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# The Tight Span of an Antipodal Metric Space: Part II—Geometrical Properties*

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**Abstract.** Suppose that $X$ is a finite set and let $\mathbb{R}^X$ denote the set of functions that map $X$ to $\mathbb{R}$. Given a metric $d$ on $X$, the *tight span* of $(X, d)$ is the polyhedral complex $T(X, d)$ that consists of the bounded faces of the polyhedron

$$P(X, d) := \{f \in \mathbb{R}^X : f(x) + f(y) \geq d(x, y)\}.$$

In a previous paper we commenced a study of properties of $T(X, d)$ when $d$ is *antipodal*, that is, there exists an involution $\sigma\colon X \to X\colon x \mapsto \overline{x}$ so that $d(x, y) + d(y, \overline{x}) = d(x, \overline{x})$ holds for all $x, y \in X$. Here we continue our study, considering geometrical properties of the tight span of an antipodal metric space that arise from a metric with which the tight span comes naturally equipped. In particular, we introduce the concept of *cell-decomposability* for a metric and prove that the tight span of such a metric is the union of cells, each of which is isometric and polytope isomorphic to the tight span of some antipodal metric. In addition, we classify the antipodal cell-decomposable metrics and give a description of the polytopal structure of the tight span of such a metric.

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## 1. Introduction

We begin by reviewing some basic definitions on polyhedral complexes from [14]. For $n \geq 1$ an integer, a *polyhedron* in $\mathbb{R}^n$ is the intersection of a finite collection of half-spaces in $\mathbb{R}^n$ and a *polytope* is a bounded polyhedron. A *face* of a polyhedron $P$ is the empty-set, $P$ itself, or the intersection of $P$ with a supporting hyperplane and, if $P$ is $d$-dimensional, then its 0-, 1- and $(d-1)$-dimensional faces are called its *vertices*, *edges* and *facets*, respectively. The collection of all faces of a polytope forms a lattice with respect to the ordering given by set inclusion, and we say that two polytopes are *polytope isomorphic* if their face-lattices are isomorphic. A *polyhedral complex* $\mathcal{C}$ is a finite collection of polytopes such that each face of a member of $\mathcal{C}$ is itself a member of $\mathcal{C}$, and the intersection of two members of $\mathcal{C}$ is a face of each. We call the members of $\mathcal{C}$ *cells*.

Now, suppose that $(X, d)$ is a *metric space*, i.e. a set $X$ together with a map $d\colon X \times X \to \mathbb{R}$ that, for all $x, y, z \in X$, satisfies (i) $d(x, y) = 0 \iff x = y$, (ii) $d(x, y) = d(y, x)$ and (iii) $d(x, y) \leq d(x, z) + d(z, y)$. We call the map $d$ a *metric*[1] and, when it is clear from the context, we will use $(X, d)$ or $d$ interchangeably. If $X$ is finite, in which case we call $(X, d)$ a *finite* metric space, we can associate a polyhedral complex $T(X, d)$ to $(X, d)$ as follows. Let $\mathbb{R}^X$ denote the set of functions that map $X$ to $\mathbb{R}$. To the pair $(X, d)$ associate the polyhedron

$$P(X, d) := \{f \in \mathbb{R}^X : f(x) + f(y) \geq d(x, y) \text{ for all } x, y \in X\},$$

and let $T(X, d)$ consist of the bounded faces of $P(X, d)$. We call $T(X, d)$ the *tight span* of $(X, d)$. This fundamental mathematical construction was introduced by Isbell in [13], and was subsequently rediscovered and studied by Dress in [4] and by Chorbak and Lamore in [2].

In [11] we commenced a study of properties of the tight span of an *antipodal* metric space, that is, a finite metric space $(X, d)$ together with an involution $\sigma\colon X \to X\colon x \mapsto \overline{x}$ so that, for all $x, y \in X$, $d(x, y) + d(y, \overline{x}) = d(x, \overline{x})$ (see also [10]). In particular, amongst other things, we proved that a finite metric space $(X, d)$ is antipodal if and only if $T(X, d)$ has a unique maximal cell, and presented a way to parameterize the facets of the tight span of such a metric space.

These results are mainly concerned with combinatorial properties of the tight span of an antipodal metric space (i.e. properties of its face-lattice). However, the tight span also has a rich geometrical structure. For example, in [4] it is shown that if $(X, d)$ is a finite metric space, then the map $d_\infty\colon T(X, d) \times T(X, d) \to \mathbb{R}$ defined, for $f, g \in T(X, d)$, by

$$d_\infty(f, g) := \max_{x \in X} |f(x) - g(x)|$$

is a metric, and also that $(X, d)$ embeds canonically and isometrically into $T(X, d)$ via the map

$$h = h^X\colon X \to T(X, d)\colon x \mapsto (h_x = h_x^X : X \to \mathbb{R} : y \mapsto d(x, y)).$$

[1] Note that $d$ is sometimes called a *proper* metric.

Here, we build on the results presented in [11], considering geometrical properties of the tight span of an antipodal metric space that arise from the metric $d_\infty$. Although we concentrate mostly on antipodal metric spaces, our main results also have implications for the structure theory of the tight span of a general finite metric space and so we now present them in a broader context.

In general, combinatorial and geometrical properties of the tight span of a finite metric space are intimately linked and difficult to understand, although much progress has been made in understanding the tight span of a *totally split-decomposable*[2] metric (see, e.g. [6]). Deriving features of the tight span of such a metric has proven especially useful and, indeed, a whole theory has been built up to deal with totally split-decomposable metrics and their applications within phylogenetic analysis (see, e.g. [1], [8], and [12]). Even though much is known about the cells and cell-structure of the tight span of a totally split-decomposable metric [6], a complete description of these structures still remains somewhat elusive. However, we have found that antipodal metric spaces will probably form an essential part of any such description. To explain why this is the case, we begin with some definitions.

Suppose that $(X, d)$ is a finite metric space and that $f$ is some element in $T(X, d)$. We denote by $[f]$ the minimal cell in $T(X, d)$ that contains $f$ (under cell inclusion) and, for $x \in X$, we call $g \in [f]$ a *gate in* $[f]$ *for* $x$ if, for all $h \in [f]$,

$$d_\infty(h_x, h) = d_\infty(h_x, g) + d_\infty(g, h).$$

If such an element $g$ exists, then it is necessarily unique, and we denote it by $f^x$. In addition, we call the cell $[f]$ $X$-*gated* if there is a gate $f^x$ in $[f]$ for every $x \in X$. In Fig. 1 we present an example of the tight span of a six-point totally split-decomposable metric space, in which the "central" three-dimensional cell is $X$-gated. In fact it can be shown that every cell in this tight span is $X$-gated and, motivated by examples such as this, we call a finite metric space $(X, d)$ *cell-decomposable* if every cell in $T(X, d)$ is $X$-gated.

The following result, that we prove in Section 3, indicates the importance of antipodal metrics to the structure theory of the tight span. In particular, it implies that every cell in the tight span of a cell-decomposable metric can be basically regarded as being the tight span of an antipodal metric space.

**Theorem 1.1.** *Suppose that* $(X, d)$ *is a finite metric space. If* $f \in T(X, d)$ *with* $[f] \neq \{f\}$ *and* $[f]$ *is* $X$*-gated, then*

$$(A := \{f^x \in [f] : x \in X\}, d' := d_\infty|_A)$$

*is an antipodal metric space. Moreover, the map*

$$\varphi\colon [f] \to T(A, d')\colon g \mapsto \tilde{g},$$

[2] A finite metric $d\colon X \times X \to \mathbb{R}$ is totally split-decomposable if, for all $t, u, v, w, x \in X$, $\alpha^d_{\{t,u\},\{v,w\}} \leq \alpha^d_{\{t,x\},\{v,w\}} + \alpha^d_{\{t,u\},\{v,x\}}$, where, for all $a, a', b, b' \in X$, $\alpha^d_{\{a,a'\},\{b,b'\}} := \frac{1}{2}(\max(d(a, a') + d(b, b'), d(a, b) + d(a', b'), d(a, b') + d(a', b)) - d(a, a') - d(b, b'))$.

