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The Extreme Points of Subsets of *s*-Concave Probabilities and a Geometric Localization Theorem

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Abstract. We prove that the extreme points of the set of *s*-concave probability measures satisfying a linear constraint are some Dirac measures and some *s*-affine probabilities supported by a segment. From this we deduce that the constrained maximization of a convex functional on the *s*-concave probability measures is reduced to this small set of extreme points. This gives a new approach to a localization theorem due to Kannan, Lovász and Simonovits which happens to be very useful in geometry to obtain inequalities for integrals like concentration and isoperimetric inequalities. Roughly speaking, the study of such inequalities is reduced to these extreme points.

1. Introduction

The localization lemma of Lovász and Simonovits [7] indicates that if one wants to prove an inequality for all $1/n$-concave measures on $\mathbb{R}^n$, it is enough to test this inequality over all $1/n$-affine measures supported by a segment. This may reduce the problem considerably. The precise statement is as follows:

Theorem [7]. *Let f and g be two lower semi-continuous Lebesgue integrable functions on $\mathbb{R}^n$ such that*

$$\int_{\mathbb{R}^n} f(x)\,dx > 0 \quad and \quad \int_{\mathbb{R}^n} g(x)\,dx > 0.$$

Then there exists a, $b \in \mathbb{R}^n$ and an affine function $\ell\colon [0, 1] \to \mathbb{R}^+$ such that

$$\int_0^1 f((1-t)a + tb)\ell(t)^{n-1}\,dt > 0 \quad and \quad \int_0^1 g((1-t)a + tb)\ell(t)^{n-1}\,dt > 0.$$

The proof given in [7] consists in using infinitely many times the "bisection method" in an algorithmic procedure to perform the construction of a sequence of measures which converges to a $1/n$-affine measure supported by a segment satisfying the same inequalities. In [6] Kannan et al. deduce from this result a theorem in terms of products of integrals over convex sets (see Corollary 1 below) that they can extend to the case of log-concave measures.

In this paper we generalize the localization lemma of Lovász and Simonovits [7] to the s-concave measures on $\mathbb{R}^n$ for $-\infty \leq s \leq \frac{1}{2}$. The case $s = 0$ corresponds to the log-concave measures. The definitions and well-known results about such measures are recalled in Section 2 of the paper. Our method consists in replacing the algorithmic procedure by the use of the Krein–Milman theorem. More precisely, in Theorem 1 we describe the extreme points of the convex hull of the set of all s-concave probabilities μ supported in a compact subset of $\mathbb{R}^n$ satisfying the constraint $\int f \, d\mu \geq 0$, where f is a fixed upper semi-continuous function. Here again, the "bisection method" is at the heart of the proof to decide which probabilities are extreme in this set. Then a standard use of the Krein–Milman theorem shows that the supremum over this set of any convex functional is attained at these extreme points, i.e. either at Dirac measures or at some s-affine probabilities supported by a segment, this is our Theorem 2. An easy example is to consider the functional $\mu \mapsto \int g \, d\mu$ for an upper semi-continuous function g and we recover the conclusion of Lovász and Simonovits for two integrals.

The main results of this paper explained above are contained in Section 3, while the proof of the main theorem is postponed to Section 4. In Section 5 of the paper we prove a generalization to the case of several constraints. Such a generalization is announced to be true by Kannan et al. in [6]. In this part we use Borsuk's theorem to find an affine subspace which simultaneously bisects several integrals.

This localization principle is very powerful to obtain dimension-free inequalities. Under different constraints, it is the main tool to prove concentration inequalities [7], [5], general isoperimetric inequalities [6] and also to study distributional inequalities for polynomials over convex bodies in $\mathbb{R}^n$ [1], [4], [8].

