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# Some Characterizations of Ellipsoids by Sections*

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**Abstract.** We characterize ellipsoids among convex bodies in $E^d$ looking at the sections parallel to two or three hyperplanes.

## 1. Introduction

Let $D$ be a convex body (i.e., a compact convex set with non-empty interior) in the $d$-dimensional Euclidean space $E^d$ ($d \geq 3$) and let $S$ be the boundary of $D$. Let $P$ be a homogeneous hyperplane in $E^d$. We say that $D$ (indistinctly, $S$) is

(i) *$P$-homothetic* if for every $u, v \in \text{Int}(D)$ the sections $(u + P) \cap S$ and $(v + P) \cap S$ are homothetic (up to translations),
(ii) *$P$-elliptic* if for every $u \in \text{Int}(D)$ the section $(u + P) \cap S$ is ellipsoidal.

Roughly speaking, $S$ is $P$-homothetic ($P$-elliptic) if all the sections of $S$ by hyperplanes parallel to $P$ are identical up to size (are ellipsoidal).

It is obvious that $S$ is an ellipsoid if it is $P$-elliptic for every homogeneous hyperplane $P$. Not so obvious, but also true, is that $S$ is an ellipsoid if it is $P$-homothetic for every $P$ (see, e.g., [6] and [8]). Improvements of these results are at the origin of an important family of characterizations of ellipsoids. An example is the proof by Aitchison [1] that it is sufficient for $S$ to be an ellipsoid, that for every $P$, property (i) holds when the hyperplanes $u + P$ and $v + P$ are sufficiently close to a parallel supporting hyperplane. Other results in this line can be found in [2]–[4], [7] and [9].

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The aim of this present paper is to provide some answers to the following question: What if $S$ is $P$-homothetic or $P$-elliptic for a limited number of hyperplanes $P$? Theorem 1 will show that only two planes are necessary to characterize ellipsoids if both properties are satisfied. Then, in Theorem 2, we will see that three planes are enough to characterize ellipsoids via the $P$-elliptic property, among centrally symmetric convex bodies.

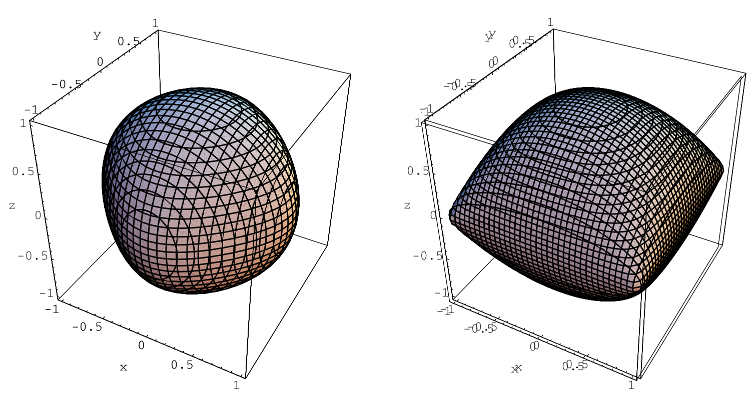
## 2. Results

**Theorem 1.** *Let $S$ be the boundary of a convex body in $E^d$ $(d \geq 3)$. If there exist two different homogeneous hyperplanes $P_1$, $P_2$, such that $S$ is $P_i$-homothetic and $P_i$-elliptic for $i = 1, 2$, then $S$ is an ellipsoid.*

Obviously, if in Theorem 1 we were to consider only the $P$-homothetic property, then $S$ need not be an ellipsoid. A simple counterexample is the cube. On the other hand, Example 1 shows that neither is the $P$-elliptic property alone sufficient to characterize ellipsoids.

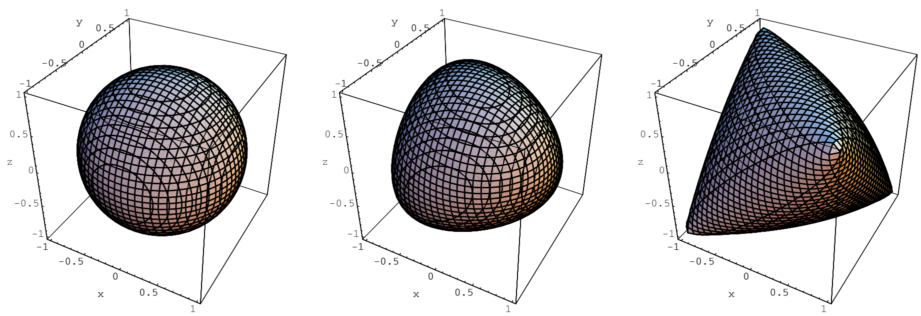
**Example 1.** The sets $A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 + x^2y^2 \leq 1\}$ and $B = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 - x^2y^2 \leq 1, \ |x| \leq 1, \ |y| \leq 1\}$ are symmetric convex bodies in $\mathbb{R}^3$ which are $P$-elliptic with respect to the planes $x = 0$ and $y = 0$. Neither $A$ nor $B$ is an ellipsoid (see Fig. 1).

Thus, if we want to characterize ellipsoids via only the $P$-elliptic property, we must add at least one hyperplane more. Nevertheless, Example 2 shows that even this is not enough.




