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Lines with Many Points on Both Sides

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Abstract. Let G be a finite set of points in the plane. A line M is a (k, k)-line if M is determined by G, and there are at least k points of G in each of the two open half-planes bounded by M. Let $f(k, k)$ denote the maximum size of a set G in the plane, which is not contained in a line and does not determine a (k, k)-line.

In this paper we improve previous results of Yaakov Kupitz ($f(k, k) \leq 3k$), Noga Alon ($f(k, k) \leq 2k + O(\sqrt{k})$), and Micha A. Perles ($f(k, k) \leq 2k + O(\log k)$). We show that $f(k, k) \leq 2k + O(\log \log k)$.

1. Introduction

Let G be a set of n points in the real affine plane. A line M is said to be determined by G if it contains two different points of G.

A line M, determined by G, is called a (k, l)-line if the two open half-planes bounded by M include at least k and l points of G, respectively.

Clearly, if the set G is contained in a line, then G has only one $(0, 0)$-line and no other spanned lines. It turns out that if G is large enough and is not contained in a line, then it must posses a (k, l)-line. In this paper we improve previous results of Alon, Kupitz [K1], [K2], and Perles regarding the upper bound for a size of a set which is not contained in a line and does not determine a (k, k)-line. The same method yields an upper bound also for the size of a set which does not determine a (k, l)-line (see the concluding remarks at the end of this paper).

Definition 1.1. Let k, l be nonnegative integers. Define $f(k, l)$ to be the maximum size of a finite set G in the plane, not contained in a line, which does not determine a (k, l)-line.

We wish to prove the following theorem.

Theorem 1.2. $f(k, k) \leq 2k + O(\log \log k)$.

A closely related problem to the one discussed in this paper is the following which was raised by Perles: let G be a finite set in the plane. How well can we evenly divide this set by a line which is determined by G? To be more precise, for every line M, determined by G, denote by $d(M)$ (the absolute value of) the difference between the number of points of G in the two open half-planes bounded by M. Define $D(G) = \min_M d(M)$. Finally define $\mu(n) = \max_{|G|=n} D(G)$. Are there any good upper bounds for $\mu(n)$?

Clearly, if G is a set of odd number of points in general position (no three on a line) in the plane, then $D(G) = 1$ which means that in general it is not always possible to divide the set of points equally. However, maybe one can always do as good as that, namely, is it possible that $\mu(n) \leq 1$ for every n?

An example by Alon [A] shows that this is not the case. Alon found a construction of a set G of 12 points so that $D(G) = 2$ (see Fig. 1). In fact based on this construction one can find arbitrary large sets G with $D(G) = 2$.

Theorem 1.2 implies that $\mu(n) = O(\log \log n)$. Indeed, let C be the absolute constant so that $f(k, k) \leq 2k + C \log \log k$. Given a set G of n points in the plane take $k = n/2 - C \log \log n$. Then $n \geq 2k + C \log \log k$. We may assume, of course, that G is not contained in a line for otherwise $D(G) = 0$. Therefore, by Theorem 1.2, there exists a line M determined by G so that in each open half-plane bounded by M, there are at least $k = n/2 - C \log \log n$ points of G. It follows that the difference between the number of points of G in the two open half-planes is at most $2C \log \log n$. In other words we showed that for every set G of n points in the plane $D(G) = O(\log \log n)$.

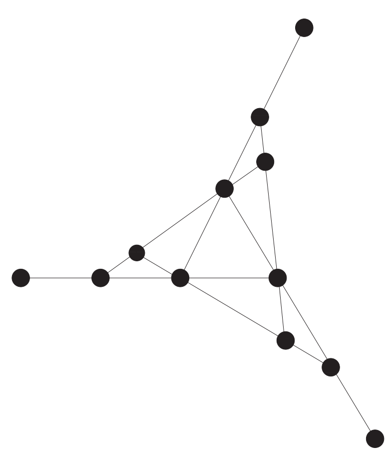

Fig. 1. A set G with $D(G) = 2$.

In Section 2 we describe the method of flip arrays, invented by Goodman and Pollack (see [GP1] and [GP2] for a survey and applications), which encodes a set of n points in the plane as a sequence of permutations on n elements. Throughout the rest of the paper we use this method to derive an upper bound for $f(k, k)$ (for other intensive uses of this method in a similar way see [U] and [PP]).

2. Flip Arrays

Let G be a set of n points in the plane. Let L be a directed line through the origin which is not perpendicular to any of the lines determined by G. We arrange the points of G in a sequence $x_1, \ldots, x_n$ by the order of their projections on L from left to right. For every $0 \le \theta \le \pi$, let $L(\theta)$ be the line through the origin which arises from L by a rotation at angle θ (in the positive direction). Let $0 < \theta_1 < \cdots < \theta_m < \pi$ be all angles in $[0, \pi]$, so that each of $L(\theta_1), \ldots, L(\theta_m)$ is perpendicular to some line determined by G. Denote for convenience, $\theta_0 = 0, \theta_{m+1} = \pi$.

For every $0 \le \theta \le \pi$ which is not one of $\{\theta_1, \ldots, \theta_m\}$, let P_θ denote the permutation on $\{1, \ldots, n\}$ so that the projections of $x_{P_\theta(1)}, \ldots, x_{P_\theta(n)}$ on $L(\theta)$ are in that order from left to right.

It is important to note that we think of a permutation P as a sequence of n elements, namely, $(P(1), \ldots, P(n))$. We then say that the *element* $P(i)$ is at the *place* i in the permutation P. The relative order of two elements i, j depends on whether $P^{-1}(i)$ is greater than or less than $P^{-1}(j)$. If $P^{-1}(i) < P^{-1}(j)$ we say that i is *to the left* of j and that j is *to the right* of i.

Let x_i and x_j be two points of G and let $\alpha < \beta$ be two angles which are not from $\{\theta_1, \ldots, \theta_m\}$. The relative order of i and j in the permutations P_α and P_β is the same iff the vector $\overrightarrow{x_i x_j}$ is not perpendicular to any of the lines $\{L(\theta) | \alpha < \theta < \beta\}$. It follows that if $0 \le k \le m$ and $\theta_k < \alpha < \beta < \theta_{k+1}$, then $P_\alpha = P_\beta$. This justifies the following notation.

Notation 2.1. For every $0 \le j \le m$, denote by Q_j the permutation P_α where α is any angle such that $\theta_j < \alpha < \theta_{j+1}$.

Clearly, Q_0 is the identity permutation, and for every two indices $1 \le i < j \le n$, the relative order between i and j changes exactly for one value of k $(0 \le k < m)$ when going from Q_k to Q_{k+1}. Eventually, Q_m is the permutation $(n, n-1, \ldots, 1)$.

Fix k such that $0 \le k < m$. We want to find out how exactly Q_{k+1} arises from Q_k. For every $1 \le i < j \le n$, the relative order between i and j in the permutation Q_{k+1} is different from that in Q_k iff the vector $\overrightarrow{x_i x_j}$ is perpendicular to $L(\theta_{k+1})$. Let M be a line spanned by points of G which is perpendicular to $L(\theta_{k+1})$. Let $x_{i_1}, \ldots, x_{i_s}$ be all the points of $G \cap M$, we assume that $i_1 < i_2 < \cdots < i_s$. Since for every $1 \le u < v \le s$ the relative order between i_u and i_v *changes* when going from Q_k to Q_{k+1}, we conclude that in Q_k the relative order of $i_1, \ldots, i_s$ is the same as in Q_0, namely, the natural order. Let j be an index which is not from $\{i_1, \ldots, i_s\}$. When going from Q_k to Q_{k+1} we do not change the relative order between j and i_1 and between j and i_s. Therefore, in the permutation Q_k, j cannot be between i_1 and i_s, or in other words either $Q_k^{-1}(j) < Q_k^{-1}(i_1)$ or

$Q_k^{-1}(j) > Q_k^{-1}(i_s)$. This shows that in Q_k the elements $i_1, \dots, i_s$ come one after the other and form a monotone increasing sequence. It then follows that in Q_{k+1} those same elements form a monotone decreasing sequence. In other words, some subsequence of consecutive elements in Q_k appear flipped in Q_{k+1}.

Definition 2.2. A *block* in a permutation P is a sequence of consecutive elements in P. We some times refer to the block as a region (containing certain *places* in a permutation) and some times we refer to its content (the *elements* which are in that region). We say that a block B is *monotone increasing* if the elements in that block form a monotone increasing sequence from left to right. We define a *monotone decreasing* block similarly.

