Discrete Comput Geom 31:655–663 (2004)
DOI: 10.1007/s00454-003-0825-4

Discrete & Computational
Geometry


# More About 4-Isosceles Planar Sets*

Changqing Xu[1] and Ren Ding[2]

[1]Mathematics Department, Shandong University,
Jinan 250100, People's Republic of China
chqxu@sina.com

[2]Mathematics Department, Hebei Normal University,
Shijiazhuang 050016, People's Republic of China
rending@heinfo.net

**Abstract.** A finite planar set is $k$-isosceles for $k \geq 3$ if every $k$-point subset of the set contains a point equidistant from the other two. In [1] Fishburn obtains several important results about isosceles planar sets and poses a series of conjectures and open questions. We disprove Conjecture 1 in [1] and provide another 34 nonsimilar 4-isosceles 8-point planar sets which answer one of the open questions in [1] negatively.

## 1. Introduction

In [1] Fishburn obtains several important results about isosceles planar sets and poses a series of conjectures and open questions. Affirmative answers to a few of those open questions are given in [2].

A finite planar set is $k$-isosceles for $k \geq 3$ if every $k$-point subset of the set contains a point equidistant from the two others. In [1] the author suggests a conjecture as follows:

**Conjecture 1.** *No* 9*-point planar set is* 4*-isosceles*.

We disprove the conjecture by showing:

**Theorem 1.** *There exist at least two* 4*-isosceles* 9*-point planar sets*.

Eleven nonsimilar 4-isosceles 8-point planar sets are given in [1]. One of the open questions posed in [1] is:

* This project was supported by the NSF of Hebei Province (No. 199174).

**Open Question.** Do these eleven 4-isosceles 8-point sets exhaust all 4-isosceles 8-point sets?

We provide another 34 nonsimilar 4-isosceles 8-point planar sets which answer the open question negatively.

## 2. Existence of 4-Isosceles 9-Point Planar Sets

Let $d(x, y)$ denote the Euclidean distance between $x$ and $y$ in $R^2$. Let $R_n$ denote the vertex set of a regular convex $n$-gon, and let $R_n^+$ be $R_n$ plus its center.

In Fig. 1(a) we construct a 9-point planar set $F = \{0, 1, 2, 3, 4, 5, a, b, c\}$ by adding three points $a, b, c$ to $R_5^+ = \{0, 1, 2, 3, 4, 5\}$ such that $b$ is the point of intersection of diagonals 14 and 35, $c$ is the point of intersection of 13 and 24, and $a$ is the center of the circle determined by the three points 0, 3, 4.

In Fig. 1(b) we construct a 9-point planar set $F = \{0, 1, 2, 3, 4, 5, a, b, c\}$ by adding three points $a, b, c$ to $R_5^+ = \{0, 1, 2, 3, 4, 5\}$ such that $b$ is the point of intersection of sides 21 and 45, $c$ is the point of intersection of 32 and 51, and $a$ is the center of the circle determined by the three points 0, 2, 5.

By the construction it is easy to prove that the 9-sets in both figures are 4-isosceles.

Motivated by [1] we pose the following conjecture:

**Conjecture 2.** *No 10-point planar set is 4-isosceles.*

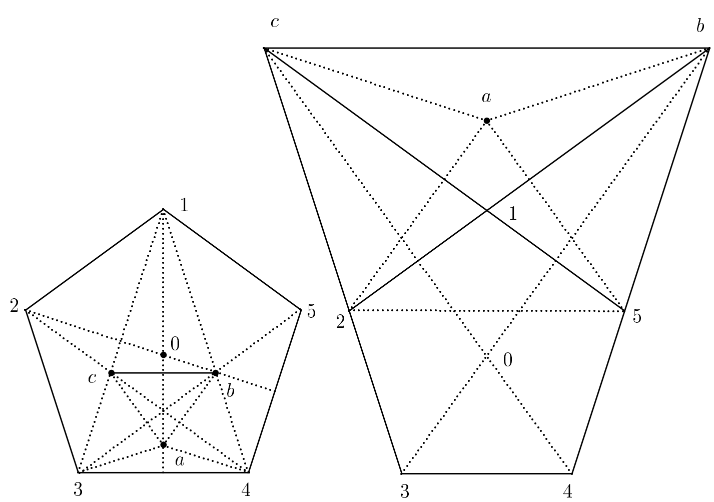
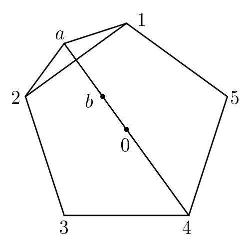