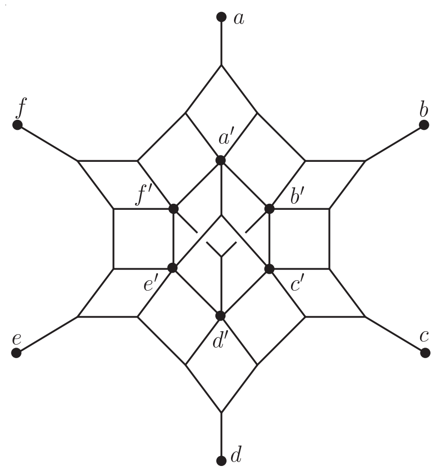


**Fig. 1.** The 1-skeleton (i.e. the union of the zero- and one-dimensional cells) of the tight span of the six-point metric on the set $X = \{a, b, c, d, e, f\}$ induced by taking the usual graph metric between the appropriately labelled vertices. The tight span consists of a "central" three-dimensional cell, that is polytope isomorphic to a 3-cube and that is bordered by twelve two-dimensional cells. Six of these two-dimensional cells are, in turn, adjacent to a one-dimensional cell or "antenna". For each element $x \in X$, the vertex $x'$ is the gate in the three-dimensional cell for $x$.

*where $\tilde{g}$ is defined, for each $a \in A$ and any $x \in X$ with $a = f^x$, by putting*

$$\tilde{g}(a) := g(x) - f^x(x),$$

*is a bijective isometry that induces a polytope isomorphism between $[f]$ and $T(A, d')$.*

In [1] it is shown that every metric on four or less points is totally split-decomposable. It is not hard to check, using the description of the structure of the tight span of a generic metric on five or less points contained in [4], that every metric on four or less points is also cell-decomposable and that a five-point metric is totally split-decomposable if and only if it is cell-decomposable. The main result of this paper states that this latter result also holds for an antipodal metric.

Before giving the precise statement of this result, we recall some definitions. Given a finite metric space $(X, d)$, the *underlying graph* $UG(X, d)$ associated with $(X, d)$ is the graph with vertex set $X$ and edge set consisting of those pairs $\{x, y\} \in \binom{X}{2}$ for which there is no $z \in X$ distinct from $x$ and $y$ with $d(x, y) = d(x, z) + d(z, y)$. In addition, we denote by $K_{3\times 2}$ the graph with six vertices that is the complement of the disjoint union of three edges, and, for $n \geq 3$ an integer, we let $C_n$ denote the $n$-cycle.

**Theorem 1.2.** *Suppose that $(X, d)$ is an antipodal metric space so that, in particular, $\#X = 2n$ with $n \geq 3$ an integer.*

(a) *If $n = 3$, then $d$ is both totally split-decomposable and cell-decomposable. Moreover, $UG(X, d)$ equals $C_6$ ($K_{3\times 2}$) if and only if $T(X, d)$ is polytope isomorphic to a 3-cube (a rhombic dodecahedron).*

(b) *If* $n \geq 4$, *then the following statements are equivalent*:
  (i) $UG(X, d)$ *equals* $C_{2n}$.
  (ii) $d$ *is cell-decomposable*.
  (iii) $d$ *is totally split-decomposable*.
  (iv) $T(X, d)$ *is polytope isomorphic to an* $n$*-cube*.

**Remark 1.3.** An explicit description of the class of antipodal totally split-decomposable metrics is given in [7].

In light of this theorem and preceding discussions, we make the following conjecture.

**Conjecture 1.4.** *A finite metric space* $(X, d)$ *is cell-decomposable if and only if it is totally split-decomposable*.

If this conjecture were true, then it would imply, for example, that the tight span of a totally split-decomposable metric space would consist of cells that are polytope isomorphic to either $n$-cubes or rhombic dodecahedra (see Remark 5.3 of [11]). This is in accordance with [1], results from which imply that these cells must be zonotopes.

The rest of the paper is organized as follows. In Section 2 we summarize some well-known results concerning the tight span. In Section 3 we prove Theorem 1.1, and in Section 4 we prove Theorem 1.2. Note that we frequently make use of results contained in [11].

## 2. Preliminaries

In this section we present a list of results concerning the tight span of a metric space that we use throughout the paper. For proofs and more detailed explanations of these results see [4], [5] and [9].

Suppose that $(X, d)$ is a finite metric space. Given a function $f$ in $T(X, d)$, we define its *tight-equality graph* to be the graph $K(f)$ with vertex set $X$ and edge set consisting of those $\{x, y\} \in \binom{X}{2}$ with $f(x) + f(y) = xy$.

(TS1) If $f \in T(X, d)$, then the minimal cell in $T(X, d)$ containing $f$ equals

$$[f] := \{g \in T(X, d) \ : \ K(f) \subseteq K(g)\}.$$

(TS2) As a consequence of (TS1), for $f, g \in T(X, d)$ we have $[g] \subseteq [f]$ if and only if $K(f)$ is a subgraph of $K(g)$.

(TS3) If $f \in T(X, d)$, then the dimension of the cell $[f]$ equals the number of connected bipartite components of $K(f)$.

(TS4) If $\#X \geq 2$, then $f \in T(X, d)$ if and only if for all $x \in X$ there is some $y \in X$ distinct from $x$ with $\{x, y\} \in E(K(f))$.

(TS5) As a consequence of (TS3) and (TS4), the dimension of $T(X, d)$ (i.e. the largest dimension for any cell in $T(X, d)$) is bounded by $\lfloor \#X/2 \rfloor$.

(TS6) If $Y \subseteq X$, and $f \in T(Y, d|_Y)$, then there exists some $g \in T(X, d)$ with $g|_Y = f$.

(TS7) If $f \in T(X, d)$ and $f(x) = 0$ for some $x \in X$, then $f = h_x$.
(TS8) For all $x \in X$ and all $f \in T(X, d)$,

$$d_\infty(f, h_x) = f(x).$$

## 3. Proof of Theorem 1.1

Suppose that $(X, d)$ is a finite metric space and $f \in T(X, d)$ with $[f] \neq \{f\}$. Since $f$ is in $T(X, d)$, for each $x \in X$ there must exist some $y \in X$ with $f(x) + f(y) = xy$ by (TS4) and (TS7). Hence, for each $x \in X$ the set

$$\rho(x) := \{y \in X \;:\; f(x) + f(y) = xy\}$$

is non-empty. We divide the rest of the proof into a series of claims from which the theorem immediately follows.

Define a map $\sigma\colon A \to A$ as follows: given $a \in A$ and $x$ any element of $X$ with $a = f^x$, put $\sigma(a) := b$ where $b := f^y$ with $y$ any element of $\rho(x)$.

**Claim 1.** *The map $\sigma$ is well defined.*

*Proof.* We begin by making some preliminary observations. Suppose $x \in X$, $y \in \rho(x)$ and $g \in [f]$. Note that, since $K(f) \subseteq K(g)$ by (TS2), by (TS8) we must have

$$d_\infty(h_x, h_y) = xy = g(x) + g(y) = d_\infty(h_x, g) + d_\infty(g, h_y).$$

In addition, since $f^x$ and $f^y$ are gates, we have

$$d_\infty(h_x, g) = d_\infty(h_x, f^x) + d_\infty(f^x, g)$$

and

$$d_\infty(g, h_y) = d_\infty(g, f^y) + d_\infty(f^y, h_y),$$

from which it follows that

$$d_\infty(f^x, g) + d_\infty(g, f^y) = d_\infty(f^x, f^y) \tag{1}$$

must hold.

Now we show that for $z \in X$ with $f^z = f^x$, we must have $\rho(z) \subseteq \{t \in X \;:\; f^t = f^y\}$. From this it immediately follows that $\sigma$ is well defined (indeed, if $z$ is such that $f^z = f^x$ and $t \in \rho(z)$, then $f^t = f^y$ follows).

Suppose $t \in \rho(z)$, then—by replacing $g$ by $f^t$ in (1)—we must have

$$d_\infty(f^x, f^t) + d_\infty(f^t, f^y) = d_\infty(f^x, f^y),$$

and—by replacing $g$ by $f^y$, $f^x$ by $f^z$ and $f^y$ by $f^t$, in (1)—we must have

$$d_\infty(f^z, f^y) + d_\infty(f^y, f^t) = d_\infty(f^z, f^t).$$

Hence, since $f^x = f^z$, by adding the last two equations we have $d_\infty(f^t, f^y) = 0$, so that $f^t = f^y$. This completes the proof of Claim 1. □

**Claim 2.** *The metric $d'$ on $A$ is antipodal with respect to the map $\sigma$.*

*Proof.* It is clear from the definition of $\sigma$ that $\sigma \circ \sigma = Id|_A$. Moreover—as can be seen by replacing $f^y$ by $\sigma(f^x)$ in (1)—for all $x \in X$ and $g \in [f]$,

$$d_\infty(f^x, g) + d_\infty(g, \sigma(f^x)) = d_\infty(f^x, \sigma(f^x)). \tag{2}$$

Now by replacing $g$ by $f^z$ for any $z \in X$ in this last equation, we immediately see that $d'$ is an antipodal metric. □

**Claim 3.** *The map $\varphi$ is a bijective isometry.*

*Proof.* First note that the map $\tilde{g}$ is well defined. Indeed, suppose $x \in X$ and $g \in [f]$. Then since $d_\infty(h_x, g) = d_\infty(h_x, f^x) + d_\infty(f^x, g)$, it immediately follows from (TS8) that $d_\infty(f^x, g) = g(x) - f^x(x)$. Hence if $x, y \in X$ with $f^x = f^y$, then $g(x) - f^x(x) = g(y) - f^y(y)$. Thus $\tilde{g}$ is well defined.