Notation 2.3. Let $1 \leq a < b \leq n$. We denote by $[a, b]$ the block which consists of the places $a, a+1, \dots, b$ in a general permutation (considered as a sequence of n elements).

In view of Definition 2.2, Q_{k+1} arises from Q_k by flipping blocks in Q_k. Every such block represents a line, determined by the points of G, which is perpendicular to $L(\theta_{k+1})$. Every pair of blocks that flip when going from Q_k to Q_{k+1} represent two parallel lines, and therefore are disjoint, so we can treat them as if they were flipped one after the other.

To summarize, given a set G of n points in the plane, we derive from it a sequence of permutations on the numbers $1, \dots, n$, with the following properties. The first permutation is the identity permutation, the last one is the permutation $(n, n-1, \dots, 1)$, and each permutation arises from its predecessor by flipping a block which is monotone increasing (right before the flip). Every such block which flips represents a line which is determined by the points of G.

3. Notation and Terminology

Let G be a set of n points in the plane. A *flip array* of G is a sequence of permutations on the elements $\{1, 2, \dots, n\}$ derived from G as described above. Each permutation arises from its predecessor by a flip T of a block B. For every element $x \in B$, we say that x *takes part in the flip T*.

Remark. The same set G can have several different flip arrays. The flip array depends on the initial choice of the direction L, the direction of rotation of L, and the order in which we flip blocks that represent parallel spanned lines.

Let S_G be a flip array of the set G. Assume $\sigma \in S_G$ is a permutation in the flip array S_G. For two elements $1 \leq x < y \leq n$, we say that x and y *change order* in σ, if, in σ, x is to the right of y (that is $\sigma^{-1}(x) > \sigma^{-1}(y)$) and in the permutations which are prior to σ in S_G, x is to the left of y.

If $P_1, P_2 \in S_G$ are two consecutive permutations so that P_2 is obtained from P_1 by a flip F, then we denote $P_F^- = P_1$ and $P_F^+ = P_2$.

We say that two elements $x, y \in \{1, 2, \dots, n\}$ *change order* in a flip F if x and y change order in P_F^+.

Let S_G be a flip array of a finite set G. For $P_1, P_2 \in S_G$, we say that P_1 is *previous* to P_2 if P_1 comes before P_2 in S_G. We then say that P_2 is later to P_1.

Similarly, we say that a flip F_1 is *previous* to a flip F_2 if $P^+_{F_1}$ is previous to $P^+_{F_2}$. We then say that F_2 is *later* to F_1. We say that a flip F *occurs between* a flip F_1 and a flip F_2 (where F_1 is previous to F_2) if F is later to F_1 and F_2 is later to F. In this case we sometimes say that F is between F_1 and F_2.

For $P_1, P_2 \in S_G$. We denote by $[P_1, P_2]$ the permutations in S_G which are not previous to P_1 and not later to P_2.

For a flip F and $P_1, P_2 \in S_G$, we say that F is *between* P_1 and P_2 if there are two consecutive permutations $\sigma, \sigma' \in [P_1, P_2]$ so that σ' is obtained from σ by the flip F.

Note the following two simple observations.

Observation 3.1. *Let S_G be a flip array of a set G. Every two elements change order at some point (permutation) in the flip array S_G. From that point on (i.e., in all permutations that come afterwards in S_G) they are always in inverted order.*

Observation 3.2. *Let S_G be a flip array of a set G of n points in the plane. If a line M, determined by G, is represented by a flip of the block $[a, b]$, then there are exactly $a - 1$ points of G in one open half-plane bounded by M, and $n - b$ points in the other half-plane bounded by M.*

4. Getting Started

Since we are interested only in asymptotic bounds we make some assumptions that will simplify the presentation of the proof and will cause a loss of a constant number of units in the bound. We thus assume that $G \subset \mathrm{R}^2$ is a finite set of points in the plane, aff $G = \mathrm{R}^2$ (i.e., G is not contained in a line), and that it does not determine a (k, k)-line. We denote by n the size of the set G and assume that $n - 2k \equiv 2 \pmod 4$. Let d denote the integer so that $n = 2k + 2(2d - 1)$. We denote $c = 2d - 1$. Therefore,

$$|G| = n = 2k + 2c = 2k + 2(2d - 1).$$

When needed we assume (without explicitly saying so) that d is large enough. Under those assumptions we will show that if G does not determine a (k, k)-line, then $k \geq 2^{2^{O(c)}}$. The proof is based on observing a flip array of G. Let S_G be such a flip array.

Notation 4.1. For a permutation $\sigma \in S_G$, let ZONE denote the block $[k + 1, n - k]$. We denote by LZONE the block $[k + 1, n/2]$ and by RZONE the block $[n/2 + 1, n - k]$.

Observe (using Observation 3.2) that the assumption that G does not determine a (k, k)-line is equivalent to assuming that no permutation in S_G arises from its predecessor by a flip whose block is included in ZONE.

Definition 4.2. A transfer is a flip whose block includes both places $n/2$ and $n/2 + 1$.

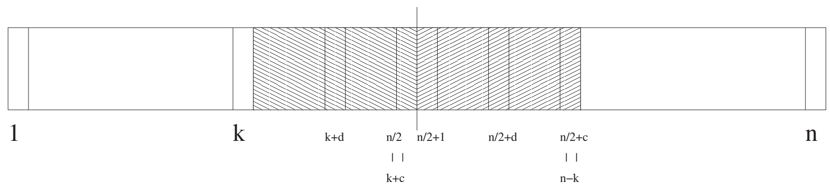

Fig. 2. Distinguished regions and places in a permutation.

Let T_i ($i \geq 0$) denote the ith transfer in the flip array S_G. We call the block $[1, n/2]$ the *left side*, and the block $[n/2 + 1, n]$ is called the *right side*. Note that the block of a flip, which is not a transfer, is fully included either in the right side or in the left side, in particular this is true for all the flips between T_i and T_{i+1}.

Note. Assume that an element x moves from the left side to the right side (or vice versa) by a flip F. Then, clearly, F must be a transfer (for the block of F must include a place from the left side as well as from the right side).

Claim 4.3. *If T is a transfer, then the block of T includes either* LZONE *or* RZONE (*or both*).

Proof. Let $B = [a, b]$ be the block of T. Since T is a transfer, $a \leq n/2$ and $b \geq n/2+1$. If B does not include RZONE, then $b < n/2+c$ and if B does not include LZONE, then $a > n/2 - c + 1$. It follows that in this case $B \subseteq$ ZONE contradicting the assumption on G. □

Remark. Similarly, if B is a block of a flip which is included in the left side (right side) and includes the place $a \in$ ZONE, then it includes the block $[k, a]$ ($[a, n-k+1]$), for otherwise B is fully contained in ZONE.

Lemma 4.4. *Let $i \geq 0$, and consider only the permutations in $[P^+_{T_i}, P^-_{T_{i+1}}]$. Then one of the following four statements is true*:

1. (a) *In $P^+_{T_i}$, LZONE is monotone decreasing and in some later permutation $\sigma \in [P^+_{T_i}, P^-_{T_{i+1}}]$ it is monotone increasing.*
   (b) *In some permutation $\sigma \in [P^+_{T_i}, P^-_{T_{i+1}}]$, LZONE is monotone decreasing and in $P^-_{T_{i+1}}$ it is monotone increasing.*
2. (a) *In $P^+_{T_i}$, RZONE is monotone decreasing and in some later permutation $\sigma \in [P^+_{T_i}, P^-_{T_{i+1}}]$ it is monotone increasing.*
   (b) *In some permutation $\sigma \in [P^+_{T_i}, P^-_{T_{i+1}}]$, RZONE is monotone decreasing and in $P^-_{T_{i+1}}$ it is monotone increasing.*

Proof. By Claim 4.3, the block of T_i includes either LZONE or RZONE. Assume that the block of T_i includes LZONE. Then in $P^+_{T_i}$, LZONE is monotone decreasing. If the block of T_{i+1} includes LZONE, then in $P^-_{T_{i+1}}$, LZONE is monotone increasing and thus 1(a) is true. Assume that the block of T_{i+1} includes RZONE. In $P^-_{T_{i+1}}$, $[n/2, n-k]$ ($\supset$ RZONE) is monotone increasing, however, in $P^+_{T_i}$, the elements at the places $n/2, n/2+1$, which we denote by x and y, respectively, are in monotone decreasing order. Therefore, either x or y must take part in a flip S which is between T_i and T_{i+1}. If x takes part in S, then the block of S must include LZONE. It follows that in P^-_S, LZONE is monotone increasing and then 1(a) is true. If y takes part in S, then the block S must include RZONE. It follows that in P^+_S, RZONE is monotone decreasing and then 2(b) is true.