2.   Preliminaries on s-Concave Measures on $\mathbb{R}^n$

Given subsets A and B of the Euclidean n-space $\mathbb{R}^n$ and $\lambda > 0$, we set $A + B = \{x + y; \ x \in A, y \in B\}$ and $\lambda A = \{\lambda x; \ x \in A\}$. For a convex set C, we denote by $\mathcal{E}\text{xt}(C)$ the set of extreme points of C. For all $s \in [-\infty, +\infty]$, we say in this paper that a measure μ in $\mathbb{R}^n$ is s-concave if the inequality

$$\mu(\lambda A + (1 - \lambda)B) \geq [\lambda \mu^s(A) + (1 - \lambda)\mu^s(B)]^{1/s} \qquad (*)$$

holds for all compact subsets $A, B \subset \mathbb{R}^n$ such that $\mu(A)\mu(B) > 0$ and all $\lambda \in [0, 1]$. The limit cases are interpreted by continuity. Thus the right-hand side of this inequality is equal to $\mu^\lambda(A) \, \mu^{1-\lambda}(B)$ for $s = 0$, to $\min(\mu(A), \mu(B))$ for $s = -\infty$, and to $\max(\mu(A), \mu(B))$ for $s = +\infty$. Notice that an s-concave measure is t-concave for all $t \leq s$. We denote by $\mathcal{P}(K)$ the set of probabilities in $\mathbb{R}^n$ supported in a convex compact K. For a probability μ, $\text{supp}(\mu)$ denotes its support.

For $\gamma \in [-\infty, +\infty]$, a function $f\colon \mathbb{R}^n \to \mathbb{R}^+$ is γ-concave if the inequality

$$f(\lambda x + (1 - \lambda)y) \geq [\lambda f^\gamma(x) + (1 - \lambda)f^\gamma(y)]^{1/\gamma}$$

holds for all x and y such that $f(x)f(y) > 0$ and all $\lambda \in [0, 1]$, where the limit cases are also interpreted by continuity. For example, the $-\infty$-concave functions are the quasi-concave ones (i.e. the functions f such that $\{f \geq t\}$ is convex for all $t \in \mathbb{R}$). The link between the s-concave probabilities and the γ-concave functions is described in the work of Borell [3]. Notice that for $s > 0$, his definition differs a little from ours because the inequality $(*)$ was not restricted to the compact sets A and B such that $\mu(A)\mu(B) > 0$. In particular, with our definition, for all $s \leq 1/d$, there exists s-concave measures supported in an affine subspace of dimension d and, for example, Dirac measures are s-concave for any $s \in [-\infty, +\infty]$.

Theorem [3]. *Let μ be a measure in $\mathbb{R}^n$, let G be the least affine subspace which contains the support of μ, set $d = \dim G$ and let m be the Lebesgue measure on G. Then for $-\infty \leq s \leq 1/d$, μ is s-concave if and only if $d\mu = \psi\, dm$, where $0 \leq \psi \in L^1_{\mathrm{loc}}(\mathbb{R}^n, dm)$ and ψ is γ-concave with $\gamma = s/(1 - sd) \in [-1/d, +\infty]$. Moreover, if $s > 1/d$, then μ is s-concave if and only if μ is a Dirac measure.*

According to this theorem, we say that a measure μ is s-affine when its density ψ satisfies that ψ^γ (or $\log \psi$ if $s = \gamma = 0$) is affine on its convex support with $\gamma = s/(1 - sd)$. We will also need the following simple lemma.

Lemma 1. *Let K be a convex set in $\mathbb{R}^n$, $\gamma \in [-\infty, 1]$. If $C\colon K \to \mathbb{R}^+$ is γ-concave and $V\colon K \to \mathbb{R}^+$ is γ-affine, then $(C - V)_+\colon K \to \mathbb{R}^+$ is γ-concave.*