*A* *B*

**Fig. 1.** $P$-elliptic with respect to two planes.



$C_0$ $C_1$ $C_2$

**Fig. 2.** $P$-elliptic with respect to three planes.

**Example 2.** Let $\alpha \in \mathbb{R}$, $|\alpha| \leq 2$. The set

$$C_\alpha = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 + \alpha xyz \leq 1, \ \max\{|x|, |y|, |z|\} \leq 1\}$$

is a convex body in $\mathbb{R}^3$ which is $P$-elliptic with respect to the planes $x = 0$, $y = 0$ and $z = 0$. If $\alpha \neq 0$, $C_\alpha$ is not ellipsoidal (see Fig. 2).

Observe that if $\alpha \neq 0$, then the sets $C_\alpha$ in Example 2 are not centrally symmetric. This leads to the following result.

**Theorem 2.** *Let $S$ be the boundary of a centrally symmetric convex body in $E^d$ $(d \geq 3)$, and let $P_1$, $P_2$, $P_3$ be three different homogeneous hyperplanes. If $S$ is $P_i$-elliptic for $i = 1, 2, 3$, then $S$ is an ellipsoid.*

It is well known that if $S$ is the boundary of a symmetric convex body in $E^d$ centered at the origin, then we can define a norm over $E^d$ such that its unit sphere is $S$. Conversely, the unit ball of any $d$-dimensional normed space is a symmetric convex body whose boundary is the unit sphere. Moreover, a $d$-dimensional normed space is an inner product space if and only if its unit sphere is ellipsoidal. Thus, the results in this paper can also be seen in the context of normed spaces.

## 3. Proofs

*Proof of Theorem* 1. Assume that $S$ is the boundary of the convex body $D$. We can consider without loss of generality that the origin of $E^d$ is an interior point of $D$.

($d = 3$) Let us suppose first that $d = 3$. Let $l_3 = P_1 \cap P_2$ and let $p$ and $q$ be the two points at which $l_3$ intersects $S$. Let $o$ be the center of the segment $pq$. We can assume that $o$ is the origin of the space. From the hypothesis we know that $C_1 = P_1 \cap S$ and $C_2 = P_2 \cap S$ are ellipses that intersect in $p$ and $q$. For $i = 1, 2$ let $l_i$ be the line in the plane $P_i$ that meets the center of $C_i$ and that has the conjugate direction of $l_3$ with respect to $C_i$. Then $l_i$ also meets $o$. Let $P_3$ be the plane generated by $l_1$ and $l_2$. Then

$$l_1 = P_1 \cap P_3, \qquad l_2 = P_2 \cap P_3, \qquad l_3 = P_1 \cap P_2.$$

In each line $l_i$, $i = 1, 2, 3$, we can take a vector $e_i$ in such a way that if $(x_1, x_2, x_3)$ are the coordinates of a point $x \in E^3$, relative to the basis $\{e_1, e_2, e_3\}$, then

$$C_1 = \{x \in E^3 : (x_1 - c_1)^2 + x_3^2 = r_1^2,\ x_2 = 0\},$$
$$C_2 = \{x \in E^3 : (x_2 - c_2)^2 + x_3^2 = r_2^2,\ x_1 = 0\},$$

for certain $c_1, c_2, r_1, r_2 \in \mathbb{R}$. In short, we can identify $E^3$ with $\mathbb{R}^3$ in such a way that $C_1$ and $C_2$ are circumferences.

We denote by $x'$ the orthogonal projection of $x \in E^3$ onto $P_3$, i.e., $x' = (x_1, x_2, 0)$. Let $D'$ be the projection of $D$ onto $P_3$ and we denote by $\mathrm{Int}_{P_3} D'$ the interior of $D'$ in $P_3$. From the hypothesis it follows that for every $x' \in \mathrm{Int}_{P_3} D'$ the sets

$$C_1(x_2) = (x_2 e_2 + P_1) \cap S, \qquad C_2(x_1) = (x_1 e_1 + P_2) \cap S$$

are circumferences. We denote by $c_1(x_2)$ and $c_2(x_1)$ the centers of these circumferences.

We perform the rest of the proof in three steps. In the first we shall see that $c_1(x_2)$ and $c_2(x_1)$ are in $P_3$ for every $x' \in \mathrm{Int}_{P_3} D'$; in the second, that all the points $c_1(x_2)$ (respectively, $c_2(x_1)$) are aligned; in the third, that $S$ is an ellipsoid.

*Step* 1. Let $x \in S$ be such that $x' \in \mathrm{Int}_{P_3} D'$ and let $H_0, \ldots, H_n$ be open rectangles in $P_3$ with sides parallel to $l_1$ and $l_2$ such that

(a) $H_i \subset D'$, $i = 0, \ldots, n$,
(b) $o \in H_0$, $x' \in H_n$,
(c) $H_i \cap H_{i-1} \neq \emptyset$, $i = 1, \ldots, n$.