We argue similarly if the block of T_i includes RZONE. □

Lemma 4.4 justifies the following definition.

Definition 4.5. If in Lemma 4.4 either 1(a) or 1(b) is true, we say that T_{i+1} is a *left*-transfer. Otherwise (then either 2(a) or 2(b) is true), we say that T_{i+1} is a *right*-transfer.

Note. Definition 4.5 does not apply for T_0.

The following Claim 4.6 shows that if T_{i+1} is a left-transfer, then there must be many (a number which is exponential in c) flips, whose block intersects with LZONE, between T_i and T_{i+1} to prepare the ground for the flip T_{i+1}. From this observation alone we can easily show that $c = O(\log n)$, which in turn implies $f(k,k) = 2k + O(\log k)$.

Claim 4.6. *Assume that T_{i+1} is a left-transfer. Then the number of flips which occur between T_i and T_{i+1}, the block of which includes the place $n/2 - s$ $(0 < s < c)$, is at least 2^{s-1}.*

Proof. We prove the claim by showing the following more general lemma.

Lemma 4.7. *Let $\sigma_1 \in S_G$. Let $m \geq 1$, $t > 0$ and assume that the block $[m, m+t]$ of σ_1 is monotone decreasing. Let $\sigma_2 \in S_G$ be the first permutation later to σ_1, in which the block $[m, m+t]$ is monotone increasing. Assume that the block of every flip F, between σ_1 and σ_2, is of the form $[a, b]$, where $a < m$ or $a > m+t$. Then the number of flips between σ_1 and σ_2, the block of which includes the place m, is at least $2^t - 1$.*

Proof. For every $\sigma \in [\sigma_1, \sigma_2]$, we define a weight function $g(\sigma)$, according to the content of the block $[m, m+t]$ in σ. Let the elements in the block $[m, m+t]$ from left to right be $x_0, \ldots, x_t$. We define $g(\sigma) = \sum_{0 \leq i < t, x_i < x_{i+1}} 2^i$. Observe that if $P_1, P_2 \in [\sigma_1, \sigma_2]$ are two consecutive permutations and F is the flip by which P_2 is obtained from P_1, then $g(P_1) \neq g(P_2)$ only if the block of F includes the place m. Also observe that in this case $g(P_2) \leq g(P_1) + 1$ (because the elements in the block of F are in monotone increasing order in P_1 and are in monotone decreasing order in P_2).

$g(\sigma_1) = 0$ because the block $[m, m+t]$ is monotone decreasing, in σ_1. In σ_2 the block $[m, m+t]$ is monotone increasing and therefore $g(\sigma_2') = 2^t - 1$. This concludes the proof. □

We now return to the proof of Claim 4.6. Since T_{i+1} is a left-transfer (recall Definition 4.5), there exist $\sigma_1, \sigma_2 \in [P^+_{T_i}, P^-_{T_{i+1}}]$, so that σ_2 is later to σ_1 and LZONE is monotone decreasing in σ_1 and is monotone increasing in σ_2. Observe that if $B = [a, b]$ is a block of a flip which occurs between T_i and T_{i+1}, then either $a > n/2$ or $a < n/2 - c + 1$. This is because no flip which occurs between T_i and T_{i+1} is a transfer and thus if $n/2 - c \le a \le n/2$, then $B \subset$ LZONE $\subset$ ZONE, contradicting our assumptions. We can now apply Lemma 4.7 with $m = n/2 - s$, $t = s$, and σ_1, σ_2, and conclude that there are at least $2^s - 1 \ge 2^{s-1}$ flips between T_i and T_{i+1}, the block of which includes the place $n/2 - s$. □

Remark. We can argue similarly when T_{i+1} is a right-transfer and prove the following analogous claim.

Claim 4.8. *Let s be an integer such that $0 < s < c$. Assume that T_{i+1} is a right-transfer. Then the number of flips between T_i and T_{i+1}, the block of which includes the place $n/2 + 1 + s$, is at least 2^{s-1}.*

Recall that $d = \lceil c/2 \rceil$. Therefore, $k + d$ is the middle place in LZONE and $n/2 + d$ is the middle place in RZONE. Observe that if B is a block of a flip which is included in the left side (right side), then the center of B is to the left of the place $k + d$ (right to the place $n/2 + d$), for otherwise $B \subseteq$ ZONE.

Definition 4.9. Let $i \ge 0$. If T_{i+1} is a left-transfer, denote by $\mathcal{A}_{i+1}$ the set of all elements which are at the place $k + d$, in some $\sigma \in [P^+_{T_i}, P^-_{T_{i+1}}]$. If T_{i+1} is a right-transfer, denote by $\mathcal{A}_{i+1}$ the set of all elements which are at the place $n/2 + d$, in some $\sigma \in [P^+_{T_i}, P^-_{T_{i+1}}]$.

Remark. The following lemmata and claims, until the end of this section, are formulated for the left side. The reader should have no difficulty in formulating and proving the analogue for the right side.

In Lemma 4.10, Corollaries 4.11 and 4.12, and Claim 4.13 we show that for every $i \ge 0$, the set $\mathcal{A}_{i+1}$, associated with the transfer T_{i+1}, contains at least 2^{d-2} elements, every two of which change order between T_i and T_{i+1}. This will encode the information that there must be many flips between every two consecutive transfers.

Lemma 4.10. *Let $i \ge 0$ and $0 \le s < d$. Consider only the flips which occur between T_i and T_{i+1}. Let a_0 denote the element at the place $k + d + s$ in $P^+_{T_i}$. Denote by a_j $(j \ge 1)$ the element at the place $k + d + s$ right after the jth flip whose block includes the place $k + d + s$. Then $\{a_j\}_{j \ge 0}$ is a strictly monotone decreasing sequence.*

Proof. Let S be the $(j + 1)$th flip whose block $B = [a, b]$ includes the place $k + d + s$. The center of B is to the left of the place $k + d$ and therefore to the left of $k + d + s$. It follows that the element which is at the place $k + d + s$ in P^+_S is smaller than the element at the place $k + d + s$ in P^-_S (for B is monotone increasing in P^-_S). In other words, $a_{j+1} < a_j$. □

Corollary 4.11. *Let $i \geq 0$ and let T_{i+1} be a left-transfer. Then $|\mathcal{A}_{i+1}| \geq 2^{d-2}$.*

Proof. By Claim 4.6 (taking $s = d-1$), there are at least 2^{d-2} flips, between T_i and T_{i+1}, the block of which includes the place $k+d$. Denote by a_j ($j \geq 0$) the element at the place $k+d$ right before the jth flip whose block includes the place $k+d$. By Lemma 4.10, $\{a_j\}_{j\geq 0}$ is a strictly monotone decreasing sequence and therefore its elements are all different. □

Corollary 4.12. *Let $i \geq 0$ and let $\sigma \in [P^+_{T_i}, P^-_{T_{i+1}}]$. Assume that in σ the element x is in the block $[k+d, n/2]$. Let $\sigma' \in [P^+_{T_i}, P^-_{T_{i+1}}]$ be any permutation later to σ, in which x is in the block $[k+d, n/2]$. Then x does not take part in any flip between σ and σ'.*

Proof. Assume to the contrary that x takes part in a flip S which is between σ and σ', and that S is the first such flip. The center of the block of S is to the left of the place $k+d$, and, in P^-_S, x is in the block $[k+d, n/2]$. Therefore, in P^+_S, x is in the block $[1, k+d-1]$, and the element y at the place $k+d$ in P^+_S satisfies $y < x$. In σ', x is in the block $[k+d, n/2]$. Hence, there must be a flip S', later to S, such that, in $P^-_{S'}$, x is in the block $[1, k+d-1]$, and, in $P^+_{S'}$, x is in the block $[k+d, n/2]$. It follows that in $P^-_{S'}$ the element y' at the place $k+d$ is greater than x. In particular $y' > y$. This is a contradiction to Lemma 4.10 with $s = 0$. □

Claim 4.13. *Let $i \geq 0$ and assume that T_{i+1} is a left-transfer. Then every two elements of $\mathcal{A}_{i+1}$ change order in some $\sigma \in [P^+_{T_i}, P^-_{T_{i+1}}]$.*

Proof. Let $x, y \in \mathcal{A}_{i+1}$ be two different elements. Let $\sigma_1 \in [P^+_{T_i}, P^-_{T_{i+1}}]$ be the first permutation in which x is at the place $k+d$. Let $\sigma_2 \in [P^+_{T_i}, P^-_{T_{i+1}}]$ be the first permutation in which y is at the place $k+d$. Without loss of generality assume that σ_1 is previous to σ_2. We claim that in σ_1, y is to the left of x. Indeed, assume to the contrary that in σ_1, y is inside the block $[k+d+1, n/2]$. By Corollary 4.12, taking $\sigma = \sigma_1$ and $\sigma' = \sigma_2$, y does not take part in any flip between σ_1 and σ_2. This is impossible because y is at the place $k+d$ in σ_2.