Proof. Let $\gamma \in (-\infty, 1] \setminus \{0\}$ and let x and y in K be such that $C(x) \geq V(x)$ and $C(y) \geq V(y)$. By Minkowski's inequality for $\|\cdot\|_\gamma$ we know that for all $\lambda \in [0, 1]$,

$$(\lambda C^\gamma(x) + (1 - \lambda)C^\gamma(y))^{1/\gamma} - (\lambda V^\gamma(x) + (1 - \lambda)V^\gamma(y))^{1/\gamma}$$
$$\geq (\lambda(C - V)^\gamma(x) + (1 - \lambda)(C - V)^\gamma(y))^{1/\gamma}$$

and as C is γ-concave and V is γ-affine, we get for all x and y in the support of $(C - V)_+$,

$$(C - V)(\lambda x + (1 - \lambda)y) \geq (\lambda(C - V)^\gamma(x) + (1 - \lambda)(C - V)^\gamma(y))^{1/\gamma}.$$

By sending γ to 0 in Minkowski's inequality, we obtain the arithmetico-geometric inequality which gives the same conclusion for log-concave functions. The case $\gamma = -\infty$ is obvious. $\square$

3. Results

Our main result is the following theorem.

Theorem 1. *Let n be a positive integer, let K be a compact convex set in $\mathbb{R}^n$, let $s \in [-\infty, \frac{1}{2}]$ and let $f\colon K \to \mathbb{R}$ be an upper semi-continuous function. We denote by P_f the set of s-concave probabilities supported in K satisfying $\int f \, d\mu \geq 0$. The set of extreme points of $\mathrm{conv}\, P_f$ is exactly:*

1. *the Dirac measures at points x such that $f(x) \geq 0$,*
2. *the probabilities ν which are s-affine, supported by a segment $[a, b] \subset K$, such that $\int f \, d\nu = 0$ and $\int_a^x f \, d\nu > 0$ on (a, b) or $\int_x^b f \, d\nu > 0$ on (a, b).*

Before proving Theorem 1, we show that it immediately extends the localization lemma of Lovász and Simonovits [7] and we prefer to see it as a constrained optimization problem on s-concave probabilities in $\mathbb{R}^n$.

Theorem 2. *Under the same assumptions as in Theorem* 1, *if $\Phi\colon \mathcal{P}(K) \to \mathbb{R}$ is a convex upper semi-continuous function, then $\sup\{\Phi(\mu); \mu \in P_f\}$ is achieved at a probability ν such that $\nu = \delta_x$ with $f(x) \geq 0$ or ν is s-affine on a segment $[a, b]$, $\int f \, d\nu = 0$ and $\int_a^x f \, d\nu > 0$ on (a, b) or $\int_x^b f \, d\nu > 0$ on (a, b).*

Proof of Theorem 2.   By Theorem 2.2 of [2], we know that the set of s-concave probabilities supported in K is w^*-compact. Since f is upper semi-continuous, the condition $\int f \, d\mu \geq 0$ is w^*-closed, therefore the set P_f is w^*-compact. By application of the Krein–Milman theorem, $\sup\{\Phi(\mu); \mu \in P_f\}$ is achieved at a probability $\nu \in \mathcal{E}\mathrm{xt}(\overline{\mathrm{conv}}^{w^*} P_f) \subset \mathcal{E}\mathrm{xt}(\mathrm{conv}\, P_f)$. The result follows by the description of the extreme points of $\mathrm{conv}\, P_f$ given in Theorem 1. $\qquad\square$

As a direct application, we deduce the following generalization of the geometric KLS localization theorem in its more popular form:

Corollary 1. *Let f_1, f_2 be two upper semi-continuous non-negative functions on $\mathbb{R}^n$ and let f_3, f_4 be two lower semi-continuous non-negative functions on $\mathbb{R}^n$. Let $-\infty \leq s \leq \frac{1}{2}$ and $\alpha, \beta > 0$. Suppose that $f_1^\alpha f_2^\beta \leq f_3^\alpha f_4^\beta$ and that for every $a, b \in \mathbb{R}^n$, for every s-affine probability ν supported by $[a, b]$,*

$$\left(\int f_1 \, d\nu \right)^\alpha \left(\int f_2 \, d\nu \right)^\beta \leq \left(\int f_3 \, d\nu \right)^\alpha \left(\int f_4 \, d\nu \right)^\beta.$$

