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An Isosceles Triangle That Tiles the Sphere in Exactly Three Ways*

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Abstract. The $(80^\circ, 60^\circ, 60^\circ)$ spherical triangle tiles the sphere, though not in an edge-to-edge fashion. We show that it tiles in exactly three different ways.

1. Introduction

In 1923 Sommerville [3] classified the spherical triangles that could tile the sphere with congruent copies in an edge-to-edge fashion, under certain restrictions. Davies obtained a complete classification of the triangles which tile the sphere edge-to-edge in 1965 [1], but left certain details to the reader. A recent paper by Ueno and Agaoka [4] verifies Davies' classification to be complete, removes certain duplications, and makes other important observations about edge-to-edge tilings.

In a recent paper [2] Dawson showed that, among the isosceles triangles, there are three sporadic triangles and one infinite family that tile the sphere but do not do so in an edge-to-edge fashion. In two of the sporadic cases, $(150^\circ, 60^\circ, 60^\circ)$ and $(100^\circ, 60^\circ, 60^\circ)$, it is very easy to show that the triangle can tile in one way only (up to reflection) [2]. The tiles in the infinite family $\{(180^\circ - 360^\circ/n, 360^\circ/n, 360^\circ/n), n \text{ odd}\}$ each tile in a large number of different ways, which can be enumerated using Burnside's theorem. In this note we prove the surprising fact that the remaining sporadic tile, $(80^\circ, 60^\circ, 60^\circ)$, tiles in exactly three different ways.

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2. Definitions and Preliminaries

For the remainder of this paper, we only consider the $(80^\circ, 60^\circ, 60^\circ)$ triangle, and therefore usually refer to it simply as a "triangle." The 80° angle is referred to as a "large" angle, and the 60° angle as a "small" angle. The opposite edges are referred to as "long"(with length L) and "short"(with length S), respectively.

There are two ways in which these angles can be selected to add to 360°: there can be six small angles, or three large angles and two small ones. We call vertices with these configurations $(0, 6)$ *vertices* and $(3, 2)$ *vertices*, respectively. If the tiling has a vertex which is on an edge of some triangle, with the remaining 180° filled with angles of other triangles, there is only one way for this to be done, namely three small angles. Such a vertex is called an $(0, 6)/2$ *split*.

The $(80^\circ, 60^\circ, 60^\circ)$ triangle has an excess of 20°, and it therefore requires exactly 36 such triangles to cover the sphere. Between them, these triangles have 36 large angles and 72 small angles. It thus follows that any tiling with these triangles must have exactly 12 $(3, 2)$ vertices, with the remaining 48 small angles accounted for by a mixture of $(0, 6)$ vertices and $(0, 6)/2$ splits. As we will see, the numbers of these can vary from tiling to tiling.

We say that a triangle's edge that contains a split vertex in its exterior *overhangs* that vertex. For future reference we state explicitly:

Lemma 1 (Overhang Lemma). *No edge can overhang a vertex that has a large angle.*

If a triangle has a specified small angle, there are two ways in which the triangle can be placed. However, once the large angle of a triangle has been specified, the location of the triangle is completely determined. We thus have, in particular:

Lemma 2 (SSL Lemma). *If two small angles and at least one large angle are determined at a vertex, the remaining angle(s) are large, and their positions are completely forced.*

One other trivial result will be needed enough to deserve a name:

Lemma 3 (Edge Lemma). *If the union of the already-determined triangles has a short or long edge with a reflex angle at each end, it must be matched by a similar edge on the other side; and if that edge is long, the position of the triangle covering that edge is completely determined.*

3. The Main Result

Theorem 1. *The* $(80^\circ, 60^\circ, 60^\circ)$ *triangle tiles the sphere in exactly three ways.*

Proof. Firstly, we look at the ways in which the triangles can meet at a $(3, 2)$ vertex. The possibilities are shown in Fig. 1. Either all three large corners come together (Fig. 1(a)–(c)), or the small corners separate two of them from the third (Fig. 1(d)–(f)).

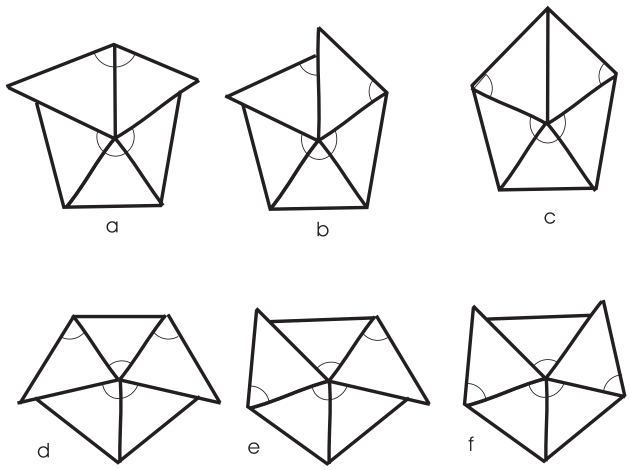

Fig. 1. The configurations for a (3, 2) vertex.