Let $H$ be any rectangle $H_i$ and let $u' = (u_1, u_2, 0) \in H$. Since $H \subset D'$, there exist two points $u, \bar{u} \in S$ whose projection onto $P_3$ is $u'$. Therefore $u = (u_1, u_2, u_3)$ for some $u_3$. The circumferences $C_1(u_2)$ and $C_2(u_1)$ meet in $u$ and $\bar{u}$. Let $P_{u'}$ be the plane parallel to $P_3$ that meets the midpoint of the segment $u\bar{u}$. All the circumferences passing through $u$ and $\bar{u}$ have their centers in $P_{u'}$. In particular, $c_1(u_2), c_2(u_1) \in P_{u'}$.

Let now $v' = (v_1, v_2, 0)$ be another point in $H$, different from $u'$, and let $v_3$ be such that $v = (v_1, v_2, v_3) \in S$. Then we have $c_1(v_2), c_2(v_1) \in P_{v'}$. Assume, without loss of generality, that $v_1 \neq u_1$ and let $w' = (v_1, u_2, 0)$. Then $w' \in H$ and $c_1(u_2), c_2(v_1) \in P_{w'}$. Bearing in mind that the planes $P_{u'}$, $P_{v'}$ and $P_{w'}$ are parallel and that

$$c_1(u_2) \in P_{u'} \cap P_{w'}, \qquad c_2(v_1) \in P_{w'} \cap P_{v'}$$

we get that $P_{u'} = P_{w'} = P_{v'}$.

From the above it follows that $P_{u'}$ is constant for every $u' \in H$. We denote that plane by $P_H$. From (b) and (c) it follows that

$$P_{x'} = P_{H_n} = P_{H_{n-1}} = \cdots = P_{H_0} = P_3.$$

In sum, if $x \in S$ is such that $x' \in \mathrm{Int}_{P_3} D'$, then the circumferences $C_1(x_2)$ and $C_2(x_1)$ have their centers in $P_3$.

*Step* 2. We return to the rectangle $H$, but assume now that $u'$ and $v'$ are such that $u_1 \neq v_1$ and $u_2 \neq v_2$. We say in that case that $u' \# v'$. We consider the circumferences $C_1(u_2)$,

$C_2(u_1)$, $C_1(v_2)$ and $C_2(v_1)$, whose centers are, respectively,

$$c_1(u_2) = (c_{11}(u_2), u_2, 0), \qquad c_2(u_1) = (u_1, c_{22}(u_1), 0),$$
$$c_1(v_2) = (c_{11}(v_2), v_2, 0), \qquad c_2(v_1) = (v_1, c_{22}(v_1), 0).$$

Let $w \in C_1(u_2) \cap C_2(v_1)$ and $t \in C_2(u_1) \cap C_1(v_2)$. Then

$$w = (v_1, u_2, w_3), \qquad t = (u_1, v_2, t_3),$$

for certain $w_3, t_3 \in \mathbb{R}$. The four circumferences we are dealing with are linked to each other by the following identities:

$$u, w \in C_1(u_2) \quad \Rightarrow \quad (u_1 - c_{11}(u_2))^2 + u_3^2 = (v_1 - c_{11}(u_2))^2 + w_3^2, \tag{1}$$
$$w, v \in C_2(v_1) \quad \Rightarrow \quad (u_2 - c_{22}(v_1))^2 + w_3^2 = (v_2 - c_{22}(v_1))^2 + v_3^2, \tag{2}$$
$$v, t \in C_1(v_2) \quad \Rightarrow \quad (v_1 - c_{11}(v_2))^2 + v_3^2 = (u_1 - c_{11}(v_2))^2 + t_3^2, \tag{3}$$
$$t, u \in C_2(u_1) \quad \Rightarrow \quad (v_2 - c_{22}(u_1))^2 + t_3^2 = (u_2 - c_{22}(u_1))^2 + u_3^2. \tag{4}$$

Summing (1)–(4) we get

$$u_1 c_{11}(u_2) + u_2 c_{22}(v_1) + v_1 c_{11}(v_2) + v_2 c_{22}(u_1)$$
$$= v_1 c_{11}(u_2) + v_2 c_{22}(v_1) + u_1 c_{11}(v_2) + u_2 c_{22}(u_1),$$

from which follows

$$\frac{c_{11}(u_2) - c_{11}(v_2)}{u_2 - v_2} = \frac{c_{22}(u_1) - c_{22}(v_1)}{u_1 - v_1}.$$

The left-hand side of the above identity depends only on $u_2$ and $v_2$, whereas the right-hand side depends only on $u_1$ and $v_1$. Therefore,

$$\frac{c_{11}(u_2) - c_{11}(v_2)}{u_2 - v_2} = \frac{c_{22}(u_1) - c_{22}(v_1)}{u_1 - v_1} = \text{constant} := a_H, \tag{5}$$

for every $u', v' \in H$, $u' \# v'$. Taking now as $H$ any $H_i$, we get again from (c) that $a_{H_n} = a_{H_{n-1}} = \cdots = a_{H_0} := a$. A straightforward calculation repeatedly applying the identity (5) shows that this identity also holds for every $u', v' \in \bigcup_{i=0}^{n} H_i$, $u' \# v'$. In particular, remembering that $o \in H_0$ and that $x' \in H_n$, we get that

$$c_{11}(x_2) = a x_2 + c_1, \qquad c_{22}(x_1) = a x_1 + c_2, \tag{6}$$

which is also true if either $x_1$ or $x_2$ are 0.