We now show that $\tilde{g} = \varphi(g)$ is contained in $T(A, d')$ for any $g \in [f]$. Since $g(x) = d_\infty(g, h_x) = d_\infty(g, f^x) + f^x(x)$ for all $x \in X$, for $g$ any element in $[f]$, we have

$$\begin{aligned}\tilde{g}(f^x) + \tilde{g}(f^y) &= g(x) - f^x(x) + g(y) - f^y(y)\\ &= d_\infty(g, f^x) + f^x(x) - f^x(x) + d_\infty(g, f^y) + f^y(y) - f^y(y)\\ &= d_\infty(f^x, g) + d_\infty(g, f^y)\end{aligned}$$

for all $x, y \in X$. Hence it immediately follows by the triangle inequality that $\tilde{g}(f^x) + \tilde{g}(f^y) \geq d_\infty(f^x, f^y)$ holds for all $x, y \in X$ and so $\tilde{g} \in P(A, d')$. By (2), it then follows that $\tilde{g}(f^x) + \tilde{g}(\sigma(f^x)) = d_\infty(f^x, \sigma(f^x))$ holds for all $x \in X$, whence, by (TS8), $\tilde{g} \in T(A, d')$ as required.

To see that $\varphi$ is surjective, suppose $q \in T(A, d')$, and define a map $q'\colon X \to \mathbb{R}$ by $q'(x) := q(f^x) + f^x(x)$. We show $q' \in [f]$ and, since $\varphi(q') = q$ clearly holds, it follows that $\varphi$ is surjective. To this end, suppose $x, y \in X$. Then

$$\begin{aligned}q'(x) + q'(y) &= q(f^x) + q(f^y) + f^x(x) + f^y(y)\\ &\geq d_\infty(f^x, f^y) + d_\infty(f^x, h_x) + d_\infty(f^y, h_y)\\ &\geq d_\infty(h_x, h_y) = xy.\end{aligned}$$

Moreover, since $(A, d')$ is antipodal, it follows from Lemma 4.1(ii) of [11] (applied to the map $q \in T(A, d')$ and the points $f^x$ and $f^y = \sigma(f^x)$) and the fact that $f^x$ and $f^y$ are gates, that equality holds in the last two inequalities for all $x, y \in X$ with $f(x) + f(y) = xy$. Hence, $q'$ is contained in $[f]$.

It is now straightforward to see that the map $\varphi$ preserves distances. From this it immediately follows that $\varphi$ is a bijective isometry. □

**Claim 4.** *The map $\varphi$ induces a polytope isomorphism.*

*Proof.* We show that if $f \in T(X, d)$, then $\varphi([f]) = [\tilde{f}]$ from which the claim follows in view of Claim 3. □

Suppose $f \in T(X,d)$. From the above considerations it follows that for all $g \in [f]$ and all $x, y \in X$ we have $d_\infty(f^x, g) + d_\infty(g, f^y) = d_\infty(f^x, f^y)$ if and only if $\tilde{g}(f^x) + \tilde{g}(f^y) = d_\infty(f^x, f^y)$ if and only if $d_\infty(h_{f^x}, \tilde{g}) + d_\infty(\tilde{g}, h_{f^y}) = d_\infty(h_{f^x}, h_{f^y})$. Hence for all $g \in [f]$, we see that $\{x, y\}$ is an edge of $K(g)$ if and only if $\{f^x, f^y\}$ is an edge of $K(\tilde{g})$. Using this, it is straightforward to check that $\varphi([f]) = [\tilde{f}]$ holds. □

**Remark.** Suppose that $(X,d)$ is a finite metric space, and that $C$ is a cell in $T(X,d)$ with dimension greater than zero. If $C$ is $X$-gated, then, by the last theorem, there exists a bijective isometry from $C$ to the tight span of the metric induced by $d_\infty$ on the gates of $C$ that induces a polytope isomorphism. However, the condition that $C$ is $X$-gated is not necessary for this conclusion to hold. For example, the conclusion holds for all one-dimensional cells in the tight span of any metric space. Note also that in case $C$ is polytope isomorphic to the tight span of a subset $S$ of its vertices with the metric induced by $d_\infty$, then $(S, d_\infty|_S)$ is necessarily antipodal by Theorem 4.2 of [11] since $T(S, d_\infty|_S)$ has only one maximal cell. It is still an open problem to determine whether in this case $C$ and $T(S, d_\infty|_S)$ are isometric.

It would also be interesting to prove a converse of Theorem 1.1. In particular, suppose that for all $x \in X$, there exists an element $g_x$ of $C$ with

$$d_\infty(h_x, g_x) = d_\infty(h_x, C) := \inf\{d_\infty(h_x, g) \,:\, g \in C\}$$

so that the metric space $(Y := \{g_x \,:\, x \in X\}, d_\infty|_Y)$ is antipodal, then does it follow that $C$ can be mapped isometrically onto $T(Y, d_\infty|_Y)$ so as to induce a polytope isomorphism? Moreover, if in addition $g_x$ is the only such element of $C$ for each $x \in X$, then does it follow that $C$ is $X$-gated?

## 4. Proof of Theorem 1.2

For clarity, we state four theorems that we prove later, and use these to prove Theorem 1.2.

**Theorem 4.1.** *Suppose that $(X,d)$ is an antipodal metric space with $\#X = 2n$, $n \geq 2$ an integer.*

(i) *If $T(X,d)$ is polytope isomorphic to an $n$-cube, then $UG(X,d)$ equals $C_{2n}$.*
(ii) *If $n = 3$ and $T(X,d)$ is polytope isomorphic to a rhombic dodecahedron, then $UG(X,d)$ equals $K_{3\times 2}$.*

**Theorem 4.2.** *Suppose that $(X,d)$ is an antipodal metric space with $\#X = 2n$, $n \geq 2$ an integer. If $UG(X,d)$ equals $C_{2n}$, then $d$ is cell-decomposable and $T(X,d)$ is polytope isomorphic to an $n$-cube.*

**Theorem 4.3.** *Suppose that $(X,d)$ is an antipodal metric space with $\#X = 6$. If $UG(X,d)$ equals $K_{3\times 2}$, then $d$ is cell-decomposable and $T(X,d)$ is polytope isomorphic to a rhombic dodecahedron.*

**Theorem 4.4.** *Suppose that $(X, d)$ is an antipodal metric space with $\#X = 2n$, $n \geq 4$ an integer. If $d$ is cell-decomposable, then $UG(X, d)$ equals $C_{2n}$.*

We now use these results to prove Theorem 1.2.

(a) In Theorem 5.1 of [11] we prove that if $(X, d)$ is an antipodal metric space with $\#X = 2n$, $n \geq 2$ an integer, then $d$ is totally split-decomposable if and only if $UG(X, d)$ is $C_{2n}$ or $n = 3$ and $UG(X, d)$ equals $K_{3\times 2}$. Moreover, in Corollary 3.3 of [11] we prove that a graph $H$ is the underlying graph of a six-point antipodal metric space if and only if $H$ is $C_6$ or $K_{3\times 2}$.

Now, suppose that $(X, d)$ is an antipodal metric space with $\#X = 6$. In view of the results just stated, it immediately follows that $d$ is totally split-decomposable and, as a consequence of Theorem 4.2 (with $n = 3$) and Theorem 4.3, that $d$ is cell-decomposable. Moreover, Theorem 4.1(i) and Theorem 4.2 (with $n = 3$) imply that $UG(X, d)$ equals $C_6$ if and only if $T(X, d)$ is polytope isomorphic to a 3-cube, and Theorem 4.1(ii) and Theorem 4.3 imply that $UG(X, d)$ equals $K_{3\times 2}$ if and only if $T(X, d)$ is polytope isomorphic to a rhombic dodecahedron.