The configuration of Fig. 1(b) is impossible, by the Overhang Lemma. Figure 1(a) is also impossible. First, the overhangs at A (in Fig. 2) are forced. Then all angles at the split vertices A must be small; then the triangles 1 must be as shown, to avoid an overhang at B; but then there are four large angles at B, which is impossible.

While the configuration of Fig. 1(c) can exist in a tiling, we can show that there is no tiling using only that (3, 2) configuration. If we attempt such a tiling, that configuration is forced at every vertex containing a large angle. Thus, beginning with the shaded configuration in Fig. 3, triangles are forced at the points A, then at the points B, then at the points C as shown. However, then there must be two more small angles at each of the points D, and four large angles come together at E, which is impossible.

It therefore follows that one of the following must occur:

Case A: *The tiling contains the configuration of Fig*. 1(d).

Case B: *The tiling contains the configuration of Fig*. 1(e).

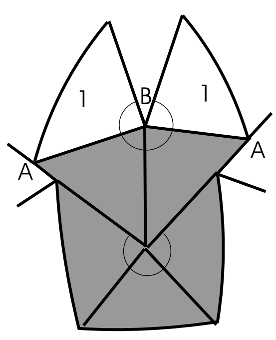

Fig. 2. An impossible configuration.

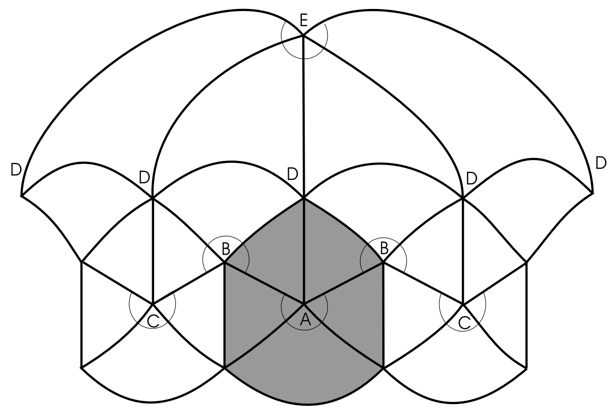

Fig. 3. The partial tiling forced by the configuration of Fig. 1(c) in the absence of other (3, 2) configurations.

Case C: *The tiling contains the configuration of Fig*.1(f) *and neither of the configurations of Fig*. 1(d) *or* 1(e).

We shall see that all of these do in fact exist, and are unique (up to reflection for tiling B).

In case A the tiling shown in Fig. 4 is forced. We start with just the initial configuration (shaded) and show that the remaining triangles are forced, a few at a time. The Overhang Lemma forces the two triangles meeting the original configuration on the edges AB to be as shown (triangle 1). Applying the same argument elsewhere forces the triangles 2 and 3, 4 and 5, and then 6 and 7. Triangle 8 is forced by the SSL Lemma.

At this point, the extended edge CC (shown heavy), of length $2S+L$, has been forced. It is easily verified that no other combination of edge lengths has this sum. Moreover,

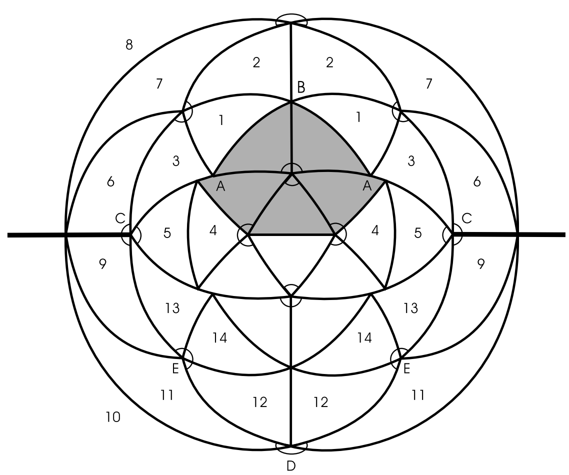

Fig. 4. Tiling A, forced by the configuration of Fig. 1(d).