We now claim that, in σ_2, x is to the left of y. Assume to the contrary that, in σ_2, x is in the block $[k+d+1, n/2]$. By Corollary 4.12, taking $\sigma = \sigma_1$ and $\sigma' = \sigma_2$, x does not take part in any flip between σ_1 and σ_2. This is impossible because x is at the place $k+d$ in σ_1. This shows that x and y change order in some $\sigma \in [P^+_{T_i}, P^-_{T_{i+1}}]$. □

The following claim will be very important in what follows.

Claim 4.14. *Let $i \geq 0$ and assume T_{i+1} is a left-transfer. Let x be an element which is inside the block $[k+d, n/2]$, in some $\sigma \in [P^+_{T_i}, P^-_{T_{i+1}}]$. If x does not take part either in T_i or in T_{i+1}, then x changes order with at least 2^{d-3} elements of $\mathcal{A}_{i+1}$, in the flips which occur between T_i and T_{i+1}.*

Proof. We consider only the flips which occur between T_i and T_{i+1}, the block of which includes the place $k+d$, and number them by order of occurrence (starting at 1). For every element y, let $T_{\rm in}(y)$ denote the number of the flip which takes y into the block $[k+d, n/2]$ (we set $T_{\rm in}(y) = 0$, if such a flip does not exist). Denote by $T_{\rm out}(y)$ the number of the flip which takes y outside the block $[k+d, n/2]$ (we set $T_{\rm out}(y) = 0$, if such a flip does not exist).

We claim that $T_{\rm in}(y)$ and $T_{\rm out}(y)$ are well defined. We show this only for $T_{\rm in}(y)$, the argument for $T_{\rm out}(y)$ is similar. Assume to the contrary that there are two flips S_1, S_2 which take y inside the block $[k+d, n/2]$. Without loss of generality, S_1 is previous to S_2. Both in $P^+_{S_1}$ and in $P^+_{S_2}$, y is in the block $[k+d, n/2]$. This contradicts Corollary 4.12, as y takes part in S_2.

Lemma 4.15. *Let y and z be two different elements and assume that both $T_{\rm in}(y)$ and $T_{\rm out}(z)$ are different from 0. If $T_{\rm in}(y) \geq T_{\rm out}(z)$, then y and z change order in some $\sigma \in [P^+_{T_i}, P^-_{T_{i+1}}]$.*

Proof. Let F_1 denote the flip whose number is $T_{\rm out}(z)$. Let F_2 denote the flip whose number is $T_{\rm in}(y)$. We know that F_2 is equal to or later to F_1. In $P^-_{F_1}$, z is inside the block $[k+d, n/2]$ and y is not, for otherwise $T_{\rm in}(y) < T_{\rm out}(z)$. In other words, in $P^-_{F_1}$, z is to the right of y. In $P^+_{F_2}$, y is inside the block $[k+d, n/2]$ and z is not, because $T_{\rm out}(z) \leq T_{\rm in}(y)$. In other words, in $P^+_{F_2}$, y is to the right of z. It follows that y and z change order in some $\sigma \in [P^+_{F_1}, P^+_{F_2}]$. □

We now go back to the proof of Claim 4.14: since T_{i+1} is a left-transfer, let $P_1 \in [P^+_{T_i}, P^-_{T_{i+1}}]$ be the first permutation in which LZONE is monotone decreasing (see Definition 4.5). Denote the elements in LZONE, from left to right, by $a_1, \ldots, a_c$. Let $P_2 \in [P^+_{T_i}, P^-_{T_{i+1}}]$ be the last permutation in which LZONE is monotone increasing. Denote the elements on LZONE in P_2 from left to right by $b_1, \ldots, b_c$. Clearly, $T_{\rm in}(b_{c-1})$ and $T_{\rm out}(a_{c-1})$ are different from 0, and $T_{\rm in}(b_{c-1}) \geq T_{\rm out}(a_{c-1})$.

If $T_{\rm in}(x) \geq T_{\rm out}(a_{c-1})$, then, by Lemma 4.15, every $y \in \mathcal{A}_{i+1}$ which satisfies $T_{\rm out}(y) \leq T_{\rm out}(a_{c-1})$, changes order with x, in some flip between T_i and T_{i+1}. Let S_1 denote the flip whose number is $T_{\rm out}(a_{c-1})$. In $P^-_{S_1}$ the block $[k+d, n/2-1]$ is monotone increasing. By Lemma 4.7, taking $\sigma_1 = P_1$, $\sigma_2 = P^-_{S_1}$, $m = k+d$, and $t = d-2$, there are at least $2^{d-2} - 1$ flips, between $P^-_{P_1}$ and S_1, the block of which includes the place $k+d$. S_1 is another such flip. Every such flip takes a unique element $y \in \mathcal{A}_{i+1}$ out from the place $k+d \in [k+d, n/2]$. Therefore, there are at least 2^{d-2} elements $y \in \mathcal{A}_{i+1}$ which satisfy $T_{\rm out}(y) \leq T_{\rm out}(a_{c-1})$.

If $T_{\rm out}(x) \leq T_{\rm in}(b_{c-1})$, then, by Lemma 4.15, every $y \in \mathcal{A}_{i+1}$ which satisfies $T_{\rm in}(y) \geq T_{\rm in}(b_{c-1})$, changes order with x at some flip between T_i and T_{i+1}. Let S_2 denote the flip whose number is $T_{\rm in}(b_{c-1})$. In $P^+_{S_2}$ the block $[k+1, n/2-1]$ is monotone decreasing. By Lemma 4.7, taking $\sigma_1 = P^+_{S_2}$, $\sigma_2 = P_2$, $m = k+d$, $t = d-2$, there are at least $2^{d-2} - 1$ flips, between $P^+_{S_2}$ and P_2, the block of which includes the place $k+d$. S_2 is another such flip. Every such flip takes a unique element $y \in \mathcal{A}_{i+1}$ into the place $k+d \in [k+d, n/2]$. Therefore, there are at least 2^{d-2} elements $y \in \mathcal{A}_{i+1}$ which satisfy $T_{\rm in}(y) \geq T_{\rm in}(b_{c-1})$.

It is enough to show that every element x which is in the block $[k+d, n/2]$, in some $\sigma \in [P^+_{T_i}, P^-_{T_{i+1}}]$, and does not take part in T_i or T_{i+1}, satisfies either $T_{\rm in}(x) \geq T_{\rm out}(a_{c-1})$ or $T_{\rm out}(x) \leq T_{\rm in}(b_{c-1})$. Recall that S_1 is the flip whose number is $T_{\rm out}(a_{c-1})$. S_1 is the flip which takes a_{c-1} out of $[k+d, n/2]$. In $P^-_{S_1}$, a_{c-1} is at the place $n/2-1$ and hence the block of S_1 includes $[k+d, n/2-1]$. If, in $P^+_{S_1}$, x is inside $[k+d, n/2]$, then either x takes part in S_1 or x is at the place $n/2$ in $P^-_{S_1}$, that is, $x = a_c$. In the first case $T_{\text{in}}(x) = T_{\text{out}}(a_{c-1})$.

Let $\pi \in [P^+_{T_i}, P^-_{T_{i+1}}]$ denote the first permutation in which x is in the block $[k+d, n/2]$. If, in $P^+_{S_1}$, x is not inside $[k+d, n/2]$, then we consider two cases.

Case 1: π *is previous to* $P^+_{S_1}$. In this case $T_{\text{out}}(x) \le T_{\text{out}}(a_{c-1}) \le T_{\text{in}}(b_{c-1})$.