*Step* 3. At this point we have that for $i = 1, 2$, all the sections of $S$ by planes parallel to $P_i$ are circles whose centers are in a line, $L_i$. It is easy to see that for every $x, y \in S$, the composition of rotations about $L_1$ with rotations about $L_2$ allow us to get a linear map that fixes $S$ and sends $x$ into $y$. Busemann [5] proved that this property is characteristic of the ellipsoids.

($d \geq 4$) Assume now that $d \geq 4$. To see that $S$ is an ellipsoid it is enough to show that if $P$ is a two-dimensional plane through the origin, then $P \cap S$ is ellipsoidal (see, e.g., [5]). If $P \subset P_i$ for $i = 1$ or 2, then $P \cap S$ is ellipsoidal because so is $P_i \cap S$. On the other hand, if $P \not\subset P_1$ and $P \not\subset P_2$, then, since $P_1 \neq P_2$, we can take $x \in P_1 \backslash (P_2 \cup P)$. Let $E = P \oplus \langle x \rangle$. Therefore, $\dim E = 3$, $\dim P_1 \cap E = \dim P_2 \cap E = 2$ and $P_1 \cap E \neq P_2 \cap E$. We shall show that $E \cap S$ is $(P_i \cap E)$-elliptic and $(P_i \cap E)$-homothetic for $i = 1, 2$. Let $u, v \in E$ be interior points of the convex body whose boundary is $S$. From the hypothesis we know that $(u + P_i) \cap S$ and $(v + P_i) \cap S$ are homothetic ellipsoids. Hence the sections by $E$, $(u + P_i) \cap S \cap E$ and $(v + P_i) \cap S \cap E$, are also homothetic ellipsoids. Since $[u + (P_i \cap E)] \cap (E \cap S) = (u + P_i) \cap S \cap E$ (similarly with $v$) we have that $E \cap S$ is $(P_i \cap E)$-elliptic and $(P_i \cap E)$-homothetic. Hence we are in the case $d = 3$ and we get that $E \cap S$ is ellipsoidal, from which it follows that $P \cap S$ is also ellipsoidal. We must note that we cannot simplify the proof taking as $E$ an arbitrary three-dimensional subspace because in that case we cannot assure that $P_1 \cap E \neq P_2 \cap E$. □

*Proof of Theorem* 2. We assume that $S$ is centered at the origin and let $C_i = P_i \cap S$, $i = 1, 2, 3$. We consider three cases.

*Case* 1. Assume that $d = 3$ and $P_1 \cap P_2 \cap P_3 = \{0\}$. Let

$$e_1 \in C_1 \cap C_3, \qquad e_2 \in C_2 \cap C_3, \qquad e_3 \in C_1 \cap C_2.$$

We take $\{e_1, e_2, e_3\}$ as basis of the space. Then the ellipses $C_1, C_2, C_3$ can be represented as the set of points whose coordinates $(x_1, x_2, x_3)$ satisfy, respectively,

$$\begin{aligned} C_1\colon &\quad x_1^2 + x_3^2 + \alpha_{13} x_1 x_3 = 1, \qquad x_2 = 0, \\ C_2\colon &\quad x_2^2 + x_3^2 + \alpha_{23} x_2 x_3 = 1, \qquad x_1 = 0, \\ C_3\colon &\quad x_1^2 + x_2^2 + \alpha_{12} x_1 x_2 = 1, \qquad x_3 = 0, \end{aligned} \tag{7}$$

for certain $\alpha_{12}, \alpha_{13}, \alpha_{23} \in \mathbb{R}$.

Let $H_0 \subset \mathrm{Int}_{P_3} D \cap P_3$ be an open rectangle centered at 0 with sides parallel to the vectors $e_1$ and $e_2$, respectively. Let $u = (u_1, u_2, u_3) \in S$ be such that $u' = (u_1, u_2, 0) \in H_0$. Then $(\pm u_1, \pm u_2, 0) \in H_0$ and $|u_1| < 1$, $|u_2| < 1$. We shall show that $u$ is a point of the quadric $Q$ of equation

$$Q\colon \quad x_1^2 + x_2^2 + x_3^2 + \alpha_{12} x_1 x_2 + \alpha_{13} x_1 x_3 + \alpha_{23} x_2 x_3 = 1. \tag{8}$$

Since $C_i \subset Q$, $i = 1, 2, 3$, we can assume without loss of generality that $u_1 \neq 0$, $u_2 \neq 0$.