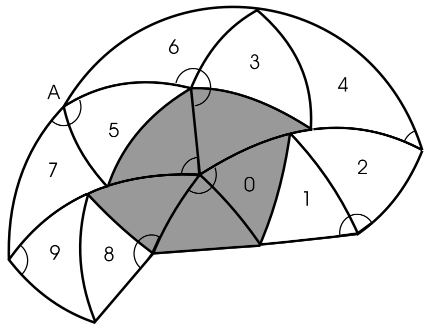

Fig. 5. Partial tiling forced by the configuration of Fig. 1(e).

the Overhang Lemma implies that the middle edge on the other side cannot be short; so the triangles 9 and 10 must be as shown. The triangles 11 are forced next.

It is now clear that the remaining two angles at D must be small. There cannot be an overhang at E, as the triangle meeting triangle 9 at the short edge CE would have to have a large angle at C or E. Thus triangle 12 is as shown, and E is a (3, 2) vertex; the remaining nine triangles follow directly.

In case B, the partial tiling shown in Fig. 5 is forced. First, the exposed long edge of triangle 0 must be covered with the long edge of another triangle 1; for there must be a small angle at the split vertex, and the Overhang Lemma prevents the new triangle from having a short edge there, ending in an overhung large angle. Then triangle 2 is forced, as its long edge may not overhang the large angle of tile 1. Triangles 3 and 4 follow in that order by similar arguments, triangle 5 is forced by the Edge Lemma, triangle 6 by the SSL Lemma, and triangles 7–9 by the Overhang Lemma.

From Fig. 5 there are two ways to put a large angle at vertex A. If we place it as triangle 10 in Fig. 6, sharing an edge with triangle 7, triangles 11 and 12 are forced by the SSL Lemma. Triangles 7 and 9–12 then form the forbidden configuration of Fig. 1(a).

If, instead, we place a tile as triangle $10'$ in Fig. 7, triangle 11 is forced, as the reflex angle at B does not permit a long edge at AB. Triangle 12 is forced by the Edge Lemma,

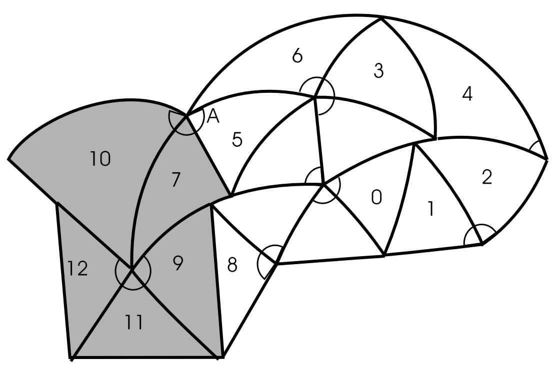

Fig. 6. An attempt that is eventually blocked.

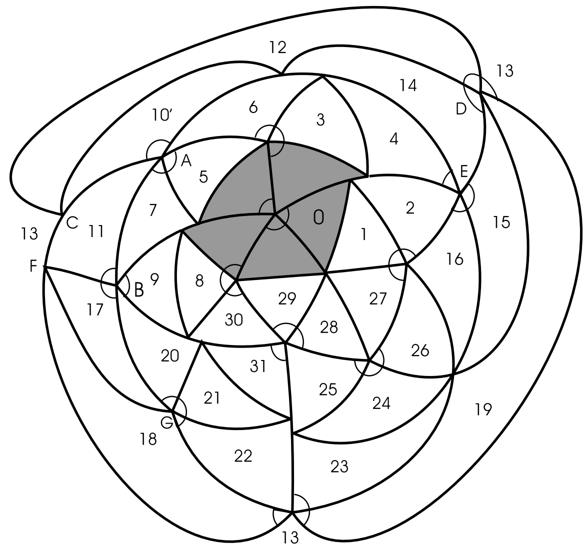

Fig. 7. Tiling B, forced by the configuration of Fig. 1(e).

and triangle 13, filling the remaining small gap at C, must have its large angle at D, and not overhang that vertex, by the Overhang Lemma. Triangle 14 follows by a similar argument. Applying the SSL Lemma at E forces triangles 15 and 16.

On the other side of the diagram (though close by on the sphere!), the short edge FB of triangle 11 must be matched by another short edge; and as F is a split vertex with no large angle, triangle 17 is as shown. Filling small gaps with due regard to the Overhang Lemma yields triangles 18–20.

Applying the SSL Lemma at G we obtain triangles 21 and 22, which in turn forces triangle 23. Triangle 24 follows from triangle 23 by the Edge Lemma. Triangles 25 and 26 fill small gaps, with their orientation determined by the Overhang Lemma; and the remaining four triangles follow in the order shown.