(b) (i) $\Leftrightarrow$ (ii) follows immediately from Theorems 4.2 and 4.4. (i) $\Leftrightarrow$ (iv) follows immediately from Theorems 4.1(i) and 4.2. To complete the proof of (b), note that (i) $\Leftrightarrow$ (iii) follows immediately from Theorem 5.1 of [11], the statement of which we gave in the proof of (a) above.

### 4.1. *Proof of Theorem* 4.1

Suppose that $(X, d)$ is an antipodal metric space with $\#X \geq 2n$, $n \geq 2$ an integer. Put $E_X := \{\{x, \overline{x}\} : x \in X\}$. In Proposition 3.2 of [11] we proved that $UG(X, d)$ must be 2-connected,[3] and in Theorem 6.1 of [11] we showed that

$$\{K(f) : [f] \text{ is a facet of } T(X, d)\} \\ = \{(X, E_X \cup \{x, y\}) : \{x, \overline{y}\} \text{ is an edge of } UG(X, d)\}$$

must hold. Thus, if $T(X, d)$ is polytope isomorphic to an $n$-cube, then $UG(X, d)$ must be a 2-connected graph with $2n$-vertices and $2n$-edges. Hence $UG(X, d)$ equals $C_{2n}$ and so (i) holds. Moreover, if $n = 3$ and $T(X, d)$ is polytope isomorphic to a rhombic dodecahedron, then $UG(X, d)$ must have six vertices and twelve edges and so $UG(X, d)$ equals $K_{3\times 2}$. □

### 4.2. *Proof of Theorem* 4.2

Suppose that $(X, d)$ is an antipodal metric space with $\#X = 2n$, $n \geq 2$ an integer, and that $UG(X, d)$ is the $2n$-cycle $x_0, x_1, \ldots, x_{2n-1}, x_{2n} = x_0$. For all $1 \leq i \leq 2n$ put $a_i := d(x_{i-1}, x_i)$, noting that $a_i = a_{n+i}$ for all $1 \leq i \leq n$.

[3] A connected graph $G = (V, E)$ with at least three vertices is called 2-*connected* if there exists no single vertex whose removal from $V$ (together with all its incident edges) results in a disconnected graph.

Consider $\mathbb{R}^n$ with the standard basis, and denote the coordinates for any $\mathbf{z} \in \mathbb{R}^n$ by $z_j$, $1 \leq j \leq n$. Define a metric $d_1$ on $\mathbb{R}^n$ by, for any $\mathbf{z}, \mathbf{w} \in \mathbb{R}^n$, putting $d_1(\mathbf{z}, \mathbf{w}) = \sum_{j=1}^{n} |z_j - w_j|$. Also, put

$$P(a_1, \ldots, a_n) := \{\mathbf{y} \in \mathbb{R}^n : 0 \leq y_j \leq a_j \text{ for } j = 1, \ldots, n\},$$

a polytope in $\mathbb{R}^n$ that is clearly polytope isomorphic to an $n$-cube.

Now, for each $0 \leq i \leq n$, define the vector $\mathbf{z}^i$ in $\mathbb{R}^n$ by

$$z_j^i := \begin{cases} a_j & \text{if } \quad 1 \leq j \leq i, \\ 0 & \text{else} \end{cases}$$

and for each $n + 1 \leq i \leq 2n - 1$, define the vector $\mathbf{z}^i$ in $\mathbb{R}^n$ by

$$z_j^i := \begin{cases} 0 & \text{if } \quad j \leq i - n, \\ a_j & \text{else.} \end{cases}$$

Put $Z := \{\mathbf{z}^0, \ldots, \mathbf{z}^{2n-1}\}$. Clearly, $Z \subseteq P$. Also, it is straightforward to see that $d_1(\mathbf{z}^i, \mathbf{z}^j) = d(x_i, x_j)$ holds for all $0 \leq i < j \leq 2n-1$, so that $(X, d)$ and $(Z, d' := d_1|_Z)$ are isometric metric spaces.

Consider the map

$$\varphi\colon\ P \to \mathbb{R}^Z\colon\ \mathbf{y} \mapsto (\varphi_{\mathbf{y}}\colon\ Z \to \mathbb{R}\colon\ \mathbf{z} \mapsto d_1(\mathbf{y}, \mathbf{z})).$$

We will show that $\varphi$ maps $P$ bijectively onto $T(Z, d')$ and induces a polytope isomorphism between $P$ and $T(Z, d')$. From this it immediately follows that $T(X, d)$ is polytope isomorphic to an $n$-cube.

**Claim 1.** *$\varphi$ maps $P$ bijectively onto $T(Z, d')$.*

*Proof.* Suppose $\mathbf{s} \in P$. The definition of $d_1$ and the triangle inequality immediately imply $\varphi_{\mathbf{s}} \in P(Z, d')$. Moreover, the definition of $Z$ implies

$$d_1(\mathbf{z}^i, \mathbf{z}^{n+i}) = \varphi_{\mathbf{s}}(\mathbf{z}^i) + \varphi_{\mathbf{s}}(\mathbf{z}^{n+i})$$

for all $i = 0, \ldots, n - 1$. Hence, $\varphi(P) \subseteq T(Z, d')$.

For $f \in T(Z, d')$, put $\psi(f)_i := (a_i + f(\mathbf{z}^{i-1}) - f(\mathbf{z}^i))/2$ for all $1 \leq i \leq n$ and define a map

$$\psi\colon\ T(Z, d') \to P\colon\ f \mapsto (\psi(f)_i)_{1 \leq i \leq n}.$$

Note that $\psi$ is well defined since, for all $1 \leq i \leq n$ and all $f \in T(Z, d')$,

$$|f(\mathbf{z}^{i-1}) - f(\mathbf{z}^i)| \leq a_i.$$

We now show that $\varphi \circ \psi$ and $\psi \circ \varphi$ equal the identity map on $T(Z, d')$ and $P$, respectively. Claim 1 then follows immediately.

We first prove that $\varphi \circ \psi$ is the identity map on $T(Z, d')$. Suppose $f \in T(Z, d')$. We show $\varphi_{\psi(f)}(\mathbf{z}) = f(\mathbf{z})$ for all $\mathbf{z} \in Z$. Since $(Z, d')$ is an antipodal metric space, it is straightforward to see that $f(\mathbf{z}^i)+f(\mathbf{z}^{n+i}) = d'(\mathbf{z}^i, \mathbf{z}^{n+i})$ holds for all $i = 0, 1, \dots, n-1$ (see Lemma 4.1(ii) of [11]). Hence, without loss of generality, we may assume $\mathbf{z} = \mathbf{z}^k$ for some $0 \le k \le n-1$. Thus, since $0 \le \psi(f)_i \le a_i$ for all $1 \le i \le n$, we have

$$|\psi(f)_i - z_i| = |\psi(f)_i - z_i^k| = \begin{cases} (a_i + f(\mathbf{z}^i) - f(\mathbf{z}^{i-1}))/2 & \text{if} \quad 1 \le i \le k, \\ (a_i + f(\mathbf{z}^{i-1}) - f(\mathbf{z}^i))/2 & \text{if} \quad k+1 \le i \le n, \end{cases}$$

and so

$$\varphi_{\psi(f)}(\mathbf{z}) = \frac{1}{2}\left[\sum_{i=1}^{n} a_i - f(\mathbf{z}^0) - f(\mathbf{z}^n)\right] + f(\mathbf{z}).$$

Thus, since $\sum_{i=1}^{n} a_i = d_1(\mathbf{z}^0, \mathbf{z}^n) = f(\mathbf{z}^0) + f(\mathbf{z}^n)$, it follows that $\varphi_{\psi(f)}(\mathbf{z}) = f(\mathbf{z})$, as required.

We now show that $\psi \circ \varphi$ is the identity map on $P$. Clearly, it suffices to show that for $\mathbf{y} \in P$, $\psi(\varphi_{\mathbf{y}})_j = y_j$ holds for all $1 \le j \le n$. However, this follows by a direct computation using the fact that for all $i \in \{1, \dots, n\}$ and for all $j \in \{1, \dots, n\} - i$, we have $z_j^{i-1} = z_j^i$ which implies

$$|y_i - z_j^{i-1}| - |y_i - z_j^i| = 0,$$

and $z_i^{i-1} = 0$, $z_i^i = a_i$ which imply

$$|y_i - z_i^{i-1}| - |y_i - z_i^i| = y_i - a_i + y_i.$$

This concludes the proof of Claim 1. □

Suppose $\mathbf{y} \in P$. Clearly, the minimal cell in $P$ that contains $\mathbf{y}$ equals

$$[\mathbf{y}] := \{\mathbf{u} \in P : u_j = y_j \text{ for all } j \in \{1, \dots, n\} \text{ with } y_j \in \{0, a_j\}\}.$$

**Claim 2.** *$\varphi$ induces a polytope isomorphism between $P$ and $T(Z, d')$.*

*Proof.* We show that for all $\mathbf{y} \in P$, the minimal cell in $T(Z, d')$ containing $\varphi_{\mathbf{y}}$ equals $\varphi([\mathbf{y}])$, that is, $[\varphi_{\mathbf{y}}] = \varphi([\mathbf{y}])$. The claim follows immediately from this and Claim 1.