By hypothesis, $C_1(u_2) = (u_2 e_2 + P_1) \cap S$ and $C_2(u_1) = (u_1 e_1 + P_2) \cap S$ are ellipses. We suppose that their equations are, respectively,

$$\begin{aligned} C_1(u_2)\colon &\quad \alpha_1 x_1^2 + \beta_1 x_3^2 + \gamma_1 x_1 x_3 + \delta_1 x_1 + \lambda_1 x_3 = 1, \qquad x_2 = u_2, \\ C_2(u_1)\colon &\quad \alpha_2 x_2^2 + \beta_2 x_3^2 + \gamma_2 x_2 x_3 + \delta_2 x_2 + \lambda_2 x_3 = 1, \qquad x_1 = u_1. \end{aligned} \tag{9}$$

Since $S$ is symmetric, the equations of the ellipses $C_1(-u_2)$ and $C_2(-u_1)$ are

$$\begin{aligned} C_1(-u_2)\colon &\quad \alpha_1 x_1^2 + \beta_1 x_3^2 + \gamma_1 x_1 x_3 - \delta_1 x_1 - \lambda_1 x_3 = 1, \qquad x_2 = -u_2, \\ C_2(-u_1)\colon &\quad \alpha_2 x_2^2 + \beta_2 x_3^2 + \gamma_2 x_2 x_3 - \delta_2 x_2 - \lambda_2 x_3 = 1, \qquad x_1 = -u_1. \end{aligned}$$

Let $(s, u_2, 0)$, and $(\hat{s}, u_2, 0)$ be the two points where $C_3$ and $C_1(u_2)$ meet. From (7) and (9) it follows that

$$\begin{aligned} s^2 + \alpha_{12} u_2 s + u_2^2 - 1 = 0, \qquad & \hat{s}^2 + \alpha_{12} u_2 \hat{s} + u_2^2 - 1 = 0, \\ \alpha_1 s^2 + \delta_1 s - 1 = 0, \qquad & \alpha_1 \hat{s}^2 + \delta_1 \hat{s} - 1 = 0. \end{aligned} \tag{10}$$

Looking at (10) as two polynomials of second degree that have the same roots, $s$ and $\hat{s}$, we get

$$\alpha_1 = \frac{1}{1 - u_2^2}, \qquad \delta_1 = \frac{\alpha_{12} u_2}{1 - u_2^2}. \tag{11}$$

In a similar way, considering $C_2 \cap C_1(u_2)$, $C_3 \cap C_2(u_1)$ and $C_1 \cap C_2(u_1)$, we obtain

$$\begin{aligned} \beta_1 = \frac{1}{1 - u_2^2}, \qquad & \lambda_1 = \frac{\alpha_{23} u_2}{1 - u_2^2}, \\ \alpha_2 = \beta_2 = \frac{1}{1 - u_1^2}, \qquad \delta_2 = \frac{\alpha_{12} u_1}{1 - u_1^2}, \qquad & \lambda_2 = \frac{\alpha_{13} u_1}{1 - u_1^2}. \end{aligned} \tag{12}$$

The same argument followed above but considering $C_1(u_2) \cap C_2(u_1)$ and $C_1(u_2) \cap C_2(-u_1)$ gives the identities

$$\beta_2(\gamma_1 u_1 + \lambda_1) = \beta_1(\gamma_2 u_2 + \lambda_2), \qquad \beta_2(\gamma_1 u_1 - \lambda_1) = \beta_1(\lambda_2 - \gamma_2 u_2),$$

that jointly with (12) give

$$\gamma_1 = \frac{\alpha_{13}}{1 - u_2^2}, \qquad \gamma_2 = \frac{\alpha_{23}}{1 - u_1^2}. \tag{13}$$

Now, bearing in mind that $u \in C_1(u_2)$, we get from (9), (11) and (12)

$$\begin{aligned} 1 &= \alpha_1 u_1^2 + \beta_1 u_3^2 + \gamma_1 u_1 u_3 + \delta_1 u_1 + \lambda_1 u_3 \\ &= \frac{u_1^2 + u_3^2 + \alpha_{13} u_1 u_3 + \alpha_{12} u_2 u_1 + \alpha_{23} u_2 u_3}{1 - u_2^2}. \end{aligned}$$

Hence, it follows from (8) that $u \in Q$.

Therefore, $S$ and $Q$ coincide at all the points that are projected onto $H_0$. Finally, similar arguments to those followed in Theorem 1 complete the proof.

*Case* 2. Assume now that $d = 3$ and that $P_1$, $P_2$ and $P_3$ meet in a line. Let $e_3 \in C_1 \cap C_2 \cap C_3$. Then there exists only one plane $P$ through the origin such that $e_3 + P$ supports $S$ in $e_3$. Let $e_1 \in C_1 \cap P$ and $e_2 \in C_2 \cap P$. We take $\{e_1, e_2, e_3\}$ as the basis of the space. Taking into account that $C_1$ and $C_2$ are centered at the origin and that $\{e_1, e_3\}$ and $\{e_2, e_3\}$ are conjugate pairs of vectors, we can represent these ellipses as the set of points whose coordinates $(x_1, x_2, x_3)$ satisfy, respectively,

$$\begin{aligned} C_1\colon \qquad x_1^2 + x_3^2 = 1, \qquad x_2 = 0, \\ C_2\colon \qquad x_2^2 + x_3^2 = 1, \qquad x_1 = 0. \end{aligned}$$

The plane $P$ meets $P_3$ in a line $(x_1, \lambda x_1, 0)$, with $\lambda \neq 0$. Then, since $C_3$ is also centered at the origin and $e_3 + P \cap P_3$ supports $C_3$ at $e_3$, we have that

$$C_3\colon \qquad \alpha x_1^2 + x_3^2 = 1, \qquad x_2 = \lambda x_1,$$

for some $\alpha > 0$.

