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The Density of Shapes in Three-Dimensional Barycentric Subdivision*

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Abstract. We prove that the infinite process of iterated barycentric subdivision, when applied to a tetrahedron, produces a dense set of shapes of smaller tetrahedra.

1. Introduction

The *barycentric subdivision* of an n-dimensional simplex Δ is a certain collection of $(n+1)!$ smaller n-simplices whose union is Δ. The construction is defined by induction on n. If $n = 0$, then Δ is a single point, and the barycentric subdivision of Δ is this same point. In general, if Δ' is one of the simplices in the barycentric subdivision of Δ, then Δ' is the convex hull of a set of the form $v \cup F'$, where v is the center of mass of Δ—i.e., the barycenter—and F' is one of the simplices in the barycentric subdivision of one of the top dimensional faces F of Δ. See p. 123 of [S] or Section 2 below for more details.

Consider the following dynamical process: Start with an n-simplex Δ and barycentrically subdivide Δ into simplices $\Delta_1, \ldots, \Delta_{(n+1)!}$. Next, subdivide Δ_j into simplices $\Delta_{j1}, \ldots, \Delta_{j(n+1)!}$, for each j, and so forth. This process produces an infinite collection C of simplices. A natural question is: *Does C consist of a dense set of shapes*? By *shape* we mean a simplex modulo similarities.

In [BBC] this question was raised and answered in the two-dimensional case. Part of the idea works in all dimensions. Let $\mathcal{T}$ be the collection of matrices of the form $T = L/|\det(L)|^{1/n}$, where L is the linear part of an affine map from Δ to a member of C. The affine naturality of barycentric subdivision forces $\mathcal{T}$ to be a semigroup of $SL_n(\boldsymbol{R})$, the group of $n \times n$ determinant-1 matrices.

When $n = 2$, a calculation in [BBC] shows that $\mathcal{T}$ contains some infinite-order elliptic elements. (In general, an *elliptic element* of $SL_n(\boldsymbol{R})$ is a matrix which generates

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a subgroup having compact closure, which happens iff the matrix is diagonalizable over $\boldsymbol{C}$ with all eigenvalues unit complex numbers.) The set of powers of an infinite-order elliptic element is dense in a compact subgroup of $SL_2(\boldsymbol{R})$ and these dense sets are used to show that $\mathcal{T}$ is dense in $SL_2(\boldsymbol{R})$. Hence, in the two-dimensional case, C contains a dense set of triangles.

Using a computer search, which we detail in the next section, we found some infinite-order elliptic elements in the three-dimensional case. This seems like a lucky accident, because the set of elliptic elements in $SL_n(\boldsymbol{R})$ has measure zero for $n \geq 3$. Using these elliptic elements, some basic Lie group theory, and Mathematica [W], we prove

Theorem 1.1. *The three-dimensional barycentric subdivision process produces a dense set of shapes of tetrahedra.*

A similar computer search failed to turn up any elliptic elements in the case $n = 4$, though we certainly would have liked to make a deeper search using a more powerful computer. We think that the density result should be true in all dimensions, whether or not $\mathcal{T}$ contains elliptic elements.

2. The Proof

Here we give a concrete description of barycentric subdivision in the three-dimensional case. Let Δ be the convex hull of points $v_0, v_1, v_2, v_3 \in \boldsymbol{R}^3$. Let S_4 be the group of permutations of the set $\{0, 1, 2, 3\}$. Given $\sigma = (i_0, i_1, i_2, i_3) \in S_4$, let c_k be the center of mass of the points $v_{i_0}, \ldots, v_{i_k}$. Let Δ_σ be the convex hull of the points c_0, c_1, c_2, c_3. The union $\bigcup_{\sigma \in S_4} \Delta_\sigma$ is the barycentric subdivision of Δ.

To begin our dynamical process, we take the initial tetrahedron Δ to be the convex hull of the vertices e_0, e_1, e_2, e_3. Here e_0 is the origin and $\{e_1, e_2, e_3\}$ is the standard basis of $\boldsymbol{R}^3$. Let A_σ be the affine map such that $A_\sigma(e_k) = c_k$ for $k = 0, 1, 2, 3$. Let L_σ be the linear part of A_σ. Finally, let $T_\sigma = L_\sigma / |\det(L_\sigma)|^{1/3}$. By construction, $A_\sigma(\Delta) = \Delta_\sigma$ and therefore $T_\sigma \subset \mathcal{T}$, the semigroup discussed in Section 1.

We order the 24 elements of S_4 lexicographically. For instance, $\sigma_1 = (0123)$ and $\sigma_2 = (0132)$. We define

$$F(i, j, k) = T_{\sigma_k} \circ T_{\sigma_j} \circ T_{\sigma_i}.$$

Say that the triple (i, j, k) is *good* if $F(i, j, k)$ is an infinite-order elliptic element. A computer search reveals 39 good sequences. Here is the list, modulo cyclic permutations:

(2, 15, 19); (5, 8, 23); (5, 19, 18); (5, 20, 16); (7, 17, 8); (8, 18, 9); (8, 18, 20);
(8, 23, 16); (9, 19, 23); (15, 19, 16); (16, 16, 19); (16, 19, 18); (19, 23, 20).