In case C, the configuration of Fig. 1(f) forces the partial tiling shown in Fig. 8.

There are three ways in which the vertex at A can be filled; two of them yield the configurations of Fig. 1(d) and (e), and the third yields another (Fig. 1(f)) (shown with heavy outline). This forces more of the tiling. Doing this four more times, we obtain the tiling of Fig. 9. □

4. Some Facts About the Tilings

First, we note that the three tilings really are different; see Fig. 10, in which the tilings A, B, and C appear, each seen from over the "North Pole" (top), the "Equator" (middle), and the "South Pole" (bottom). Tiling A has eight split vertices, and the others have 12 each.

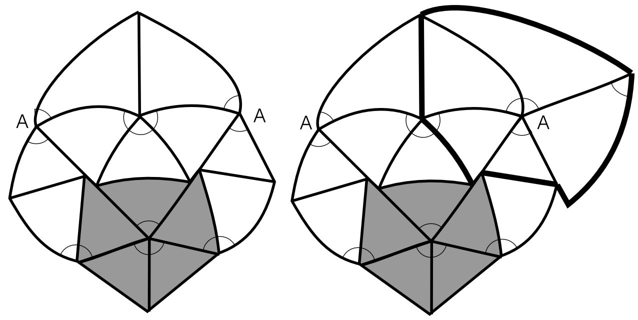

Fig. 8. Partial tiling forced by the configuration of Fig. 1(f) in the absence of the configurations of Fig. 1(d) and 1(e).

We can also see that the symmetry groups differ. Tiling A has a symmetry group generated by two orthogonal reflections. Tiling B has only one symmetry, a 180° rotation about the axis that is seen end-on in the top and bottom views, and would be vertical in the middle view. Tiling C has a symmetry group of order 12, generated (for instance) by a 120° rotation, a reflection in a plane through the axis of the rotation, and a point inversion.

While the equatorial view of Tiling A superficially resembles the polar views of Tiling C, this is slightly misleading. In the latter tiling the "hexagons" are antipodally placed, whereas in the former case they are not.

Tiling C is made up of six "kites", each made up of six triangles. A complete kite can be seen at the top of the middle view of that tiling. Tiling A contains eight kites, which overlap in various ways; and tiling B has six, also overlapping.

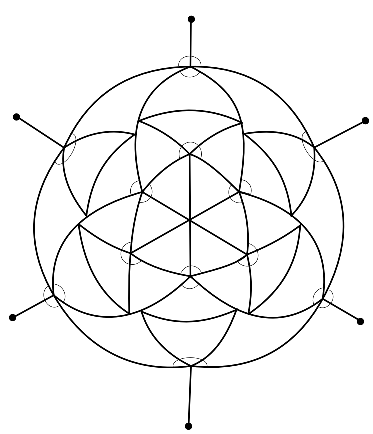

Fig. 9. Tiling C, forced by the configuration of Fig. 1(f) in the absence of the configurations of Fig. 1(d) and 1(e).

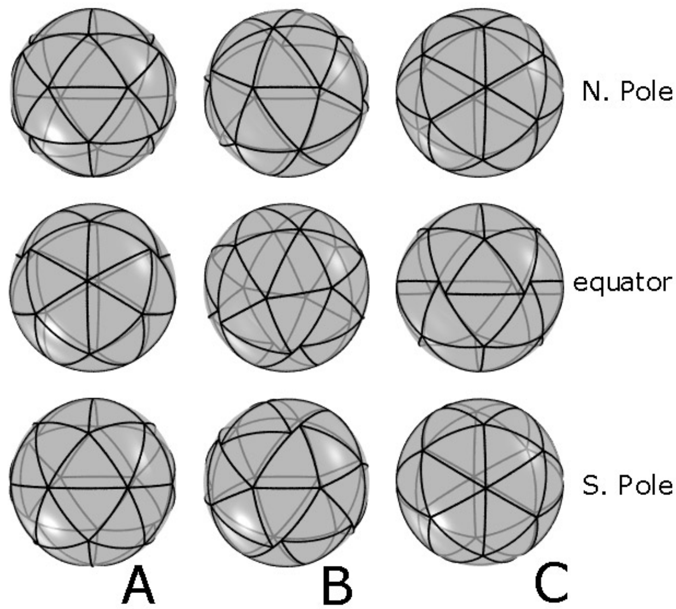

Fig. 10. Views of the three tilings.

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