It is easy to see that there is only one quadric $Q$ that contains the ellipses $C_1$, $C_2$ and $C_3$. It is given by

$$Q\colon \qquad x_1^2 + x_2^2 + x_3^2 + \left(\frac{\alpha - 1 - \lambda^2}{\lambda}\right) x_1 x_2 = 1. \tag{14}$$

We shall show that $S = Q$.

Let $u = (u_1, u_2, u_3) \in S$. To see that $u \in Q$, we assume first that

$$|u_1| < \frac{1}{2 + 3|\lambda|}. \tag{15}$$

We need this assumption to be sure that some of the ellipses that we consider meet.

Our aim is to see that the ellipse $C_2(u_1) = (u_1 e_1 + P_2) \cap S$ coincides with the section of $Q$ determined by the plane $u_1 e_1 + P_2$. Naming this section $C$, we have

$$C\colon \ \left(\frac{1}{1 - u_1^2}\right) x_2^2 + \left(\frac{1}{1 - u_1^2}\right) x_3^2 + \left(\frac{(\alpha - 1 - \lambda^2) u_1}{\lambda(1 - u_1^2)}\right) x_2 = 1, \qquad x_1 = u_1. \tag{16}$$

Assume that

$$C_2(u_1)\colon \qquad \alpha_2 x_2^2 + \beta_2 x_3^2 + \gamma_2 x_2 x_3 + \delta_2 x_2 + \varepsilon_2 x_3 = 1, \qquad x_1 = u_1. \tag{17}$$

Since $S$ is symmetric, the equation of the ellipse $C_2(-u_1) = (-u_1 e_1 + P_2) \cap S$ is

$$C_2(-u_1)\colon \qquad \alpha_2 x_2^2 + \beta_2 x_3^2 + \gamma_2 x_2 x_3 - \delta_2 x_2 - \varepsilon_2 x_3 = 1, \qquad x_1 = -u_1. \tag{18}$$

Let now $P_3'$ be the plane parallel to $P_3$ that contains the point $(u_1, -\lambda u_1, 0)$. Hence, $P \cap P_3'$ is the line $(x_1, \lambda(x_1 - 2u_1), 0)$. From (15) it follows that $|u_1| + |\lambda u_1| < 1$, which means that the point $(u_1, -\lambda u_1, 0)$ lies inside the parallelogram of vertices $(\pm 1, 0, 0)$ and $(0, \pm 1, 0)$. Since $S$ is convex, it follows that $P_3'$ intersects $S$. By hypothesis $S \cap P_3'$ is an ellipse that we name $C_3'$. We assume that

$$C_3'\colon \qquad \alpha_3 x_1^2 + \beta_3 x_3^2 + \gamma_3 x_1 x_3 + \delta_3 x_1 + \varepsilon_3 x_3 = 1, \qquad x_2 = \lambda(x_1 - 2u_1). \tag{19}$$

Next we consider what information the intersections of the above ellipses give us.

The ellipses $C_1$ and $C_2(u_1)$ meet in the points $(u_1, 0, \pm\sqrt{1 - u_1^2})$, from which it follows that

$$\beta_2(1 - u_1^2) \pm \varepsilon_2 \sqrt{1 - u_1^2} = 1,$$

and hence

$$\beta_2 = \frac{1}{1 - u_1^2}, \qquad \varepsilon_2 = 0. \tag{20}$$

From (15) and the fact that $C_3$ is symmetric with respect to the third coordinate, we get that $C_3$ and $C_2(u_1)$ meet in $(u_1, \lambda u_1, \pm z_1)$ for some $z_1 \neq 0$. Then

$$\alpha u_1^2 + z_1^2 = 1$$

and

$$\alpha_2 \lambda^2 u_1^2 + \left(\frac{1}{1 - u_1^2}\right) z_1^2 \pm \gamma_2 \lambda u_1 z_1 + \delta_2 \lambda u_1 = 1.$$

Therefore,

$$\gamma_2 = 0 \tag{21}$$

and

$$\lambda^2 u_1^2 \alpha_2 + \lambda u_1 \delta_2 = \frac{(\alpha - 1) u_1^2}{1 - u_1^2}. \tag{22}$$

Again from (15) it follows that $C_2$ meets $C_3'$ in the points $(0, -2\lambda u_1, \pm\sqrt{1 - 4\lambda^2 u_1^2})$. Therefore,

$$\beta_3 (1 - 4\lambda^2 u_1^2) \pm \varepsilon_3 \sqrt{1 - 4\lambda^2 u_1^2} = 1,$$

and hence

$$\beta_3 = \frac{1}{1 - 4\lambda^2 u_1^2}, \qquad \varepsilon_3 = 0. \tag{23}$$

Similarly, $C_1$ meets $C_3'$ in $(2u_1, 0, \pm\sqrt{1 - 4u_1^2})$. Therefore,

$$4\alpha_3 u_1^2 + \frac{1 - 4u_1^2}{1 - 4\lambda^2 u_1^2} \pm 2\gamma_3 u_1 \sqrt{1 - 4u_1^2} + 2\delta_3 u_1 = 1,$$

from which it follows that

$$\gamma_3 = 0 \tag{24}$$

and

$$2u_1^2 \alpha_3 + u_1 \delta_3 = \frac{2u_1^2 (1 - \lambda^2)}{1 - 4\lambda^2 u_1^2}. \tag{25}$$