We had hoped to see a divine pattern in this list, but did not.

Our density proof uses only the elements

$$S = F(23, 20, 19); \qquad M_1 = F(5, 20, 16); \qquad M_2 = F(20, 16, 5).$$

Another triple of elements from the list would probably work just as well. In the Appendix we include a short Mathematica program which computes

$$S = \frac{1}{24}\begin{bmatrix} 54 & 48 & 39 \\ -6 & -32 & -35 \\ -78 & -32 & -23 \end{bmatrix}; \qquad M_1 = \frac{1}{72}\begin{bmatrix} -60 & -68 & -27 \\ 36 & 12 & 81 \\ -60 & 4 & 27 \end{bmatrix};$$

$$M_2 = \frac{1}{24}\begin{bmatrix} 18 & 12 & 21 \\ -54 & -68 & -71 \\ 54 & 52 & 43 \end{bmatrix}.$$

Lemma 2.1. *S, M_1, and M_2 are infinite-order elliptic elements of $SL_3(\boldsymbol{R})$.*

Proof. The eigenvalues of S and M_j respectively are $\{1, \alpha, \bar{\alpha}\}$ and $\{1, \beta, \bar{\beta}\}$, where $\alpha = -25/48 + i\sqrt{1679}/48$ and $\beta = -31/48 + i\sqrt{1343}/48$. Both α and β have norm 1, so S and M_j are elliptic. If S had finite order, then α would be a primitive nth root of unity for some n. Then α would have $\varphi(n)$ distinct Galois conjugates, where φ is the Euler phi-function. Since α is a quadratic irrational, we have $\varphi(n) = 2$. The forces $n \leq 6$. Clearly, α is not an nth root of unity for $n \leq 6$. Hence S has infinite order. The same argument works for M_j. □

Let $\langle S\rangle$ be the closure of the semigroup generated by S. Since S is infinite-order elliptic, $\langle S\rangle$ is a closed one-parameter compact subgroup. Let $G \subset SL_3(\boldsymbol{R})$ be the closed subgroup generated by the eight compact subgroups $G_{ij} = M_i^j \langle S\rangle M_i^{-j}$. Here $i \in \{1, 2\}$ and $j \in \{1, 2, 3, 4\}$.

Lemma 2.2. $G = SL_3(\boldsymbol{R})$.

Proof. The lie algebra to $SL_3(\boldsymbol{R})$ is $\mathfrak{sl}_3(\boldsymbol{R})$, the space of traceless 3×3 matrices. Below we justify the claim that

$$\mathfrak{s} = \begin{bmatrix} 70 & 54 & 57 \\ -114 & -107 & -104 \\ 18 & 52 & 37 \end{bmatrix} \in \mathfrak{sl}_3(\boldsymbol{R})$$

generates $\langle S\rangle$. By this we mean that

$$\langle S\rangle = \{\exp(t\mathfrak{s}) \mid t \in \boldsymbol{R}\}.$$

For i and j as above we define $\mathfrak{g}_{ij} = M_i^j \mathfrak{s} M_i^{-j}$. By construction

$$G_{ij} = \{\exp(t\mathfrak{g}_{ij}) \mid t \in \boldsymbol{R}\}.$$

Let $\mathfrak{G}$ be the vector space spanned by the eight vectors $\mathfrak{g}_{ij}$.

For any lie algebra vectors $\mathfrak{a}$ and $\mathfrak{b}$ we have the well-known formula

$$\exp(\mathfrak{a} + \mathfrak{b}) = \lim_{k\to\infty}\left(\exp\left(\frac{\mathfrak{a}}{k}\right) \cdot \exp\left(\frac{\mathfrak{b}}{k}\right)\right)^k.$$

(See Exercise 8.38 of [FH].) This formula easily implies that $\exp(\mathfrak{G}) \subset G$. Since $\dim(\mathfrak{sl}_3(\boldsymbol{R})) = 8$, all we need to prove is that $\dim(\mathfrak{G}) = 8$. There is a natural map

$P\colon \mathfrak{sl}_3(\boldsymbol{R}) \to \boldsymbol{R}^8$. We simply string out the coordinates of a trace-zero matrix $\mathfrak{g}$, leaving off $\mathfrak{g}(3,3)$. It is easy to see that P is a vector space isomorphism. Let M be the 8×8 matrix whose rows are $P(\mathfrak{g}_{ij})$. We compute

$$\det(M) = \frac{15746793376867188813314629943901117}{1597399996853114634 24} \neq 0.$$