Then for every s-concave probability μ on $\mathbb{R}^n$,

$$\left(\int f_1 \, d\mu \right)^\alpha \left(\int f_2 \, d\mu \right)^\beta \leq \left(\int f_3 \, d\mu \right)^\alpha \left(\int f_4 \, d\mu \right)^\beta.$$

Proof.   By adding a small constant to f_3, we may assume that $f_3 > 0$ on $\mathbb{R}^n$. Let μ be an s-concave compactly supported probability in $\mathbb{R}^n$ and denote its support by K. Define

$$f = f_1 - \left(\frac{\int f_1 \, d\mu}{\int f_3 \, d\mu} \right) f_3, \qquad \Phi(\theta) = \left(\frac{\int f_1 \, d\mu}{\int f_3 \, d\mu} \right)^{\alpha/\beta} \left(\int f_2 \, d\theta \right) - \left(\int f_4 \, d\theta \right)$$

for every probability $\theta \in \mathcal{P}(K)$ and P_f as in Theorem 1. By the assumptions on f_1, f_2, f_3 and f_4, the functions f and Φ are upper semi-continuous and Φ is affine. Since $\mu \in P_f$, by Theorem 2, there exists a probability ν which is either a Dirac or s-affine and supported by a segment $[a, b]$ such that $\Phi(\mu) \leq \Phi(\nu)$ and $\int f \, d\nu \geq 0$. This gives

$$\left(\frac{\int f_1 \, d\mu}{\int f_3 \, d\mu} \right)^{\alpha/\beta} \int f_2 \, d\mu - \int f_4 \, d\mu \leq \left(\frac{\int f_1 \, d\mu}{\int f_3 \, d\mu} \right)^{\alpha/\beta} \int f_2 \, d\nu - \int f_4 \, d\nu$$

$$\leq \left(\frac{\int f_1 \, d\nu}{\int f_3 \, d\nu} \right)^{\alpha/\beta} \int f_2 \, d\nu - \int f_4 \, d\nu \leq 0.$$

By a standard approximation this gives the result for any s-concave probability in $\mathbb{R}^n$. $\square$

4. Proof of Theorem 1

Obviously the Dirac measures at points x such that $f(x) \geq 0$ are extreme points of $\operatorname{conv} P_f$ thus, from now on, we describe the extreme points of $\operatorname{conv} P_f$ which are not Dirac measures.

In steps 1–3 we fix a measure ν, extreme in $\operatorname{conv} P_f$, which is not a Dirac measure and prove that it satisfies the properties quoted in (2) of Theorem 1. We denote by G the least affine subspace which contains the support of ν.

Step 1: $\dim G = 1$.

Proof. If $\dim G \geq 2$ there exists x_0 in the relative interior of $\operatorname{supp} \nu$ and a two-dimensional subspace of $\mathbb{R}^n$, E such that $x_0 + E \subset G$. For all $u \in S_1(E)$, the unit circle in E, let $H_u = \{x \in G; \langle x - x_0, u \rangle = 0\}$, $H_u^+ = \{x \in G; \langle x - x_0, u \rangle \geq 0\}$ and $H_u^- = \{x \in G; \langle x - x_0, u \rangle \leq 0\}$ and define $\phi \colon S_1(E) \to \mathbb{R}$ by $\phi(u) = \int_{H_u^+} f \, d\nu - (\int f \, d\nu)/2$. For all $u \in S_1(E)$, H_u is a hyperplane in G, hence $\nu(H_u) = 0$. This implies that $\phi(-u) = -\phi(u)$ and that ϕ is continuous. Hence there exists $u_0 \in S_1(E)$ such that $\phi(u_0) = 0$. By the choice of x_0, it is clear that $\nu(H_{u_0}^+) > 0$ and $\nu(H_{u_0}^-) > 0$, thus we may define $\nu_1 = \nu_{|H_{u_0}^+} / \nu(H_{u_0}^+)$ and $\nu_2 = \nu_{|H_{u_0}^-} / \nu(H_{u_0}^-)$. Then $\nu = \nu(H_{u_0}^+)\nu_1 + \nu(H_{u_0}^-)\nu_2$ with ν_1 and $\nu_2 \in P_f \backslash \{\nu\}$, which means that ν is not extreme in $\operatorname{conv} P_f$.