Suppose $\mathbf{y} \in P$. We first show $[\varphi_{\mathbf{y}}] \subseteq \varphi([\mathbf{y}])$. Suppose $f \in [\varphi_{\mathbf{y}}]$. By Claim 1, there exists some $\mathbf{s} \in P$ with $\varphi_{\mathbf{s}} = f$, and so it suffices to show $\mathbf{s} \in [\mathbf{y}]$. Suppose not. Then there exists some $1 \le k \le n$ with $y_k \in \{0, a_k\}$ and $s_k \ne y_k$. If $y_k = 0$, then let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ with $u_i = a_i$ if $i < k$ and $u_i = 0$ else, and $v_i = 0$ if $i \le k$ and $v_i = a_i$ else. Clearly, $\mathbf{u}, \mathbf{v} \in P$ and it is straightforward to show that $\{\mathbf{u}, \mathbf{v}\}$ is an edge of $K(\varphi_{\mathbf{y}})$. By assumption, $\varphi_{\mathbf{s}} \in [\varphi_{\mathbf{y}}]$ and so, by (TS2), $\{\mathbf{u}, \mathbf{v}\} \in E(K(\varphi_{\mathbf{s}}))$. However, then

$$|0 - s_k| + |0 - s_k| = |u_k - s_k| + |v_k - s_k| = |u_k - v_k| = 0$$

and so $s_k = 0$, a contradiction. Similar arguments show that if $y_k = a_k$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ with $u_i = a_i$ if $i \le k$ and $u_i = 0$ else and $v_i = 0$ if $i < k$ and $v_i = a_i$ else, then we also arrive at a contradiction.

We now show $\varphi([\mathbf{y}]) \subseteq [\varphi_{\mathbf{y}}]$. Suppose $f \in \varphi([\mathbf{y}])$. Then there exists some element $\mathbf{s} \in [\mathbf{y}]$ with $f = \varphi_{\mathbf{s}}$. We will show $E(K(\varphi_{\mathbf{y}})) \subseteq E(K(\varphi_{\mathbf{s}}))$. It then immediately follows by (TS2) that $[f] = [\varphi_{\mathbf{s}}] \subseteq [\varphi_{\mathbf{y}}]$, and so, in particular, $f \in [\varphi_{\mathbf{y}}]$, as required.

Suppose $\mathbf{u}, \mathbf{v} \in Z$ with $\{\mathbf{u}, \mathbf{v}\}$ an edge of $K(\varphi_{\mathbf{y}})$. Note first that $u_j \neq v_j$, for all $j \in \{1, \ldots, n\}$ with $y_j \notin \{0, a_j\}$. Indeed, if there exists some $j \in \{1, \ldots, n\}$ with $u_j = v_j$ and $y_j \notin \{0, a_j\}$, then since $\{\mathbf{u}, \mathbf{v}\} \in E(K(\varphi_{\mathbf{y}}))$,

$$0 = |u_j - v_j| = |u_j - y_j| + |y_j - u_j|$$

and thus $v_j = u_j = y_j$ which, in turn, implies $u_j = v_j \notin \{0, a_j\}$, a contradiction. Next note $y_j = s_j$ for all $j \in \{1, \ldots, n\}$ with $y_j \notin \{0, a_j\}$ and $|u_j - v_j| = a_j = |u_j - s_j| + |v_j - s_j|$ for all $y_j \notin \{0, a_j\}$. Showing $\{\mathbf{u}, \mathbf{v}\} \in E(K(\varphi_{\mathbf{s}}))$ is now straightforward using the definition of the tight-equality graph. This concludes the proof of Claim 2. □

To show that $d$ is cell-decomposable we introduce a new concept. If $(S, \rho)$ is a metric space, then we call a sequence of elements $s_1, \ldots, s_m \in S$, $m \geq 2$ an integer, a *geodesic* if $\rho(s_1, s_m) = \sum_{i=1}^{m-1} \rho(s_i, s_{i+1})$. We now show that $\varphi$ maps certain geodesics in $P$ to geodesics in $T(Z, d')$.

**Claim 3.** *If* $\mathbf{z}, \mathbf{s} \in Z$ *and* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in P$ *with* $\mathbf{z}, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{s}$ *a geodesic in* $P$, *then* $\varphi_{\mathbf{z}}, \varphi_{\mathbf{u}}, \varphi_{\mathbf{v}}, \varphi_{\mathbf{w}}, \varphi_{\mathbf{s}}$ *is a geodesic in* $T(Z, d')$.

*Proof.* We begin by observing that if $\mathbf{z}, \mathbf{x} \in Z$ and $\mathbf{y} \in P$ are such that

$$|d_1(\mathbf{x}, \mathbf{z}) - d_1(\mathbf{y}, \mathbf{x})| = \max_{\mathbf{s} \in Z} |d_1(\mathbf{s}, \mathbf{z}) - d_1(\mathbf{y}, \mathbf{s})|,$$

then

$$\begin{aligned} d_1(\mathbf{z}, \mathbf{y}) &= \sum_{j=1}^{n} |z_j - y_j| \\ &= \sum_{j=1}^{n} (|x_j - z_j| - |x_j - y_j|) \\ &= |d_1(\mathbf{x}, \mathbf{z}) - d_1(\mathbf{y}, \mathbf{x})| \\ &= \max_{\mathbf{s} \in Z} |d_1(\mathbf{s}, \mathbf{z}) - d_1(\mathbf{y}, \mathbf{s})| \\ &= d_\infty(\varphi_{\mathbf{z}}, \varphi_{\mathbf{y}}). \end{aligned}$$

Now suppose $\mathbf{z}, \mathbf{s} \in Z$ and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in P$ with $\mathbf{z}, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{s}$ a geodesic in $P$. Then using the observation just mentioned together with the fact that for all $\mathbf{a}, \mathbf{b} \in P$ we have $d_\infty(\varphi_{\mathbf{a}}, \varphi_{\mathbf{b}}) \leq d_1(\mathbf{a}, \mathbf{b})$ by the triangle-inequality, it is straightforward to check that $\varphi_{\mathbf{z}}, \varphi_{\mathbf{u}}, \varphi_{\mathbf{v}}, \varphi_{\mathbf{w}}, \varphi_{\mathbf{s}}$ is a geodesic in $T(Z, d')$. This concludes the proof of Claim 3. □

We now prove that $d$ is cell-decomposable. We must show that any cell in $T(Z, d')$ is $Z$-gated. Suppose $F \subseteq T(Z, d')$ is a cell, $f$ is contained in $T(Z, d')$ with $F = [f]$, $g \in [f]$ and $\mathbf{z} \in Z$. By Claim 1, there exist vectors $\mathbf{y}, \mathbf{v} \in P$ with $\varphi_{\mathbf{y}} = f$ and $\varphi_{\mathbf{v}} = g$ and, by Claim 2, $\mathbf{v} \in [\mathbf{y}]$.