**Fig. 1.** Existence of 4-isosceles 9-point planar sets. (a) $d(a, 0) = d(a, b) = d(a, 4) = d(a, 3) = d(a, c)$; $d(1, 5) = d(1, b) = d(1, c) = d(1, 2)$; $d(0, b) = d(0, c)$; $d(b, c) = d(b, 4) = d(b, 5) = d(c, 2) = d(c, 3)$. (b) $d(a, 0) = d(a, 2) = d(a, 5) = d(a, b) = d(a, c)$; $d(b, 1) = d(b, 5) = d(c, 1) = d(c, 2)$; $d(0, b) = d(0, c)$; $d(b, c) = d(b, 2) = d(b, 4) = d(c, 3) = d(c, 5)$.

## 3. More 4-Isosceles 8-Point Planar Sets

Our answer to the above-mentioned open question in [1] is that the eleven 4-isosceles 8-point sets in [1] do not exhaust the 4-isosceles 8-point planar sets. We provide thirty-four 4-isosceles 8-point sets. By adding two points $a, b \in R^2$ to $R_5^+$ in 34 different ways, we obtain 34 nonsimilar 4-isosceles 8-point sets $F = R_5^+ \cup \{a, b\}$ as shown in Figs. 2–35. In each figure's caption we describe the way of adding two points $a$, $b$ under the title "construction" and we list all deduced equalities by which we can prove that the set $F = R_5^+ \cup \{a, b\}$ is 4-isosceles. Let $A$ be any 4-point subset of $F$, we prove that $A$ contains an isosceles triangle. Since any three points from $R_5^+$ form an isosceles triangle, we need only consider $A = \{a, b, x, y\}$ where $x, y \in R_5^+$. We have $C_2^6 = 15$ choices for $x, y$. For each case we can prove the existence of an isosceles triangle in a way similar to that of Theorem 1.