This is only possible if the vectors $P(\mathfrak{g}_{ij})$ span $\boldsymbol{R}^8$. □

Let $\bar{\mathcal{T}}$ be the closure of $\mathcal{T}$ in $SL_3(\boldsymbol{R})$. By construction $\langle S\rangle \subset \bar{\mathcal{T}}$. Since M_j is an infinite-order elliptic element, $M_i^{\pm j} \in \bar{\mathcal{T}}$ for all relevant i and j. Therefore the *group* G_{ij} is contained in the *semigroup* $\bar{\mathcal{T}}$. This implies that $G \subset \bar{\mathcal{T}}$. However, $G = SL_3(\boldsymbol{R})$. Therefore $\mathcal{T}$ is dense in $SL_3(\boldsymbol{R})$. Our theorem follows immediately from this.

Our only piece of unfinished business is to justify the formula for $\mathfrak{s}$. By computing the eigenspaces of S we find that the matrix

$$U = \begin{bmatrix} -21 & 0 & 2 \\ -34 & -1 & -3 \\ 58 & 2 & 0 \end{bmatrix}$$

conjugates S to block triangular form:

$$U^{-1}SU = \begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix}; \qquad B = \frac{1}{48}\begin{bmatrix} -14 & -60 \\ 30 & -36 \end{bmatrix}.$$

Note that $B \in SL_2(\boldsymbol{R})$ is infinite-order elliptic. Let $\langle B\rangle$ be the closure of the group generated by B. We claim that the matrix

$$\mathfrak{b} = 48B - 24\,\mathrm{trace}(B)I = \begin{bmatrix} 11 & -60 \\ 30 & -11 \end{bmatrix} \in \mathfrak{sl}_2(\boldsymbol{R})$$

generates $\langle B\rangle$ in the sense that $\langle B\rangle = \{\exp(t\mathfrak{b}) \mid t \in \boldsymbol{R}\}$. To prove this, we note that $\mathfrak{b}$ and B commute, when multiplied together as matrices. Hence, for any $t \in \boldsymbol{R}$ the element $\beta_t = \exp(t\mathfrak{b})$ commutes with any element of $\langle B\rangle$. As is well known $SL_2(\boldsymbol{R})$ acts isometrically on the hyperbolic plane $\boldsymbol{H}^2$ by linear fractional transformations. The group $\langle B\rangle$, which consists entirely of elliptic elements, acts as the group of isometric rotations about some fixed point $x \in \boldsymbol{H}^2$. Since β_t commutes with all elements of $\langle B\rangle$, it must also act as an isometric rotation about x. Hence $\beta_t \subset \langle B\rangle$ for all t. Our claim now follows easily.

Since $\mathfrak{b}$ generates $\langle B\rangle$,

$$\mathfrak{s} = U\begin{bmatrix} 0 & 0 \\ 0 & \mathfrak{b} \end{bmatrix}U^{-1}$$

generates $\langle S\rangle$ in the sense of Lemma 2.1. Expanding this product gives the formula for $\mathfrak{s}$ used in Lemma 2.1.

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Appendix. A Mathematica File

We refer the reader to [W] for details on the implementation of Mathematica. A copy of this file produced our calculations.

```
e[0]={0,0,0}; e[1]={1,0,0}; e[2]={0,1,0}; e[3]={0,0,1};
S4=Permutatations[{0,1,2,3}];

T[n_]:=(sigma=S4[[n]];
c0=(e[sigma[[1]]])/1;
c1=(e[sigma[[1]]]+e[sigma[[2]]])/2;
c2=(e[sigma[[1]]]+e[sigma[[2]]]+e[sigma[[3]]])/3;
c3=(e[sigma[[1]]]+e[sigma[[2]]]+e[sigma[[3]]]
    +e[sigma[[4]]])/4;
L=Transpose[c1-c0,c2-c0,c3-c0];
L/Power[Abs[Det[L]],1/3])

F[a_,b_,c_]:=Simplify[T[a].T[b].T[c]]
S=F[23,20,19]; M1=F[5,20,16]; M2=F[20,16,5];
s={{70, 54, 57}, {-114, -107, -104}, {18, 52, 37}}
U={{-21, 0, 2}, {-34, -1, -3}, {58, 2, 0}}

Ad[x_,y_]:=x.y.Inverse[x];

g11=Ad[M1,s];
g12=Ad[M1.M1,s];
g13=Ad[M1.M1.M1,s];
g14=Ad[M1.M1.M1.M1,s];
g21=Ad[M2,s];
g22=Ad[M2.M2,s];
g23=Ad[M2.M2.M2,s];
g24=Ad[M2.M2.M2.M2,s];

P[x_]:=Take[Flatten[x],8]
M={P[g11],P[g12],P[g13],P[g14], P[g21],P[g22],P[g23],P[g24]}
Det[M]
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