Consider the vectors $\mathbf{u}, \mathbf{w} \in \mathbb{R}^n$ defined by

$$u_j := \begin{cases} z_j & \text{if} \quad y_j \notin \{0, a_j\}, \\ y_j & \text{else,} \end{cases} \qquad \text{and} \qquad w_j := \begin{cases} a_j - z_j & \text{if} \quad y_j \notin \{0, a_j\}, \\ y_j & \text{else,} \end{cases}$$

$1 \leq j \leq n$, and the vector $\mathbf{s} \in \mathbb{R}^n$ defined by putting $s_j := a_j - z_j$ for all $1 \leq j \leq n$. A straightforward computation shows that $\mathbf{z}, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{s}$ is a geodesic in $P$. Hence, by Claim 3, $\varphi_{\mathbf{z}}, \varphi_{\mathbf{u}}, \varphi_{\mathbf{v}}, \varphi_{\mathbf{w}}, \varphi_{\mathbf{s}}$ is a geodesic in $T(Z, d')$. It follows that $\varphi_{\mathbf{u}}$ is a gate for $\mathbf{z}$ in $[f] = [\varphi_y]$. Indeed, if this was not the case, then, since $\varphi_{\mathbf{z}} = h_{\mathbf{z}}$ and $\varphi_{\mathbf{v}} \in [f]$,

$$\begin{aligned} d_\infty(\varphi_{\mathbf{z}}, \varphi_{\mathbf{s}}) &= d_\infty(\varphi_{\mathbf{z}}, \varphi_{\mathbf{u}}) + d_\infty(\varphi_{\mathbf{u}}, \varphi_{\mathbf{v}}) + d_\infty(\varphi_{\mathbf{v}}, \varphi_{\mathbf{w}}) + d_\infty(\varphi_{\mathbf{w}}, \varphi_{\mathbf{s}}) \\ &> d_\infty(\varphi_{\mathbf{z}}, \varphi_{\mathbf{v}}) + d_\infty(\varphi_{\mathbf{v}}, \varphi_{\mathbf{w}}) + d_\infty(\varphi_{\mathbf{w}}, \varphi_{\mathbf{s}}) \\ &\geq d_\infty(\varphi_{\mathbf{z}}, \varphi_{\mathbf{s}}), \end{aligned}$$

which is impossible. This concludes the proof of the theorem. □

## 4.3. *Proof of Theorem* 4.3

We use two results to prove Theorem 4.3. The first one allows us to show that an antipodal metric is cell-decomposable by considering the facets of its tight span.

**Proposition 4.5.** *Suppose that $(X, d)$ is an antipodal metric space. Then $d$ is cell-decomposable if and only if for every facet $F$ of $T(X, d)$*,

(i) *$F$ is $X$-gated*, *and*
(ii) *the metric induced by $d_\infty$ on the gates of $X$ in $F$ is cell-decomposable.*

*Proof.* Suppose that $d$ is cell-decomposable. Then, by definition, every facet of $T(X, d)$ is $X$-gated and thus (i) holds.

We now show that (ii) holds. Note that if the facets of $T(X, d)$ are vertices of $T(X, d)$, then (ii) clearly holds. Now, suppose $F \subseteq T(X, d)$ is a facet of $T(X, d)$ with dimension greater than zero, and that $C \subseteq F$ is a cell in $F$. Let $x \in X$. Then, by assumption, there exists a gate $f_C^x$ for $x$ in $C$ and a gate $f_F^x$ for $x$ in $F$. Thus, if $g$ is in $C$, then since $C$ is $X$-gated, we have

$$d_\infty(g, h_x) = d_\infty(g, f_C^x) + d_\infty(f_C^x, h_x),$$

and since $F$ is $X$-gated and $C \subseteq F$, we have

$$d_\infty(f_C^x, h_x) = d_\infty(f_C^x, f_F^x) + d_\infty(f_F^x, h_x)$$

as well as

$$d_\infty(g, h_x) = d_\infty(g, f_F^x) + d_\infty(f_F^x, h_x).$$

It follows that

$$d_\infty(g, f_F^x) = d_\infty(g, f_C^x) + d_\infty(f_C^x, f_F^x)$$

holds for all $g \in C$. Hence, defining $A := \{f_F^x \in F : x \in X\}$ and letting $\varphi$: $F \to T(A, d' := d_\infty|_A)$ be the bijection given by Theorem 1.1, it follows that $\varphi(C)$ is $A$-gated. Thus $d'$ is cell-decomposable and so (ii) holds.

We now prove the converse. Suppose (i) and (ii) hold. Note that $T(X, d)$ has a unique maximal cell, by Theorem 4.2 of [11], and so this cell is clearly $X$-gated (for all $x \in X$ the gate $x$ is $h_x$). Moreover, every vertex in $T(X, d)$ is clearly $X$-gated. Hence it suffices to show that any cell $C \subseteq T(X, d)$ with dimension not equal to zero or to the dimension of $T(X, d)$ is $X$-gated.

In order to see this, suppose that $C$ is such a cell, $x \in X$ and $g \in C$. Then there must exist some facet $F \subseteq T(X, d)$ containing $C$, and hence by (i) there exists some gate $f_F^x \in F$ for $x$. In view of Theorem 1.1 and the fact that $C$ is a subset of $F$, there must exist some $f_C^x \in C$ with

$$d_\infty(f_F^x, g) = d_\infty(f_F^x, f_C^x) + d_\infty(f_C^x, g).$$

Now since $F$ is $X$-gated by (i) and $C \subseteq F$ we have

$$d_\infty(h_x, g) = d_\infty(h_x, f_F^x) + d_\infty(f_F^x, g)$$

and

$$d_\infty(h_x, f_C^x) = d_\infty(h_x, f_F^x) + d_\infty(f_F^x, f_C^x).$$

Hence

$$d_\infty(h_x, g) = d_\infty(h_x, f_C^x) + d_\infty(f_C^x, g)$$

holds for all $g \in C$, so that $C$ is $X$-gated, as required. □

The second result, whose proof if straightforward and stated without proof, describes the possible tight-equality graphs for a vertex of the tight span of a six-point antipodal metric.

**Lemma 4.6.** *Suppose that $(X, d)$ is an antipodal six-point metric space so that, in particular, $G := UG(X, d)$ equals $C_6$ or $K_{3\times 2}$. Put $X = \{x, y, z, \overline{x}, \overline{y}, \overline{z}\}$ and $E_X := \{\{x, \overline{x}\}, \{y, \overline{y}\}, \{z, \overline{z}\}\}$. Suppose also that $f$ is a vertex of $T(X, d)$.*

(i) *If $G$ equals $C_6$ so that, without loss of generality, $G$ equals $x, y, z, \overline{x}, \overline{y}, \overline{z}, x$, then either*

$$E(K(f)) = E_X \cup E$$

*with $E$ equal to either $\{\{x, \overline{y}\}, \{\overline{y}, z\}, \{z, x\}\}$ or $\{\{y, \overline{x}\}, \{\overline{x}, \overline{z}\}, \{\overline{z}, y\}\}$, or*

$$E(K(f)) = E_X \cup \{\{a, b\} : b \in X\} \cup \{\{u, v\}\}$$

*for some $a \in X$ and $u, v$ the two vertices that are adjacent to $a$ in $G$ (in which case $f = h_a$ holds).*

(ii) *If $G$ equals $K_{3\times 2}$, so that the edges not contained in $G$ are precisely $\{x, \overline{x}\}$, $\{y, \overline{y}\}$ and $\{z, \overline{z}\}$, then either*

$$E(K(f)) = E_X \cup \{\{a, b\}, \{b, c\}, \{c, a\}\}$$

*with* $a, b, c \in X$ *distinct*, $b, c \in X - \{a, \overline{a}\}$ *and* $b \neq c, \overline{c}$, *or*

$$E(K(f)) = E_X \cup \{\{a, b\} : b \in X\},$$

*for some* $a \in X$ (*in which case* $f = h_a$ *holds*).

*Proof of Theorem* 4.3. Suppose that $(X, d)$ is an antipodal metric space with $\#X = 6$ and $UG(X, d)$ equals $K_{3\times 2}$. Put $X := \{x, y, z, \overline{x}, \overline{y}, \overline{z}\}$, so that the edges not contained in $UG(X, d)$ are precisely $\{x, \overline{x}\}$, $\{y, \overline{y}\}$ and $\{z, \overline{z}\}$.

We first prove that $d$ is cell-decomposable using Proposition 4.5. To do this we first describe the facets of $T(X, d)$. Suppose that $F$ is a facet of $T(X, d)$ and $f$ is some element in $T(X, d)$ with $F = [f]$. In view of the description of the tight-equality graphs of the facets of an antipodal metric space given in Theorem 6.1 of [11] (see the proof of Theorem 4.1 for the precise statement of this result), we can assume without loss of generality that $K(f)$ is the disjoint union of the path $\overline{x}, x, z, \overline{z}$ and the edge $\{y, \overline{y}\}$. Thus $[f]$ consists precisely of those maps $g\colon X \to \mathbb{R}_{\geq 0}$ that satisfy

$$\begin{aligned} g(\overline{x}) &= x\overline{x} - g(x), & g(z) &= xz - g(x), \\ g(\overline{v}) &= x\overline{z} + g(x), & g(\overline{y}) &= x\overline{x} - g(y), \end{aligned} \tag{3}$$

and

$$xy \leq g(x) + g(y) \leq xz + zy, \qquad zy - xz \leq g(x) - g(y) \leq xy. \tag{4}$$

This follows since the inequalities

$$2x\overline{x} - \overline{x}\,\overline{y} = 2x\overline{x} - xy \geq xz + zy, \qquad xy \geq y\overline{z} - \overline{z}x \quad \text{and} \quad xy \geq xz - zy$$

hold, and since (4) implies $0 \leq g(x) \leq xz$ and

$$0 < \frac{xy + zy - xz}{2} \leq g(y) \leq \frac{xz + zy + xy}{2} < x\overline{x}.$$

Now, let $g_1, g_2 \in [f]$ be defined by (3), (4) and

$$\begin{aligned} g_1(x) &= \frac{xy + xz - zy}{2}, & g_1(y) &= \frac{zy + xy - xz}{2}, \\ g_2(x) &= \frac{xy - xz - zy}{2}, & g_2(y) &= \frac{zy + xy + xz}{2}. \end{aligned}$$

Then it immediately follows from (4) that $[f]$ is the convex hull of $\{g_1, g_2, h_x, h_z\}$.