Case 2: π *is later to* $P^-_{S_1}$. Then $T_{\text{in}}(x) > T_{\text{out}}(a_{c-1})$.

Therefore, we may conclude that either $T_{\text{in}}(x) \ge T_{\text{out}}(a_{c-1})$ or $T_{\text{out}}(x) \le T_{\text{in}}(b_{c-1})$ or $x = a_c$. We show that if none of the two first cases happens, then $x = a_c = b_c$. Indeed, assume that $x = a_c$. Recall that S_2 is the flip whose number is $T_{\text{in}}(b_{c-1})$. S_2 is the flip which takes b_{c-1} into the block $[k+d, n/2]$ (to the place $n/2-1$). The block of S_2 includes $[k+d, n/2-1]$. In P_1, x is inside the block $[k+d, n/2]$. It follows from Corollary 4.12 that either, in $P^+_{S_2}$, x is not inside $[k+d, n/2]$ or if it is, then x does not take part in S_2, in which case x is at the place $n/2$, in $P^-_{S_2}$. In the first case $T_{\text{out}}(x) \le T_{\text{in}}(b_{c-1})$. In the latter case $x = b_c$. Therefore, either $T_{\text{in}}(x) \ge T_{\text{out}}(a_{c-1})$ or $T_{\text{out}}(x) \le T_{\text{in}}(b_{c-1})$ or $x = a_c = b_c$.

However, in the latter case x takes part either in T_i or in T_{i+1}. Indeed, this follows from Definition 4.5, as either $a_1, \dots, a_c$ are the element in LZONE in $P^+_{T_i}$ or $b_1, \dots, b_c$ are the elements in LZONE in $P^-_{T_{i+1}}$. □

5. Studying Bad Transfers

Definition 5.1. For every element $x \in \{1, 2, \dots, n\}$, define

$$\eta_i(x) = max\{j \le i \mid \text{x takes part in } T_j\}.$$

If x does not take part in T_j for any $j \le i$, define $\eta_i(x) = -1$.

Notation 5.2. It will be convenient to denote $M(c) = 2^{d-3}$, as this number will be used extensively in the rest of the proof. Observe that $M(c)$ indeed depends on c, since $c = 2d - 1$.

Remark. In view of the above notation we should note that (by Corollary 4.11) for every transfer T_{i+1}, $|\mathcal{A}_{i+1}| \ge M(c)$. Moreover, by Claim 4.14, if T_{i+1} is a left-transfer (right-transfer), then every element x which is inside the block $[k+d, n/2]$ $([n/2+1, n/2+d])$ in some $\sigma \in [P^+_{T_i}, P^-_{T_{i+1}}]$ and does not take part in T_{i+1} or T_i, changes order with at least $M(c)$ elements of $\mathcal{A}_{i+1}$.

Definition 5.3. Let $0 \le a \le b$ be two integers. $L_{a,b}$ is defined to be the set of all elements that do not take part in any transfer T_v for $v = a, a+1, \dots, b$. In other words, $L_{a,b} = \{x \mid \eta_b(x) < a\}$.

Definition 5.4. Let $i \geq 0$. We say that T_{i+1} is *good* if $|\mathcal{A}_{i+1} \cap L_{0,i}| \geq \frac{1}{2}M(c)$. If T_{i+1} is not good it is said to be *bad*.

Definition 5.5. If T_i is a bad transfer. We define $Q_i = \mathcal{A}_i \setminus L_{0,i}$ and $I_i = \{\eta_{i-1}(x) \mid x \in Q_i\}$.

Let T_i be any bad transfer. If $x, y \in Q_i$, then clearly $\eta_{i-1}(x), \eta_{i-1}(y) \geq 0$. We claim that $\eta_{i-1}(x) \neq \eta_{i-1}(y)$. To see this, assume to the contrary that $\eta_{i-1}(x) = \eta_{i-1}(y) = a \leq i-1$. This means that x and y take part (and hence also change order) in T_a. This is a contradiction for they are both in $\mathcal{A}_i$ and hence, by Claim 4.13, change order in some flip between T_{i-1} and T_i. Since $|Q_i| \geq \frac{1}{2}M(c)$, it follows now that $|I_i| \geq \frac{1}{2}M(c)$.

Definition 5.6. If T_i is a bad transfer, then we define p_i to be the $(\frac{1}{4}M(c)+1)$th largest number in I_i, and $Y_i = \{x \in Q_i \mid \eta_{i-1}(x) \leq p_i\}$. Clearly, $Y_i \subset Q_i \subset \mathcal{A}_i$ and $|Y_i| \geq \frac{1}{4}M(c)$.

The idea behind the definition of a good transfer is that if T_i is a good transfer (say left), then many (at least $\frac{1}{2}M(c)$) elements from $\{1, 2, \ldots, n/2\}$ are in $\mathcal{A}_i$. This is very good for us because we know that eventually all these elements should be in the right side and therefore there must be many transfers which carry them there (every transfer may carry at most one element from $\mathcal{A}_i$ to the right side, as every two elements from $\mathcal{A}_i$ change order already before T_i). However, then we get more transfers and more sets $\mathcal{A}_j$ and so forth. We will eventually see that the portion of the good transfers among all transfers should be very small. It will follow that most of the transfers must be bad transfers, but for these we will obtain a nice upper bound in terms of the total number of transfers, roughly $(r/M(c))\log r$ where r is the total number of transfers. It then follows that $\log r$ should be as big as $M(c)$. This shows roughly that $r > 2^{2^c}$, but we know that $r < \binom{n}{2}$, as every transfer represents a line which is determined by G. This will prove Theorem 1.2.

From now until the end of Claim 5.10, T_i is a fixed bad left-transfer. (However, in what follows one can easily state and prove the corresponding statements when T_i is a bad right-transfer. Actually, there is a complete symmetry between left and right in this paper and we eventually use it without any loss of generality.)

Claim 5.7. *Assume $p_i < j < i$, $\eta_{i-1}(x) = j$, $y \in Y_i$. If x and y change order in some flip between T_j and T_i, then, in $P^-_{T_i}$, y is to the right of x.*

Proof. In $P^+_{T_j}$ the elements which take part in T_j are to the right of all other elements in the left side. Let $s = \eta_{i-1}(y)$. y takes part in T_s but not in any $T_{s'}$ for $s < s' < i$ and it follows that y is in the left side in every $\sigma \in [P^+_{T_s}, P^-_{T_i}]$. $y \in Y_i$ and hence $s \leq p_i$. Since $s \leq p_i < j < i$, then in particular, in $P^+_{T_j}$, y is in the left side. Therefore, in $P^+_{T_j}$, x is to the right of y. If x and y change order at some point after T_j, then clearly y moves to the right of x. □

Claim 5.8. *Let $p_i < j < i-1$ and assume $\eta_i(x) = j$ and that, in $P^+_{T_i}$, x is in the left side. Then x changes order with at least $\frac{1}{4}M(c)$ elements of Y_i, throughout the flips which occur between T_j and T_i.*

Proof. Since $\eta_i(x) = j$ and, in $P^+_{T_i}$, x is in the left side, it follows that x is in the left side for every $\sigma \in [P^+_{T_j}, P^+_{T_i}]$. If x is inside the block $[k+d, n/2]$, in some $\sigma \in [P^+_{T_{i-1}}, P^-_{T_i}]$, then, by Claim 4.14, x changes order with at least $M(c)$ elements of $\mathcal{A}_i$ and therefore with at least $\frac{1}{2}M(c)$ elements of Q_i and thus with at least $\frac{1}{4}M(c)$ elements of Y_i, throughout the flips which occur between T_{i-1} and T_i.

Assume then that x is outside the block $[k+d, n/2]$, for every $\sigma \in [P^+_{T_{i-1}}, P^-_{T_i}]$. x is one of the elements in the block of T_j. In $P^+_{T_j}$, those elements are the rightmost elements in the left side. Let $y \in Y_i$ and $s = \eta_{i-1}(y)$. y takes part in T_s and not in any $T_{s'}$ for $s < s' < i$. Therefore, in every $\sigma \in [P^+_{T_s}, P^-_{T_{i-1}}]$, y is in the left side. Since $s \leq p_i < j < i-1$, then in particular, in $P^+_{T_j}$, y is in the left side, to the left of x. $y \in Y_i \subset \mathcal{A}_i$ and therefore y visits the place $k+d$ at some point between T_{i-1} and T_i. When this happens, y is to the right of x. It follows that x and y change order in some flip between T_j and T_i. This is true for every $y \in Y_i$ and $|Y_i| \geq \frac{1}{4}M(c)$. □

Combining Claims 5.7 and 5.8, we immediately deduce the following corollary.