From step 1 and Borell's characterization, the support of ν is a segment $[a, b]$, $d\nu = \psi \, dm$, where m is the Lebesgue measure on $[a, b]$, $0 \leq \psi \in L^1_{\text{loc}}(\mathbb{R}^n, dm)$ and ψ is γ-concave with $\gamma = s/(1 - s)$. We have $-\infty \leq s \leq \frac{1}{2}$, hence $\gamma \in [-1, 1]$.

Step 2: $\int f \, d\nu = 0$ and $\int_a^x f \, d\nu > 0$ on (a, b) or $\int_x^b f \, d\nu > 0$ on (a, b).

Proof. Notice that the function $x \mapsto \int_a^x f \, d\nu$ is continuous on $[a, b]$. If there exists $c \in (a, b)$ such that $\int_a^c f \, d\nu = 0$, define $\nu_1 = \nu_{|[a,c]} / \nu([a, c])$ and $\nu_2 = \nu_{|[c,b]} / \nu([c, b])$, then $\nu = \nu([a, c])\nu_1 + \nu([c, b])\nu_2$ with ν_1 and $\nu_2 \in P_f \backslash \{\nu\}$. This means that ν is not extreme in $\operatorname{conv} P_f$. If $\int f \, d\nu > 0$, then there exists $c \in (a, b)$ with $\int_a^c f \, d\nu = (\int f \, d\nu)/2 > 0$. Defining ν_1 and ν_2 in the same way we get the same result.

Step 3: ν is s-affine.

Proof. By step 2 we may assume that $\int_a^x f \, d\nu > 0$, for all $x \in (a, b)$. Let $c \in (a, b)$, $u = (b - a)/\|b - a\|_2$ and for $\alpha \in \mathbb{R}$, define $\varphi_\alpha \colon [a, b] \to \mathbb{R}$ by $\varphi_\alpha(x) = (\psi(c)/2)(1 + \gamma \alpha \langle x - c, u \rangle)_+^{1/\gamma}$ extending it by continuity for $\gamma = 0$. It is clear that φ_α is γ-affine on $[a, b]$. We consider the following measures with support included in $[a, b]$:

$$d\mu_\alpha = (\psi - \varphi_\alpha)_+ \, dm, \qquad d\nu_\alpha = \inf(\psi, \varphi_\alpha) \, dm.$$

Since $\gamma \in [-1, 1]$, ψ is γ-concave and φ_α is γ-affine on $[a, b]$, we get from Lemma 1 that $(\psi - \varphi_\alpha)_+$ is γ-concave so μ_α is s-concave. It is clear that $\inf(\psi, \varphi_\alpha)$ is also γ-concave so ν_α is also s-concave. By Lebesgue's theorem, the function $\alpha \mapsto \int f \, d\nu_\alpha$ is continuous on $\mathbb{R}$ and

$$\lim_{\alpha \to -\infty} \int f \, d\nu_\alpha = \int_a^c f \, d\nu > 0, \qquad \lim_{\alpha \to +\infty} \int f \, d\nu_\alpha = \int_c^b f \, d\nu < 0$$

since $\int f \, d\nu = 0$. Hence there exists α_0 such that $\int f \, d\nu_{\alpha_0} = 0$. Since $\nu = \mu_{\alpha_0} + \nu_{\alpha_0}$ this gives $\int f \, d\mu_{\alpha_0} = 0$ and taking $\lambda = \nu_{\alpha_0}([a, b]) \in (0, 1)$, we get $\nu = (1 - \lambda)\nu_1 + \lambda\nu_2$ where $\nu_1 = \mu_{\alpha_0}/(1 - \lambda)$ and $\nu_2 = \nu_{\alpha_0}/\lambda$. The probabilities ν_1 and ν_2 belong to P_f and since ν is extreme, we deduce that $\nu_1 = \nu_2 = \nu$ which means that $\psi = \varphi_{\alpha_0}/\lambda$ and ν is s-affine.