We now show that $[f]$ satisfies Proposition 4.5(i) and (ii), from which it immediately follows that $d$ is cell-decomposable. Since the metric induced on $\{g_1, g_2, h_x, h_z\}$ is antipodal and

$$\begin{aligned} x\overline{x} &= d_\infty(h_{\overline{x}}, h_z) + d_\infty(h_z, h_x), \\ v\overline{z} &= d_\infty(h_{\overline{z}}, h_x) + d_\infty(h_x, h_z), \\ y\overline{y} &= \frac{zy + xy - xz}{2} + xz + \left(y\overline{y} - \frac{zy + xy + xz}{2}\right) \\ &= d_\infty(h_y, g_1) + d_\infty(g_1, g_2) + d_\infty(g_2, h_{\overline{z}}), \end{aligned}$$

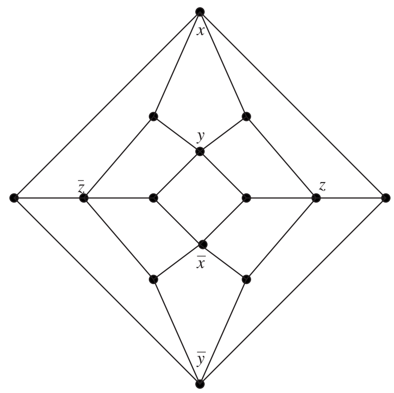


**Fig. 2.** A Schlegel diagram for the rhombic dodecahedron adapted from Fig.1.5B on p.8 of [3]. This is also a Schlegel diagram for the tight span of a six-point antipodal metric on the set $\{x, y, z, \overline{x}, \overline{y}, \overline{z}\}$ with the underlying graph equal to $K_{3\times 2}$.

it follows that $[f]$ is gated with respect to $X$. Thus $[f]$ satisfies (i). Moreover, (ii) clearly holds since there are four gates for $X$ in $[f]$ and, as remarked in the Introduction, any metric on a 4-set is cell-decomposable.

To complete the proof of the theorem, we need to show that $T(X, d)$ is polytope isomorphic to a rhombic dodecahedron. By Theorem 4.2 of [11] $T(X, d)$ is a three-dimensional polytope. Moreover, using Theorem 6.1 of [11] we can find the tight-equality graphs corresponding to the facets of $T(X, d)$, of which there are $\#E(K_{3\times 2}) = 12$, and, using Lemma 4.6, we can find the tight-equality graphs corresponding to the vertices of $T(X, d)$, of which there are 14. It follows by Euler's formula [14, p. 877] that $T(X, d)$ has $12 + 14 - 2 = 24$ edges. Using this, the tight-equality graphs corresponding to the facets and vertices of $T(X, d)$, and (TS2), it is now straightforward to find the tight-equality graphs corresponding to edges of $T(X, d)$ and then use (TS2) to check that the face-lattice of $T(X, d)$ is isomorphic to the face-lattice of the rhombic dodecahedron with a Schlegel diagram as pictured in Fig. 2. This completes the proof of the theorem. □

### 4.4. *Proof of Theorem* 4.4

Suppose that $(X, d)$ is an antipodal metric space with $\#X \geq 2n$, $n \geq 4$ an integer, and that $d$ is cell-decomposable. We will prove that $UG(X, d)$ equals $C_{2n}$ using induction on $n$.

We first show that the theorem holds for $n = 4$. Put $X = \{x, y, u, v, \overline{x}, \overline{y}, \overline{u}, \overline{v}\}$. We must prove that $UG(X, d)$ equals $C_8$. We begin with a claim concerning the structure of the tight-equality graph of elements in $T(X, d)$.

**Claim 1.** *There is no function $f \in T(X, d)$ with*

$$E(K(f)) = \{\{x, \overline{x}\}, \{y, \overline{y}\}, \{u, \overline{u}\}, \{v, \overline{v}\}, \{x, y\}, \{y, v\}, \{v, x\}\}.$$

*Proof.* Suppose to the contrary that there exists some $f \in T(X, d)$ with $E(K(f))$ as stated. Then since $x, y, v, x$ is a 3-cycle in $K(f)$, it follows for any $g \in [f]$ that

$$g(x) = \frac{xy + xv - yv}{2}.$$

Since $d$ is cell-decomposable there is a gate $f^x$ for $x$ in $[f]$. However, then if $g$ is any element of $[f]$ we have

$$\begin{aligned} f^x(x) &= \frac{xy + xv - yv}{2} \\ &= g(x) \\ &= d_\infty(h_x, g) \\ &= d_\infty(h_x, f^x) + d_\infty(f^x, g) \\ &= f^x(x) + d_\infty(f^x, g), \end{aligned}$$

and hence $f^x = g$. Thus $[f]$ must be a vertex of $T(X, d)$ and hence, by (TS3), $K(f)$ is connected and non-bipartite, a contradiction. This completes the proof of Claim 1. □

We now consider what happens when $d$ is restricted to a 6-subset of $X$ and gives rise to an antipodal metric.

**Claim 2.** *For every* 6*-subset $Y$ of $X$ with $d|_Y$ antipodal, there is a* 6*-subset $U \neq Y$ of $X$ with $d|_U$ antipodal and $UG(U, d|_U)$ equal to $C_6$.*

*Proof.* Let $Y$ be a 6-subset of $X$ with $d' := d|_Y$ antipodal, so that $UG(Y, d')$ equals either $C_6$ or $K_{3\times 2}$. Without loss of generality, we assume $Y = \{x, y, v, \overline{x}, \overline{y}, \overline{v}\}$.

Consider $T(Y, d')$. Since $T(Y, d')$ is a three-dimensional polytope by Theorem 4.2 of [11], it follows that it must have a vertex $f$ that is distinct from $h_y^Y$ for all $y \in Y$. Moreover, by Lemma 4.6, without loss of generality we can assume

$$E(K(f)) = \{\{x, \overline{x}\}, \{y, \overline{y}\}, \{v, \overline{v}\}, \{x, y\}, \{x, v\}, \{y, v\}\}.$$

Hence, by (TS6), $T(X, d)$ contains a vertex $g$ with $g|_Y = f$ for which the graph induced by $K(g)$ on $Y$ equals $K(f)$.

We now claim that there exists a pair of distinct elements $t_1, t_2$ in $\{x, y, v\}$ with $\{\overline{u}, t_2\}, \{u, t_1\} \in E(K(g))$. We prove this in two steps: we show (a) there exist distinct elements $t_1, t_2$ in $Y$ with $\{\overline{u}, t_2\}, \{u, t_1\} \in E(K(g))$, and then (b) $t_1, t_2 \in \{x, y, v\}$.

