of Ξ'_τ, contained in $\pi_\ell \cap \Pi$, for some $\ell \in L_\tau$. Clearly, no other hyperplane of L_τ can fully contain the 3-face, so $p_{\ell'}$ is involved in just one containment incidence, for a total of $O(n/r)$ joints of this type. Summing over all cells τ of Ξ, the overall number of such joints is $O(r^4 \log r \cdot (n/r)) = O(nr^3 \log r)$.

Putting It All Together. Adding the bounds obtained in the preceding analysis steps, we obtain a grand total of

$$O\left(\frac{n^2}{s} + nr^3 t \log r + \frac{n^3 s}{r^8 t^{17/4} \log^2 r} + \frac{n^2}{r^2} + \frac{n^2 s}{r^4 t^2 \log r} + ns + r^6 \log r\right)$$

joints. We now choose

$$r = \frac{n^{13/69}}{\log^{3/23} n}, \qquad s = r^2 = \frac{n^{26/69}}{\log^{6/23} n}, \qquad t = \frac{n}{r^5 \log n} = \frac{n^{4/69}}{\log^{8/23} n},$$

to obtain that the overall number of joints is $O(n^{112/69} \log^{6/23} n)$. (This choice of parameters equalizes the first four terms in the above bound; the last three terms are dominated by the first four.)

We thus obtain the main result of this paper:

Theorem 2.3. *The number of joints of a set of n lines in* 3*-space is* $O(n^{112/69} \log^{6/23} n) = O(n^{1.6232})$.

2.5. *Discussion*

There are two natural conjectures concerning $J(n)$. The first (in view of the best known lower bound) is that $J(n) = \Theta(n^{3/2})$. The second, and somewhat weaker conjecture, is that $J(n) \approx O(n^{8/5})$. There are several informal reasons for the second conjecture. For example, observe that the two stages of decomposition end up with about r^8 subproblems, each involving about n/r^5 lines, which leads to a recurrence relation, whose basic solution is about $n^{8/5}$. Of course, the subproblems are different from the original one, since joints are "lost" there. Still, the general characteristics of the decomposition suggest this bound.

We strongly believe that at least the second conjecture is true. There are two weak spots in our analysis. The first is the handling of regulus-edges of the dual cuttings. We can handle well reguli that contain many lines of L'_τ, but it seems that we handle the "lighter" reguli in a suboptimal manner. At any rate, the term that the analysis of these light reguli yields, namely $O(nr^3 t \log r)$, is one of the causes for our bound to be weaker than $O(n^{8/5})$. The second cause is the way we handle the subproblems at the second partitioning stage: we bound there the number of relevant joints simply by the product of the sizes of the two corresponding sets of lines. We suspect that this is a gross overestimate, and that sharper bounds can be obtained using a more careful analysis.

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References

1. B. Aronov, V. Koltun, and M. Sharir, Incidences between points and circles in three and higher dimensions, *Proc*. 18*th Annu*. *ACM Sympos*. *Comput*. *Geom*. 2002, pp. 116–122. Also accepted for *Discrete Comput*. *Geom*.
2. B. Aronov, V. Koltun, and M. Sharir, Cutting triangular cycles of lines in space, *Proc*. 35*th Annu*. *ACM Sympos*. *Theory Comput*., 2003, pp. 547–555. Also accepted for *Discrete Comput*. *Geom*.
3. B. Aronov, M. Pellegrini, and M. Sharir, On the zone of a surface in a hyperplane arrangement, *Discrete Comput*. *Geom*. **9** (1993), 177–186.
4. B. Chazelle, H. Edelsbrunner, L. Guibas, R. Pollack, R. Seidel, M. Sharir, and J. Snoeyink, Counting and cutting cycles of lines and rods in space, *Comput*. *Geom*. *Theory Appl*. **1** (1992), 305–323.
5. B. Chazelle, H. Edelsbrunner, L.J. Guibas, M. Sharir, and J. Stolfi, Lines in space: combinatorics and algorithms, *Algorithmica* **5** (1996), 428–447.
6. B. Chazelle and J. Friedman, A deterministic view of random sampling and its use in geometry, *Combinatorica* **10** (1990), 229–249.
7. K. Clarkson, H. Edelsbrunner, L. Guibas, M. Sharir, and E. Welzl, Combinatorial complexity bounds for arrangements of curves and spheres, *Discrete Comput*. *Geom*. **5** (1990), 99–160.
8. S. Har-Peled, Constructing cuttings in theory and practice, *SIAM J*. *Comput*. **29** (2000), 2016–2039.
9. S. Har-Peled and M. Sharir, On-line point location in planar arrangements and its applications, *Discrete Comput*. *Geom*. **26** (2001), 19–40.
10. J. Pach and P.K. Agarwal, *Combinatorial Geometry*, Wiley-Interscience, New York, 1995.
11. M. Sharir, On joints in arrangements of lines in space and related problems, *J*. *Combin*. *Theory Ser. A* **67** (1994), 89–99.
12. M. Sharir and E. Welzl, Point-line incidences in space, *Combin*. *Probab. Comput*. **13** (2004), 203–220.
13. A. Solan, Cutting cycles of rods in space, *Proc*. 14*th Annu*. *ACM Sympos*. *Comput*. *Geom*., 1998, pp. 135–142.
14. D.M.Y. Sommerville, *Analytic Geometry of Three Dimensions*, Cambridge University Press, Cambridge, 1934.
15. E. Szemerédi and W.T. Trotter, Extremal problems in discrete geometry, *Combinatorica* **3** (1983), 381–392.

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