From (20), (21), (23) and (24) we get that $C_2(u_1)$ and $C_3'$ are symmetric with respect to the third coordinate. Hence they meet in $(u_1, -\lambda u_1, \pm z_2)$ for some $z_2 \neq 0$, from which it follows that

$$\lambda^2 u_1^2 \alpha_2 + \left(\frac{1}{1 - u_1^2}\right) z_2^2 - \lambda u_1 \delta_2 = 1 \tag{26}$$

and

$$u_1^2 \alpha_3 + \left(\frac{1}{1 - 4\lambda^2 u_1^2}\right) z_2^2 + u_1 \delta_3 = 1. \tag{27}$$

Finally, from (15) we have also that $C_2(-u_1)$ meets $C_3'$ in $(-u_1, -3\lambda u_1, \pm z_3)$ for some $z_3 \neq 0$. Therefore,

$$9\lambda^2 u_1^2 \alpha_2 + \left(\frac{1}{1-u_1^2}\right) z_3^2 + 3\lambda u_1 \delta_2 = 1 \tag{28}$$

and

$$u_1^2 \alpha_3 + \left(\frac{1}{1-4\lambda^2 u_1^2}\right) z_3^2 - u_1 \delta_3 = 1. \tag{29}$$

The six equations (22) and (25)–(29) form a non-singular linear system with the unknowns $\alpha_2$, $\delta_2$, $\alpha_3$, $\delta_3$, $z_2^2$ and $z_3^2$. Solving this system, we get in particular

$$\alpha_2 = \frac{1}{1-u_1^2}, \qquad \delta_2 = \frac{(\alpha - 1 - \lambda^2)u_1}{\lambda(1-u_1^2)},$$

that jointly with (20) and (21) give that $C$ coincides with $C_2(u_1)$ as we wished to show.

At this point we have seen (remember assumption (15)) that the slice of $S$ defined by the planes $x_1 = \pm 1/(2 + 3|\lambda|)$ coincides with the corresponding slice of $Q$. Similar arguments to those followed in the proof of Theorem 1 give that the whole of $S$ coincides with $Q$.

*Case* 3. Finally, assume that $d \geq 4$. We shall prove by induction on $d$ that $S$ is an ellipsoid. So let us assume that the theorem is true for $(d-1)$-dimensional spaces and let $E$ be a $(d-1)$-dimensional subspace such that $P_1 \cap P_2 \cap P_3 \not\subset E$. Then $\dim(E \cap P_i) = d - 2$ for $i = 1, 2, 3$, $E \cap P_i \neq E \cap P_j$ for $i \neq j$ and $E \cap S$ is $(E \cap P_i)$-elliptic for $i = 1, 2, 3$. Applying the hypothesis of induction we get that $E \cap S$ is an ellipsoid. Density arguments give that $E \cap S$ is an ellipsoid for any $E$, $\dim E = d - 1$, and therefore $S$ is an ellipsoid. □

## 4. Remarks

**Remarks on Example 1.** The only thing not trivial in Example 1 is the convexity of the sets $A$ and $B$. To see that $A$ is convex it is enough to observe that the set $A' = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 + x^2y^2 < 1\}$ is an open convex subset of $(-1, 1) \times (-1, 1)$ and that the Hessian matrix of the function $z(x, y) = -\sqrt{1 - x^2 - y^2 - x^2y^2}$ is positive definite for every $(x, y) \in A'$. The first is easy. The second follows from the fact that

$$\frac{\partial^2 z}{\partial x^2} = \frac{1 - y^4}{(1 - x^2 - y^2 - x^2y^2)^{3/2}} > 0,$$

$$\begin{vmatrix} \dfrac{\partial^2 z}{\partial x^2} & \dfrac{\partial^2 z}{\partial x\, \partial y} \\ \dfrac{\partial^2 z}{\partial x\, \partial y} & \dfrac{\partial^2 z}{\partial y^2} \end{vmatrix} = \frac{(1 - x^2y^2)^2 + x^2(1 - y^2)^2 + y^2(1 - x^2)^2}{(1 - x^2 - y^2 - x^2y^2)^2} > 0,$$

for every $(x, y) \in A'$. The convexity of $B$ follows in a similar way.

It is interesting to note that with $B$ being $P$-elliptic with respect to two planes, it is not strictly convex.