*Proof of* (a). Since $g$ is a vertex of $T(X, d)$, (TS3) implies that $K(g)$ is connected and so there exists some $t_1 \in Y$ with $\{u, t_1\} \in E(K(g))$. Moreover, there must exist some $t_2 \in Y$ with $\{\overline{u}, t_2\}$ in $E(K(g))$ for otherwise there would be some $\varepsilon > 0$ for which the map $g' \colon X \to \mathbb{R}_{\geq 0}$ defined by

$$z \mapsto \begin{cases} g(u) + \varepsilon & \text{if} \quad z = u, \\ g(\overline{u}) - \varepsilon & \text{if} \quad z = \overline{u}, \\ g(z) & \text{if} \quad z \in Y \end{cases}$$

is contained in $T(X, d)$, which is impossible in view of Claim 1 and the fact that $K(g')$ is the disjoint union of $K(f)$ and $\{u, \overline{u}\}$. In addition, $t_1$ and $t_2$ are distinct, since otherwise

$$\overline{t_1}\overline{u} = g(\overline{t_1}) + g(\overline{u}) \quad \text{and} \quad \overline{t_1}u = g(\overline{t_1}) + g(u),$$

which implies $g(\overline{t_1}) = 0$ and hence, by (TS7), $g = h^X_{\overline{t_1}}$ and so $h^Y_{\overline{t_1}} = h^X_{\overline{t_1}}|_Y = g|_Y = f$, which is impossible. This completes the proof of (a). □

*Proof of* (b). Suppose that (b) does not hold. Without loss of generality we can assume $t_1 = \overline{x}$ and, since $t_1$ and $t_2$ are distinct elements of $Y$, that $t_2 \in \{x, y, \overline{y}\}$. Put $U := \{x, y, u, \overline{x}, \overline{y}, \overline{u}\}$ and $g' := g|_U$. Note, by Lemma 4.1(iii) of [11], that $g' \in T(U, d|_U)$. Since the metric space $(U, d|_U)$ is antipodal and thus consists of a unique maximal cell, by (TS2) the tight-equality graph $K(g')$ must be a subgraph of the tight-equality graph of one of the vertices of $T(U, d|_U)$. Thus, by Lemma 4.6, $K(g')$ does not contain three vertices all having degree one and thus $K(g')$ is a subgraph of $K(h^U_w)$ for some $w \in U$. However, then $K(g')$ must have a vertex with degree one, and this vertex must be $\overline{y}$. Therefore, $K(g')$ is a subgraph of $K(h^U_y)$, but this is impossible since $\{\overline{x}, u\} \notin E(K(h^U_y))$. This concludes the proof of (b). □

Without loss of generality, $t_1 = x$ and $t_2 = y$. However, then $x, y, \overline{u}, u, x$ is a 4-cycle in $K(g)$ and so

$$xu + y\overline{u} = g(x) + g(u) + g(y) + g(\overline{u}) = xy + u\overline{u} = xy + uy + y\overline{u},$$

which implies $xu = xy + yu$. Hence, $\{x, u\}$ is not an edge of $UG(U, d|_U)$. Since $UG(U, d|_U)$ equals $K_{3\times2}$ or $C_6$ and $\{z, \overline{z}\}$ is not an edge of $UG(U, d|_U)$ for any $z \in U$, it follows that $UG(U, d|_U)$ equals $C_6$. This concludes the proof of Claim 2. □

We now complete the proof of the base case. Since there are four 6-subsets $Y$ of $X$ with $(Y, d|_Y)$ an antipodal metric space, it follows by Claim 2 that there must exist two 6-subsets $Y, Z$ of $X$ with $(Y, d|_Y)$ and $(Z, d|_Z)$ antipodal metric spaces and $UG(Y, d|_Y)$ and $UG(Z, d|_Z)$ equal to $C_6$.

Without loss of generality, we either have (i) $UG(Y, d|_Y) = x, y, u, \overline{x}, \overline{y}, \overline{u}, x$ and $UG(Z, d|_Z) = x, u, v, \overline{x}, \overline{u}, \overline{v}, x$, or (ii) $UG(Y, d|_Y) = x, y, u, \overline{x}, \overline{y}, \overline{u}, x$ and $UG(Z, d|_Z) = x, y, v, \overline{x}, \overline{y}, \overline{v}, x$. Before considering these cases, we observe that if $\{a, b\}$ is an edge in $UG(X, d)$ for some $a, b \in X$ and $W$ is a subset of $X$ containing $\{a, b\}$, then $\{a, b\}$ must also be an edge of $UG(W, d|_W)$.

Now, suppose (i) holds. Since $UG(X, d)$ is 2-connected [11, Proposition 3.2], it immediately follows from the observation just mentioned that $UG(X, d)$ contains the cycle $x, y, u, v, \overline{x}, \overline{y}, \overline{u}, \overline{v}, x$. Now suppose that $\{y, \overline{v}\}$ is an edge in $UG(X, d)$. Put $W := \{x, y, v, \overline{x}, \overline{y}, \overline{v}\}$. Then, since $UG(W, d|_W)$ is either $C_6$ or $K_{3\times2}$ and it contains the edge $\{y, \overline{v}\}$, it follows that $UG(W, d|_W)$ equals $K_{3\times2}$. Hence $\{x, v\}$ is an edge of $UG(W, d|_W)$ and it is not an edge of $UG(X, d)$ (as it is not an edge in $UG(Z, d|_Z)$). Hence either $d(x, v) = d(x, u) + d(u, v)$ or $d(x, v) = d(x, \overline{u}) + d(\overline{u}, v)$. This contradicts the fact that $\{x, v\}$ is not an edge in $UG(Z, d|_Z)$. Hence, $\{y, \overline{v}\}$ is not an edge of $UG(X, d)$ and so, by symmetry, it immediately follows that $UG(X, d)$ equals $C_8$.

Suppose that (ii) holds. Consider the set $W := \{x, u, v, \overline{x}, \overline{u}, \overline{v}\}$. Then $UG(W, d|_W)$ is either $C_6$ or $K_{3\times 2}$. If it equals $C_6$, then we are in case (i). So suppose that it equals $K_{3\times 2}$. Then, using similar arguments to case (i), it follows that $\{x, v\}$ is not an edge of $UG(X, d)$ and it is an edge of $UG(W, d|_W)$. However, this implies that $\{x, v\}$ is an edge of $UG(Z, d|_Z)$. This contradiction completes the proof of case (ii) and the base case.

To complete the proof, suppose that for any cell-decomposable antipodal metric space $(X, d)$ with $\#X = 2k$, $4 \leq k < n$, the underlying graph $UG(X, d)$ equals $C_{2k}$. The inductive step relies on the following claim.

**Claim 3.** *Suppose that $(X, d)$ is an antipodal cell-decomposable metric space. If $Z \subseteq X$ with $(Z, d' := d|_Z)$ an antipodal metric space, then $d'$ is cell-decomposable.*

*Proof.* We show that for $[f] \subseteq T(Z, d')$ any cell, the set

$$D := \{g \in T(X, d) : g|_Z \in [f]\}$$

(which is non-empty by (TS6)) is a cell in $T(X, d)$. From this it is straightforward to show that for all $z \in Z$, if $f^z \in T(X, d)$ is a gate for $z$ in $D$ (which exists since $d$ is cell-decomposable by assumption), then $f^z|_Z$ is a gate for $z$ is $[f]$, and so $d'$ is cell-decomposable, as required.

To see that $D$ is a cell in $T(X, d)$ choose some $g'$ in $D$ with $\#E(K(g'))$ minimal. We show $D = [g']$.

Suppose $h \in [g']$. Then, by Lemma 4.1(iii) of [11], $h|_Z \in T(Z, d')$. Moreover, since $K(g') \subseteq K(h)$ by (TS2), it follows that $K(g'|_Z) \subseteq K(h|_Z)$. Hence, since $K(f) \subseteq K(g'|_Z)$ it follows that $h|_Z \in [f]$ and so $h \in D$. Thus $[g'] \subseteq D$.

To see that the converse set inclusion holds, suppose $h \in D$. Consider the function $g^* := (g' + h)/2$. Then $g^*|_Z$ is contained in $[f]$ as every cell in $T(Z, d')$ is convex (since $d'$ is antipodal and so $T(Z, d')$ is a polytope). Moreover, $g^*(x) + g^*(y) = d(x, y)$ if and only if $g'(x) + g'(y) = d(x, y)$ and $h(x) + h(y) = d(x, y)$. Hence, $E(K(g^*)) \subseteq E(K(g'))$, and so $E(K(g^*)) = E(K(g'))$, by the minimality of $\#E(K(g'))$. However, then $E(K(g')) \subseteq E(K(h))$ and so $h \in [g']$. Thus $D \subseteq [g']$. This completes the proof of Claim 3. □

Now suppose $x \in X$ and denote the set of neighbours of $x$ in $UG(X, d)$ by $U$. In addition, suppose $y \in X - \{x, \overline{x}\}$ and put $Z := X - \{y, \overline{y}\}$. By induction and Claim 3, it follows that $UG(Z, d|_Z)$ is a $2(n-1)$-cycle. Hence, there exists some $u \in X - \{x, \overline{x}\}$ with $x$ not adjacent to either $u$ or $\overline{u}$ in $UG(X, d)$, so that, in particular, $u, \overline{u} \notin U$. Now consider $W := X - \{u, \overline{u}\}$. Then, using Claim 3 and induction once more, it follows that $UG(W, d|_W)$ is a $2(n-1)$-cycle, and hence $\#U \leq 2$. However, by Proposition 3.2 of [11] $UG(X, d)$ is 2-connected, and so $\#U = 2$. It follows that $UG(X, d)$ is a $2n$-cycle. This concludes the proof the theorem. □

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