In step 4 we prove that a probability satisfying properties (2) of Theorem 1 is extreme in $\mathrm{conv} \, P_f$.

Step 4: *if a probability ν is s-affine on a segment $[a, b] \subset \mathbb{R}^n$, $\int f \, d\nu = 0$ and $\int_a^x f \, d\nu > 0$ on (a, b) or $\int_x^b f \, d\nu > 0$ on (a, b), then ν is extreme in $\mathrm{conv} \, P_f$.*

Proof. We define $F \colon [a, b] \to \mathbb{R}$ by $F(x) = \int_a^x f \, d\nu$, we may and do assume that $F > 0$ on (a, b). We denote by ψ the density of ν with respect to the Lebesgue measure m on $[a, b]$. It is a γ-affine function on $[a, b]$. Suppose that ν is the convex combination of measures $\mu_1, \ldots, \mu_p \in P_f \setminus \{\nu\}$:

$$\nu = \sum_{i=1}^p \lambda_i \mu_i, \qquad 0 < \lambda_i \le 1, \qquad \sum_{i=1}^p \lambda_i = 1. \tag{1}$$

For all $i = 1, \ldots, p$, $\int f \, d\mu_i \ge 0$, since $\mu_i \in P_f$, and by hypothesis on ν, $\int f \, d\nu = 0 = \sum_{i=1}^p \lambda_i \int f \, d\mu_i$ so $\int f \, d\mu_i = 0$. By (1), for each $i = 1, \ldots, p$, μ_i is absolutely continuous with respect to ν and since $\mu_i \in P_f$, it has a γ-concave density ψ_i with respect to m on $[a, b]$. Therefore, if we denote by ρ_i the density of μ_i with respect to ν, we have

$$\sum_{i=1}^p \lambda_i \rho_i = 1 \quad \text{on } [a, b] \qquad \text{and} \qquad \rho_i = \frac{d\mu_i}{d\nu} = \frac{\psi_i}{\psi}$$

is the quotient of a γ-concave function by a γ-affine function. It is easily checked that these properties imply that ρ_i is quasi-concave, non-negative and continuous on its support. Moreover a quasi-concave function on a segment is either monotone or first non-decreasing then non-increasing on this segment. Since $\sum_{i=1}^p \lambda_i \rho_i = 1$ on $[a, b]$, at least one of the functions ρ_i which are non-zero at a is non-increasing in a neighbourhood

of a. We may assume that one of these functions is ρ_1. Hence ρ_1 is non-increasing on its support $[a, c]$, where $c \in (a, b)$. Integrating by parts, we get

$$\int f \, d\mu_1 = 0 = \int_a^c \rho_1 \, dF = [\rho_1 F]_a^c - \int_a^c F \, d\rho_1 = \rho_1(c)F(c) - \int_a^c F \, d\rho_1.$$

Since $\rho_1(c)F(c) \geq 0$, ρ_1 is non-increasing and $F > 0$ on (a, c), this implies that ρ_1 is constant on $[a, c]$ and $F(c) = 0$. Therefore $c = b$ and since $d\mu_1 = \rho_1 \, d\nu$ with μ_1 and ν probabilities on $[a, b]$, $\rho_1 = 1$ on $[a, b]$ hence $\mu_1 = \nu$; this is absurd.

This ends the proof of Theorem 1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5. Generalization to Several Constraints

We do not know the exact characterization of the extreme points of the set of s-concave probabilities satisfying several constraints but we can still establish the following generalization of Theorem 1.