Corollary 5.9. *Let $p_i < j < i-1$ and assume $\eta_i(x) = j$ and that, in $P^+_{T_i}$, x is in the left side. Then in $P^-_{T_i}$ at least $\frac{1}{4}M(c)$ elements of Y_i, which already changed order with x, are to its right, in the left side.*

Claim 5.10. *Let $0 < s < \frac{1}{4}M(c) - d$ and assume that T_{i+s} is a bad left-transfer. Then $p_i - p_{i+s} \geq \frac{1}{4}M(c) - s - 1$.*

Proof. Fix j such that $p_i < j < i-1$, we show that $j \notin I_{i+s}$. Assume to the contrary that $j \in I_{i+s}$, then there is an element $x \in \mathcal{A}_{i+s}$ such that $\eta_{i+s-1}(x) = j$. Since $j < i$ it follows that $\eta_i(x) = j$ and that x is in the left side in $P^+_{T_i}$. By Corollary 5.9, in $P^-_{T_i}$, at least $\frac{1}{4}M(c)$ elements of Y_i, which changed order with x before T_i, are to the right of x on the left side. Denote the set of these elements by W. For every l such that $i \leq l \leq i+s$, at most one element of Y_i takes part in T_l, for every two elements of $Y_i \subset \mathcal{A}_i$ change order before T_i. It follows that in $P^+_{T_{i+s-1}}$ there are at least $\frac{1}{4}M(c) - s > d$ elements of W (to the right of x) in the left side. Since $x \in \mathcal{A}_{i+s}$, x is at the place $k+d$, in some $\sigma \in [P^+_{T_{i+s-1}}, P^-_{T_{i+s}}]$. In σ every element of W which is in the left side must be to the right of x. This is impossible since when x is at the place $k+d$ there are only $d-1$ places in the left side to the right of x.

We thus showed that if $p_i < j < i-1$, then $j \notin I_{i+s}$. In I_{i+s} there are $\frac{1}{4}M(c)$ of the numbers $p_{i+s}+1, p_{i+s}+2, \dots, i+s-1$. It follows that $p_i - p_{i+s} \geq \frac{1}{4}M(c) - s - 1$. □

Lemma 5.11. *Let $1 \leq q \leq p$ be given integers and let W be a set of m elements from $\{1, \dots, n\}$. Assume $P_1, P_2 \in S_G$ and in every $\sigma \in [P_1, P_2]$ the elements of W are all* *in the block* $[1, p]$. *Moreover, assume that for every* $x \in W$ *there is* $P_x \in [P_1, P_2]$ *such that* x *is inside the block* $[q, p]$, *in* P_x. *Then at least* $\binom{m-(p-q)}{2}$ *pairs of elements from* W *change order in one of the permutations in* $[P_1, P_2]$.

Proof. For every $x \in W$, let $P_x \in [P_1, P_2]$ denote the last permutation in which x is inside the block $[q, p]$. Write the elements of W in a sequence $a_1, a_2, \dots, a_m$ according to the order of $P_{a_1}, \dots, P_{a_m}$ in S_G. That is, if $i < j$, then P_{a_i} is not later to P_{a_j}.

Fix t such that $1 \le t \le m$. a_t must change order (in some $\sigma \in [P_1, P_2]$) with every element which is later to a_t in the list and is inside the block $[1, q-1]$, in P_{a_t}. The number of these elements is at least $m - t - (p - q)$. The number of pairs from $\{a_j\}_{j \ge 1}$ which change order in one of the permutations in $[P_1, P_2]$ is thus at least

$$(m - 1 - (p - q)) + (m - 2 + (p - q)) + \cdots = \binom{m - (p - q)}{2}. \qquad \square$$

Lemma 5.12. *Assume* $u \le t_1 < t_2 < \cdots < t_s$ *and for every* $1 \le i \le s$, T_{t_i} *is a left-transfer. For every* $1 \le i \le s$, *let* $H_i \subset \mathcal{A}_{t_i} \cap L_{u,t_i}$. *Then for every* $t \ge t_1$,

$$\left|\left(\bigcup_{t_i \le t} H_i\right) \cap L_{u,t}\right| \ge \sum_{t_i \le t} (|H_i| - d) - (t - t_1)d. \tag{1}$$

Proof. We prove the lemma by induction on t. For $t = t_1$ the claim just says that $|H_1| \ge |H_1| - d$. Assume the lemma is true for $t - 1$, we prove it for t. We need the following auxiliary claim:

Claim 5.13. *Let* $m > u$ *be an integer. At most* d *elements from* $(\bigcup_{t_i \le m-1} H_i) \cap L_{u,m-1}$ *may take part in* T_m, *and at most* d *elements from this set may belong to* $\mathcal{A}_m$.

Proof. Let W be any subset of $(\bigcup_{t_i \le m-1} H_i) \cap L_{u,m-1}$ which consists of $d+1$ elements. W satisfies the conditions of Lemma 5.11 with $p = n/2$, $q = k + d$, $P_1 = P^+_{T_u}$, and $P_2 = P^-_{T_{m-1}}$. Therefore, by Lemma 5.11, there is at least

$$\binom{d + 1 - (p - q)}{2} = \binom{2}{2} = 1$$

pair of elements from W which change order *before* T_{m-1}. Those two elements cannot both take part in T_m and cannot both be in $\mathcal{A}_m$. $\square$

By Claim 5.13, at most d elements of $(\bigcup_{t_i \le t-1} H_i) \cap L_{u,t-1}$ take part in T_t. Therefore,

$$\left|\left(\bigcup_{t_i \le t-1} H_i\right) \cap L_{u,t-1} \Big\backslash \left(\bigcup_{t_i \le t-1} H_i\right) \cap L_{u,t}\right| \le d. \tag{2}$$

We consider two cases.

Case 1: $t = t_a$ *for some* $1 \le a \le s$. By Claim 5.13,

$$\left| \mathcal{A}_{t_a} \cap \left(\bigcup_{t_i \le t_a - 1} H_i \right) \cap L_{u, t_a - 1} \right| \le d.$$

$H_a \subseteq \mathcal{A}_{t_a}$ and $L_{u,t_a} \subseteq L_{u,t_a-1}$. Hence

$$\left| H_a \cap \left(\bigcup_{t_i \le t_a - 1} H_i \right) \cap L_{u, t_a} \right| \le d.$$

It follows that

$$\left| \left(\bigcup_{t_i \le t_a} H_i \right) \cap L_{u, t_a} \right| \ge \left| \left(\bigcup_{t_i \le t_a - 1} H_i \right) \cap L_{u, t_a} \right| + |H_a| - d. \tag{3}$$

Using (2), we get

$$\begin{aligned} \left| \left(\bigcup_{t_i \le t_a - 1} H_i \right) \cap L_{u, t_a} \right| &= \left| \left(\bigcup_{t_i \le t_a - 1} H_i \right) \cap L_{u, t_a - 1} \right| \\ &\quad - \left| \left(\bigcup_{t_i \le t_a - 1} H_i \right) \cap L_{u, t_a - 1} \Bigg\backslash \left(\bigcup_{t_i \le t_a - 1} H_i \right) \cap L_{u, t_a} \right| \\ &\ge \left| \left(\bigcup_{t_i \le t_a - 1} H_i \right) \cap L_{u, t_a - 1} \right| - d. \end{aligned}$$

Combining this with (3), we conclude that

$$\left| \left(\bigcup_{t_i \le t_a} H_i \right) \cap L_{u, t_a} \right| \ge \left| \left(\bigcup_{t_i \le t_a - 1} H_i \right) \cap L_{u, t_a - 1} \right| - d + |H_a| - d.$$

Using now the induction hypothesis, we obtain

$$\begin{aligned} \left| \left(\bigcup_{t_i \le t_a} H_i \right) \cap L_{u, t_a} \right| &\ge \left(\sum_{t_i \le t_a - 1} |H_i| - d \right) - (t_a - 1 - t_1) d \\ &\quad - d + |H_a| - d \\ &= \left(\sum_{t_i \le t_a} |H_i| - d \right) - (t_a - t_1) d. \end{aligned}$$