**Remarks on Example 2.** As above, the only difficulty in this example is to prove the convexity of $C_\alpha$. Assume that $\alpha \neq 0$ and consider the function

$$F(x, y, z) = x^2 + y^2 + z^2 + \alpha xyz.$$

We consider the norm $\|(x, y, z)\| = \max\{|x|, |y|, |z|\}$. It is easy to see that $F$ has the following property:

$$\left.\begin{array}{r} F(x, y, z) < 1 \\ \|(x, y, z)\| \leq 1 \end{array}\right\} \quad \Rightarrow \quad \|(x, y, z)\| < 1. \tag{30}$$

Now, let

$$D_\alpha = \{(x, y, z) \in \mathbb{R}^3 : F(x, y, z) < 1, \ \|(x, y, z)\| < 1\}.$$

To prove that $C_\alpha$ is convex we see first that $C_\alpha = \overline{D_\alpha}$ and then that $D_\alpha$ is convex. Since $C_\alpha$ is closed and $D_\alpha \subset C_\alpha$, we have $\overline{D_\alpha} \subset C_\alpha$. Conversely, let $(\bar{x}, \bar{y}, \bar{z}) \in C_\alpha$. If $F(\bar{x}, \bar{y}, \bar{z}) < 1$, then from (30) it follows that $(\bar{x}, \bar{y}, \bar{z}) \in D_\alpha$. On the other hand, assume that $F(\bar{x}, \bar{y}, \bar{z}) = 1$ and let $f$ be the function defined by

$$f(\lambda) = F(\lambda\bar{x}, \lambda\bar{y}, \lambda\bar{z}), \qquad \lambda \in \mathbb{R}.$$

Then

$$f(1) = 1, \qquad , f'(1) = 2 + \alpha\bar{x}\bar{y}\bar{z} \geq 0, \qquad f''(1) = 2 + 4\alpha\bar{x}\bar{y}\bar{z}.$$

If $f'(1) > 0$, then $f$ is strictly increasing in a neighborhood of 1 and taking $\lambda_n \to 1$, $\lambda_n < 1$, we get that

$$\|(\lambda_n\bar{x}, \lambda_n\bar{y}, \lambda_n\bar{z})\| \leq \lambda_n < 1, \qquad F(\lambda_n\bar{x}, \lambda_n\bar{y}, \lambda_n\bar{z}) = f(\lambda_n) < 1,$$

from which it follows that $(\bar{x}, \bar{y}, \bar{z}) \in \overline{D}_\alpha$. If $f'(1) = 0$, then $f''(1) = -6$. Therefore $f$ has a local maximun at the point $\lambda = 1$, and we can conclude as above.

We proceed to show that $D_\alpha$ is convex. Let $u_1 = (x_1, y_1, z_1)$ and $u_2 = (x_2, y_2, z_2)$ be two different points of $D_\alpha$. Since $u_1$ and $u_2$ are inside the cube of radius 1, we have that $\|\lambda u_1 + (1 - \lambda)u_2\| < 1$ for every $0 \leq \lambda \leq 1$, and also that there exist $\lambda_1 < 0$ and $\lambda_2 > 1$ such that

$$\|\lambda_1 u_1 + (1 - \lambda_1)u_2\| = \|\lambda_2 u_1 + (1 - \lambda_2)u_2\| = 1.$$

From (30) it follows that

$$F(\lambda_1 u_1 + (1 - \lambda_1)u_2) \geq 1, \qquad F(\lambda_2 u_1 + (1 - \lambda_2)u_2) \geq 1.$$

We define now

$$g(\lambda) = F(\lambda u_1 + (1 - \lambda)u_2), \qquad \lambda \in \mathbb{R}.$$

The function $g$ is a polynomial function of degree $\leq 3$ such that

$$g(0) = F(u_2) < 1, \qquad g(1) = F(u_1) < 1,$$

$$g(\lambda_1) = F(\lambda_1 u_1 + (1 - \lambda_1)u_2) \geq 1, \qquad g(\lambda_2) = F(\lambda_2 u_1 + (1 - \lambda_2)u_2) \geq 1.$$

This clearly forces that $g(\lambda) < 1$ for every $0 \leq \lambda \leq 1$, and the proof of convexity is complete.

We leave it to the reader to verify that $\mathrm{Int} C_\alpha = D_\alpha$ and that the boundary points of $C_\alpha$ are those at which $F(x, y, z) = 1$.

It is also interesting to note here that the sets $C_2$ and $C_{-2}$ are not strictly convex.

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## References

1. P.W. Aitchison, A characterisation of the ellipsoid, *J. Austral. Math. Soc.* **11** (1970), 385–394.
2. G. Bianchi and P.M. Gruber, Characterizations of ellipsoids, *Arch. Math.* (4) **49** (1987), 344–350.
3. G.R. Burton, Sections of convex bodies, *J. London. Math. Soc.* (2) **12** (1976), 331–336.
4. G.R. Burton and P. Mani, A characterization of the ellipsoid in terms of concurrent sections, *Comment. Math. Helv.* I **53** (1978), 485–507.
5. H. Busemann, *The Geometry of Geodesics*, Academic Press, New York, 1955.
6. T. Kubota, On the theory of closed convex surface, *Proc. London Math Soc.* (2) **14** (1915), 230–239.
7. L. Montejano, Convex bodies with homothetic sections, *Bull. London. Math. Soc.* **23** (1991), 381–386.
8. S. Nakagawa, On theorems regarding ellipsoid, *Tôhoku Math. J.* **8** (1915), 11–13.
9. C.M. Petty, Ellipsoid, in *Convexity and Its Applications* (P. M. Gruber and J. M. Wills, eds.), Birkhäuser Verlag, Basel, 1983, pp. 264–276.

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