Theorem 3. *Let n be a positive integer, let K be a compact convex set in $\mathbb{R}^n$, let $p \in \{1, \ldots, n\}$ and let $-\infty \leq s \leq 1/(p + 1)$. For $f = (f_1, \ldots, f_p) : K \to \mathbb{R}^p$, with f_i upper semi-continuous functions, we denote by P_f the set of s-concave probabilities supported in K satisfying $\int f_i \, d\mu \geq 0, \forall i$. Let ν be an extreme point of $\mathrm{conv}\, P_f$ and denote by G the least affine subspace which contains its support. Then $\dim G \leq p$ and if $\dim G = p$, then ν is s-affine on its support, $\int f \, d\nu = 0$ and for all x_0 in the relative interior of $\mathrm{supp}(\nu)$, for all u in the unit sphere of G, $\int_{\langle x - x_0, u \rangle \geq 0} f \, d\nu \neq 0$. Moreover, if $\Phi: \mathcal{P}(K) \to \mathbb{R}$ is a convex upper semi-continuous function then $\sup_{\mu \in P_f} \Phi(\mu)$ is achieved at an extreme point of $\mathrm{conv}\, P_f$ as described above.*

Proof. Fix a measure ν, extreme in $\mathrm{conv}\, P_f$ and let G be the least affine subspace which contains the support of ν, we can assume that $\dim G \geq p$.

<u>Step 1</u>: $\dim G = p$.

Proof. If $\dim G \geq p + 1$ there exists x_0 in the relative interior of $\mathrm{supp}\, \nu$ and a $(p + 1)$-dimensional subspace of $\mathbb{R}^n$, E such that $x_0 + E \subset G$. For all $u \in S_p(E)$, the unit sphere in E, let $H_u = \{x \in G; \langle x - x_0, u \rangle = 0\}$, $H_u^+ = \{x \in G; \langle x - x_0, u \rangle \geq 0\}$ and $H_u^- = \{x \in G; \langle x - x_0, u \rangle \leq 0\}$ and define $\phi : S_p(E) \to \mathbb{R}^p$ by $\phi(u) = \int_{H_u^+} f \, d\nu - (\int f \, d\nu)/2$. For all $u \in S_p(E)$, H_u is a hyperplane in G, hence $\nu(H_u) = 0$. This implies that $\phi(-u) = -\phi(u)$ and that ϕ is continuous. Hence, from Borsuk's theorem, there exists $u_0 \in S_p(E)$ such that $\phi(u_0) = 0$. By the choice of x_0, it is clear that $\nu(H_{u_0}^+) > 0$ and $\nu(H_{u_0}^-) > 0$, thus we may define $\nu_1 = \nu_{|H_{u_0}^+}/\nu(H_{u_0}^+)$ and $\nu_2 = \nu_{|H_{u_0}^-}/\nu(H_{u_0}^-)$. Then $\nu = \nu(H_{u_0}^+)\nu_1 + \nu(H_{u_0}^-)\nu_2$ with ν_1 and $\nu_2 \in P_f \backslash \{\nu\}$, which means that ν is not extreme in $\mathrm{conv}\, P_f$.

<u>Step 2</u>: *for all x_0 in the interior of $\mathrm{supp}(\nu)$, for all u in the unit sphere of G, $\int_{\langle x - x_0, u \rangle \geq 0} f \, d\nu \neq 0$ and $\int f \, d\nu = 0$.*

Proof. Assume first that there exists x_0 in the interior of supp(ν), and u in the unit sphere of G, such that $\int_{\langle x-x_0, u\rangle \geq 0} f \, d\nu = 0$. We define $H^+ = \{x \in G; \ \langle x - x_0, u\rangle \geq 0\}$, $H^- = \{x \in G; \ \langle x - x_0, u\rangle \leq 0\}$, $\nu_1 = \nu_{|H^+}/\nu(H^+)$ and $\nu_2 = \nu_{|H^-}/\nu(H^-)$. Then $\nu = \nu(H^+)\nu_1 + \nu(H^-)\nu_2$ with ν_1 and $\nu_2 \in P_f \setminus \{\nu\}$, which means that ν is not extreme in conv P_f.