Case 2: $t \neq t_a$ *for every* $1 \leq a \leq s$. Similar to the previous case we have

$$\left|\left(\bigcup_{t_i \leq t} H_i\right) \cap L_{u,t}\right| = \left|\left(\bigcup_{t_i \leq t-1} H_i\right) \cap L_{u,t}\right| = \left|\left(\bigcup_{t_i \leq t-1} H_i\right) \cap L_{u,t-1}\right|$$
$$- \left|\left(\bigcup_{t_i \leq t-1} H_i\right) \cap L_{u,t-1} \Big\backslash \left(\bigcup_{t_i \leq t-1} H_i\right) \cap L_{u,t}\right|$$
$$\geq \left|\left(\bigcup_{t_i \leq t-1} H_i\right) \cap L_{u,t-1}\right| - d.$$

Using the induction hypothesis we get

$$\left|\left(\bigcup_{t_i \leq t} H_i\right) \cap L_{u,t}\right| \geq \left(\sum_{t_i \leq t-1} |H_i| - d\right) - (t-1-t_1)d - d$$
$$= \left(\sum_{t_i \leq t} |H_i| - d\right) - (t - t_1)d. \qquad \square$$

6. The Final Analysis

In this section we conclude the proof of Theorem 1.2. The crucial step is to give a good upper bound to the number of bad transfers. We will show (Lemma 6.1 and Claims 6.2 and 6.3) that the number of bad transfers is at most (roughly) $(r/M(c)) \log r$, where r is the total number of transfers. We will also bound the number of good transfers and show roughly that it is at most $r/M(c)$.

Let $r > 0$ be the number of transfers in the flip array S_G. We define a matrix $A = (a_{i,j})_{i,j}$ of size $r \times r$ in the following way. Whenever T_t $(0 < t \leq r-1)$ is a bad left-transfer we set $a_{t,p_t} = 1$. We set A to be 0 at every place which is not set to be 1. It follows from the definition of p_t that if T_t is a bad transfer, then $p_t + \frac{1}{4}M(c) \leq t$. Hence all the 1-entries in A are located within the lower triangle of A of height $r - \frac{1}{4}M(c)$. Let $Lower(A)$ denote that part of A, in other words, $Lower(A) = \{(i, j) \in A \mid i - \frac{1}{4}M(c) \geq j\}$.

Lemma 6.1. *Assume that* $a_{i,j} = 1$, *and let* B *be a rectangular sub-matrix of height* h *which is included in* $Lower(A)$, *and whose lower right corner is* $(i-1, j-1)$. *Then there are at most* $10hd/\frac{1}{4}M(c)$ 1-*entries in* B.

Proof. Since $a_{i,j} = 1$, T_i is a bad left-transfer and $p_i = j$. Therefore there is an element $x \in Q_i \subset \mathcal{A}_i$ such that $\eta_{i-1}(x) = j$. Let $g(B)$ denote the number of 1-entries inside B. Let $t_1 < t_2 < \cdots < t_{g(B)}$ be all the indices of the lines in B (as a sub-matrix of A) which include 1 entries.

Fix s such that $1 \leq s \leq g(B)$. Since $B \subset Lower(A)$ and the place (t_s, p_{t_s}) is within B, it follows that $p_{t_s} < j < t_s - 1$ (recall that $(i-1, j-1)$ is the lower right corner of B). Moreover, we have $j < t_s \leq i-1$ and thus $\eta_{i-1}(x) = j$ implies also that $\eta_{t_s}(x) = j$. Since $x \in \mathcal{A}_i$ it follows that x is in the left side in $P^-_{T_i}$ and therefore it is in the left side also in $P^-_{t_s}$ (as $\eta_{i-1}(x) = j < t_s$). We can now use Corollary 5.9 by which, in $P^-_{T_{t_s}}$, at least $\frac{1}{4}M(c)$ elements of Y_{t_s}, which have already changed order with x, are to the right of x in the left side. Denote that set of elements by H_s. For every $1 \le s \le g(B)$, $p_{t_s} < j$ and therefore $Y_{t_s} \subset L_{j,s-1}$. This implies $H_s \subset L_{j,s-1}$, for every $1 \le s \le g(B)$.

$H_1, \ldots, H_{g(B)}$ satisfy the conditions of Lemma 5.12 for $T_{t_1}, \ldots, T_{t_s}$ and $u = j$. Using Lemma 5.12 with $t = i - 1$ we obtain

$$\begin{aligned} \left|\left(\bigcup_{t_s \le i-1} H_s\right) \cap L_{j,i-1}\right| &\ge \sum_{t_s \le i-1} (|H_s| - d) - (i - 1 - t_1)d \\ &\ge g(B)(\tfrac{1}{4}M(c) - d) - hd. \end{aligned} \tag{4}$$

Since $x \in Y_i \subset \mathcal{A}_i$, x is at the place $k + d$ in some $\sigma \in [P^+_{T_{i-1}}, P^-_{T_i}]$. In $P^+_{T_{i-1}}$ all the elements of the set $(\bigcup_{t_s \le i-1} H_s) \cap L_{j,i-1}$ are to the right of x on the left side, and they have already changed order with x. Since this is also true in σ, in which x is at the place $k + d$, we must have $|(\bigcup_{t_s \le i-1} H_s) \cap L_{j,i-1}| \le d - 1$.

Therefore, using (4),

$$g(B)(\tfrac{1}{4}M(c) - d) - hd \le d - 1.$$

In other words,

$$g(B) \le \frac{hd + d - 1}{\frac{1}{4}M(c) - d} \le \frac{10hd}{\frac{1}{4}M(c)}.$$

□

Claim 6.2. *No two* 1*-entries in A are included within a sub-square of A of size* $\frac{1}{8}M(c)$.

Proof. This is an immediate corollary of Claim 5.10. Indeed, let (t, p_t) and (s, p_s) be the coordinates of any two different 1-entries in A, representing two bad left-transfers T_t and T_s. If $|t - s| < \frac{1}{8}M(c)$, then, by Claim 5.10, $|p_t - p_s| \ge \frac{1}{4}M(c) - |t - s| - 1 \ge \frac{1}{8}M(c)$. □

Denote $N(c) = \frac{1}{8}M(c)$. We introduce a new matrix $D = (d_{i,j})_{i,j}$ of size $\lceil r/N(c)\rceil \times \lceil r/N(c)\rceil$. We set $d_{i,j} = 1$ iff there is a 1-entry in the corresponding sub-square of A of size $N(c) \times N(c)$, namely, the one whose upper left corner is $(iN(c), jN(c))$. Otherwise we set $d_{i,j} = 0$.

Let *Lower*(D) denote the lower triangle of D without the main diagonal and the two diagonals beneath it. In other words, *Lower*$(D) = \{(i, j) \mid 0 \le i, j < \lceil r/N(c)\rceil, j + 2 < i\}$.

By Claim 6.2, the number of 1-entries in A equals the number of 1-entries in D. The following is just a reformulation, in terms of the matrix D, of what we already know about the matrix A:

1. All 1-entries in D are either in *Lower*(D) or on the diagonal above it.
2. If $d_{i,j} = 1$, then every rectangular sub-matrix of height h which is included in *Lower*(D) and whose right lower corner is $(i - 1, j - 1)$ has at most $5d(h + 1) \le 10dh$ 1-entries (follows from Lemma 6.1).

Claim 6.3. *Let B be an $m \times l$ rectangular sub-matrix included in $Lower(D)$, then inside B there are at most $(10d+1)m+l$ 1-entries.*

Proof. We prove the claim by induction on $m+l$. If $m=1$ or $l=1$ the claim is obvious. We think of B as an $m \times l$ matrix. Let j be the rightmost column which includes a 1-entry in B. Let (i, j) be the coordinates of the lowest 1-entry in the jth column of B. Let $B' \subset B$ be the rectangular sub-matrix whose lower right corner is $(i-1, j-1)$ and whose upper left corner is $(0,0)$. Let $B'' \subset B$ be the rectangular sub-matrix whose upper left corner is $(i,0)$ and whose lower right corner is $(m-1, j-1)$. Clearly, all 1-entries in B are included either in B' or in B'' or in the jth column of B. B' is of size $i-1 \times j-1$. Since $B \subset LowerD$ and there is a 1-entry at the place (i, j) in B, then there are at most $10d(i-1)$ 1-entries in B'. By the induction hypothesis there are at most $(10d+1)(m-(i-1))+j-1$ 1-entries in B''. Moreover, there are at most $i+1$ 1-entries in the jth column of B. We conclude that there are at most $10d(i-1)+(10d+1)(m-(i-1))+j-1+i+1=(10d+1)m+j+1 \leq (10d+1)m+l$ 1-entries in B. □

In what follows log stands for $\log_2$. Recall that r is the number of transfers in the flip array S_G. Let $r' = 2^{\lceil \log r \rceil}$. r' is the integer power of 2 which satisfies $r \leq r' < 2r$.

Claim 6.4. *There are at most $20d\lceil r'/N(c)\rceil(\log\lceil r'/N(c)\rceil+1)$ 1-entries in D.*

Proof. We first show that $Lower(D)$ includes at most $(10d+2)\lceil r'/N(c)\rceil(\log\lceil r'/N(c)\rceil+1)$ 1-entries. Let D' be the $(\lceil r/N(c)\rceil-3) \times (\lceil r/N(c)\rceil-3)$ lower triangular matrix whose lower triangle equals $Lower(D)$. In Claim 6.3 we proved the following property of D':

(*) If B is a sub-matrix of D' of size $a \times b$ which is included in the lower triangle of D', then B has at most $(10d+1)a+b$ 1-entries.

Let $g(m)$ denote the maximum number of 1-entries in a matrix D' of size $m \times m$ which satisfies (*). Let E be any $2m \times 2m$ lower triangular matrix which satisfies (*). Let E_1 denote the $m \times m$ sub-matrix of E whose lower right corner is $(m-1, m-1)$ and whose upper left corner is $(0,0)$. Let E_2 denote the $m \times m$ sub-matrix of E whose lower right corner is $(2m-1, 2m-1)$ and whose upper left corner is (m, m). Let B denote the $m \times m$ sub-matrix of E whose lower right corner is $(m-1, 2m-1)$ and whose upper left corner is $(0, m)$. B is included in the lower triangle of E and therefore, by (*), it has at most $(10d+1)m+m=(10d+2)m$ 1-entries. Clearly, E_1 and E_2 also satisfy (*), because the lower triangles of both matrices are included in the lower triangle of E. Since every 1-entry inside E is either in E_1 or in E_2 or in B, we conclude that

$$g(2m) \leq 2g(m)+(10d+2)m.$$

Using the boundary condition $g(1)=1$, it follows that if m is a power of 2, then $g(m) \leq (2+10d)m(\log m+1)$. Since $g(m)$ is monotone in m, then the number of 1-entries in D' (which is the same as the number of 1-entries in $Lower(D)$) is at most $(2+10d)\lceil r'/N(c)\rceil(\log\lceil r'/N(c)\rceil+1)$.

Every 1-entry in D is either in $Lower(D)$ or in the three diagonals above it. Therefore, the number of 1-entries in D is at most

$$(2+10d)\left\lceil\frac{r'}{N(c)}\right\rceil\left(\log\left\lceil\frac{r'}{N(c)}\right\rceil+1\right)+3\left\lceil\frac{r'}{N(c)}\right\rceil$$
$$\leq 20d\left\lceil\frac{r'}{N(c)}\right\rceil\left(\log\left\lceil\frac{r'}{N(c)}\right\rceil+1\right). \qquad \square$$

Recall that r (the size of A) is the number of transfers in the flip array. We claim that $r > 1$. For assume to the contrary that there is only one transfer T_0 in the flip array, then it must be that T_0 takes all the elements $\{1, 2, \ldots, n/2\}$ to the right side and all the elements $\{n/2+1, n/2+2, \ldots, n\}$ to the left side. Therefore the block of T_0 consists of the whole n elements. This implies that G is contained in a line, contradicting our assumption.

Without loss of generality assume that T_1 is a left transfer. The set $\mathcal{A}_1$ consists of at least $M(c)$ elements. We claim that at most one of these elements is not from $\{1, 2, \ldots, n/2\}$. Indeed, every element $x > n/2$, which is in the left side in $P^-_{T_1}$, must take part in T_0, hence every two such elements already change order in T_0. If those elements were in $\mathcal{A}_1$, they would also change order at some point between T_0 and T_1, by Claim 4.13.

In $P^+_{T_{r-1}}$ the elements $1, 2, \ldots, n/2$ are in the right side, because an element can move from the left side to the right side only by a transfer, the last of which is T_{r-1}. In each transfer T_i $(i > 1)$ at most one element of $\mathcal{A}_1$ takes part. It follows that there must be at least $M(c)$ transfers in S_G. Hence $r \geq M(c)$ and in particular $\lceil r'/N(c)\rceil = r'/N(c)$.

By Claim 6.4 there are at most $20d(r'/N(c))(\log(r'/N(c)) + 1)$ 1-entries in D, and hence at most that number of 1-entries in A. This shows that there are at most $20d(r'/N(c))(\log(r'/N(c)) + 1)$ bad left-transfers in the flip array S_G. By symmetry, this is also true for the number of bad right-transfers. (This is because all claims and lemmata proved above also apply to the right side. One way to see this is by reflecting the set G through a line perpendicular to initial direction L with which we defined the flip array. All right-transfers become left-transfers and vice versa.) We conclude that the number of bad transfers in the flip array is at most $40d(r'/N(c))(\log(r'/N(c)) + 1)$.

Let a and b denote the number of good transfers and the number of bad transfers, respectively. Without loss of generality we assume that there are at least $\frac{1}{2}a$ good left-transfers.

Definition 6.5. Let $T_{i_1}, \ldots, T_{i_g}$ be all the good left-transfers in S_G. For every $1 \leq j \leq g$ define $E_j = \mathcal{A}_{i_j} \cap L_{0,i_j}$. By the definition of a *good* transfer, $|E_j| \geq \frac{1}{2}M(c)$ for every j.

The sets $E_1, E_2, \ldots, E_g$ satisfy the conditions of Lemma 5.12 with $u = 0$. Using Lemma 5.12 with $t = r - 1$, we get

$$\begin{aligned}\left|\left(\bigcup_s E_s\right)\cap L_{0,r-1}\right| &\geq \sum_s(|E_s|-d)-(r-1)d\\ &\geq \tfrac{1}{2}a(\tfrac{1}{2}M(c)-d)-rd\\ &\geq aN(c)-rd. \qquad (5)\end{aligned}$$

In $P^{+}_{T_{r-1}}$ the elements $1, 2, \ldots, n/2$ are in the right side. Therefore $(\bigcup_s E_s) \cap L_{0,r-1} = \emptyset$. Consequently, by (5),

$$r \geq \frac{aN(c)}{d}, \tag{6}$$

$a + b = r - 1$, and

$$b \leq 40d\frac{r'}{N(c)}\left(\log\frac{r'}{N(c)} + 1\right) < 40d\frac{2r}{N(c)}\left(\log\frac{2r}{N(c)} + 1\right).$$

Therefore,

$$a \geq r - 40d\frac{2r}{N(c)}\left(\log\frac{2r}{N(c)} + 1\right). \tag{7}$$

Using (6) and (7) we obtain

$$r \geq \frac{N(c)}{d}\left(r - 40d\frac{2r}{N(c)}\left(\log\frac{2r}{N(c)} + 1\right)\right),$$

from which we get

$$\log\frac{2r}{N(c)} + 1 \geq \frac{(1 - d/N(c))N(c)}{80d}.$$

We recall that $N(c) = \frac{1}{8}M(c) = 2^{d-5}$ and $c = 2d - 1$. From here we can easily conclude that $c < O(\log\log r)$. The number of lines determined by G is at most $\binom{n}{2}$. Every transfer represents a line which is determined by G. Therefore,

$$\binom{n}{2} = \binom{2k + 2c}{2} \geq r.$$

Hence, $c < O(\log\log k)$. □

7. Concluding Remarks

In this paper we gave an upper bound for $f(k, k)$, however, using the same proof except the final analysis one can give an upper bound for $f(k, l)$, namely,

$$f(k, l) = O(\log(|k - l| + \log(k + l))). \tag{8}$$

For small l (for example, $l = 0$) this bound is asymptotically tight, as was shown by Kupitz and Perles [KP].

The following conjecture of Kupitz and Perles is thus still open.

Conjecture 7.1. $f(k, k) \leq 2k + C$, *where C is an absolute constant.*

Perles showed, by construction, that $f(k, k) \geq 2k + 4$, which is the best known lower bound.

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