Now we prove that $\int f \, d\nu = 0$. Assume that $\int f \, d\nu \neq 0$. We identify G with $\mathbb{R}^p$ and we denote by e_{p+1} a vector orthogonal to G in $\mathbb{R}^n$. For every $u \in S^p$, the unit sphere of $G \oplus \mathbb{R}e_{p+1}$, let $H_u^+ = \{x \in G; \ \langle x - e_{p+1}, u\rangle \geq 0\}$ and $H_u^- = \{x \in G; \ \langle x - e_{p+1}, u\rangle \leq 0\}$. This is a parametrization of the half-spaces of G by S^p. As in step 1, we define $\phi: S^p \to \mathbb{R}^p$ by $\phi(u) = \int_{H_u^+} f \, d\nu - (\int f \, d\nu)/2$. The same use of Borsuk's theorem gives a vector $u_0 \in S^p$ such that $\phi(u_0) = 0$. We conclude as in step 1 that ν is not extreme in conv P_f.

Observe that the first part is also valid when dim $G < p$ while in the second part, the use of Borsuk's theorem requires dim $G = p$.

Step 3: ν is s-affine.

Proof. By the characterization of Borell and steps 1 and 2, ν is absolutely continuous with respect to the Lebesgue measure m on G, $d\nu/dm = \psi$ is γ-concave on G with $\gamma = s/(1-sp)$ and $\int f \, d\nu = 0$. As in step 2, we identify G with $\mathbb{R}^p$, we denote by e_{p+1} a vector orthogonal to G in $\mathbb{R}^n$ and by S^p the unit sphere of $G \oplus \mathbb{R}e_{p+1}$. Let c be within the relative interior of supp(ν). For every $u \in S^p$ such that $\langle u, e_{p+1}\rangle \neq 0$, we define a γ-affine function $\varphi_u: K \to \mathbb{R}$ by

$$\varphi_u(x) = \frac{\psi(c)}{2}\left(1 + \gamma \frac{\langle x - c, u\rangle}{\langle e_{p+1}, u\rangle}\right)_+^{1/\gamma},$$

extending it by continuity for $\gamma = 0$. We also define $\phi: S^p \to \mathbb{R}^p$ by

$$\phi(u) = \begin{cases} \int f(\psi - \varphi_u)_+ \, dm & \text{if} \quad \langle u, e_{p+1}\rangle > 0, \\ \int f \inf(\psi, \varphi_u) \, dm & \text{if} \quad \langle u, e_{p+1}\rangle < 0, \\ \int_{\langle x-c, u\rangle \geq 0} f \, d\nu & \text{if} \quad \langle u, e_{p+1}\rangle = 0. \end{cases}$$

It is easily checked that ϕ is continuous on S^p and $\phi(u) + \phi(-u) = \int f \, d\nu = 0$. Hence, from Borsuk's theorem, there exists $u_0 \in S_p$ such that $\phi(u_0) = 0$. From step 2, $\langle u_0, e_{p+1}\rangle \neq 0$. Taking $\lambda = \int \inf(\psi, \varphi_{u_0}) \, dm \in (0, 1)$, we define ν_1 and ν_2 by $d\nu_1/dm = (\psi - \varphi_{u_0})_+/(1 - \lambda)$ and $d\nu_2/dm = \inf(\psi, \varphi_{u_0})/\lambda$. We have $-\infty \leq s \leq 1/(p+1)$, hence $\gamma = s/(1 - sp) \in [-1, 1]$. By Lemma 1, it is clear that $\nu_1, \nu_2 \in P_f$. Since $\nu = (1 - \lambda)\nu_1 + \lambda\nu_2$ and ν is extreme, we deduce that $\nu_1 = \nu_2 = \nu$ which means that $\psi = \varphi_{u_0}/\lambda$ and ν is s-affine.

The proof of the part with the function Φ is exactly the same as in Theorem 2. This ends the proof of Theorem 3. □

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