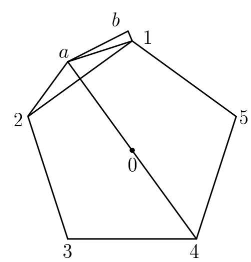



Construction: $0a = 01, a1 = ab$

Properties: $54 = 5b, 23 = 3b,$

$ab = a2, a3 = a5, 34 = 3b$

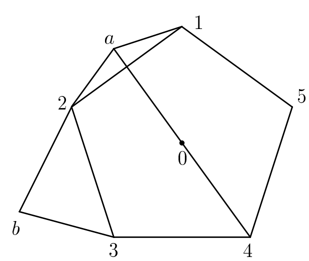
**Fig. 2**




Construction: $0a = 01, a1 = ab, 4a = 4b$

Properties: $ab = a2, a3 = a5$

**Fig. 3**

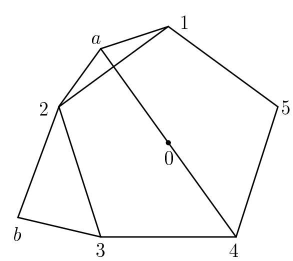



Construction: $0a = 01, ab = a3,$

$4a = 4b$

Properties: $ab = a5, a1 = a2$

**Fig. 4**

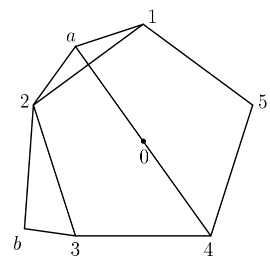



Construction: $0a = 01, ab = a3,$

$14 = 4b$

Properties: $ab = a5, a1 = a2$

**Fig. 5**

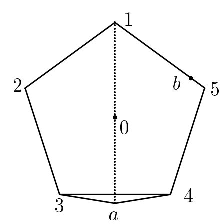


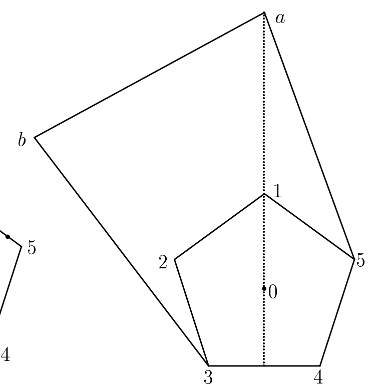

Construction: $0a = 01, ab = a3,$

$ab = 4b$

Properties: $ab = a5, a1 = a2$

**Fig. 6**

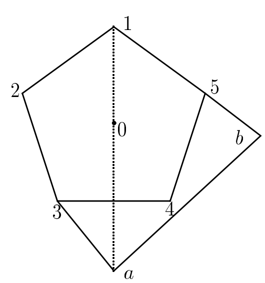


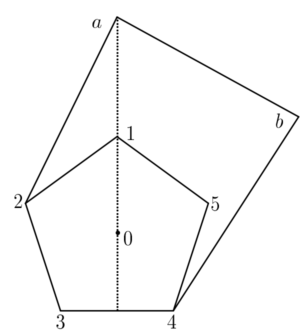




Construction: $14 = 1a, ab = a2, 0a = 0b$

Properties: $ab = a5, 13 = 1a, a3 = a4$

**Fig. 7** **Fig. 8**

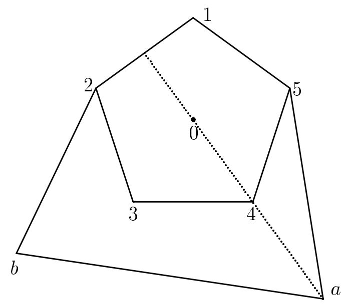



Construction: $0a = 0b, ab = a2, ab = a5, 14 = 1b$

Properties: $13 = 1b, a3 = a4$

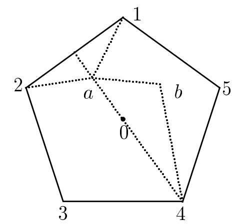
**Fig. 9** **Fig. 10**




Construction: $34 = 4a, ab = a1, 0a = 0b$

Properties: $ab = a2, 45 = 4a, a3 = a5$

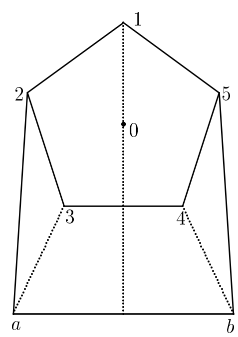
**Fig. 11**

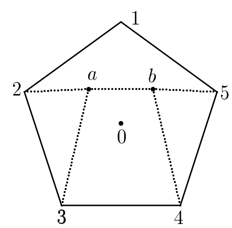



Construction: $0a = 0b, ab = a1,$
$ab = a2, 34 = 4b$

Properties: $45 = 4b, a3 = a5$

**Fig. 12**

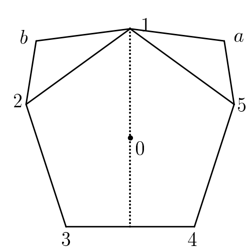


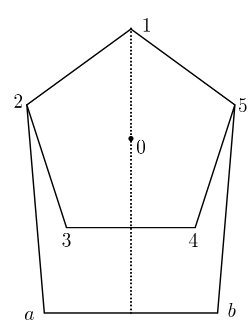




Construction: $0a = 0b, 1a = 1b, ab = a2, 34 = 3a$

Property: $ab = b5$

**Fig. 13** **Fig. 14**

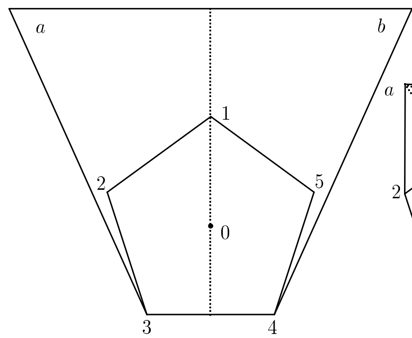



Construction: $0a = 0b, 1a = 1b, ab = a4, 2a = 25$

Property: $ab = b3$

**Fig. 15** **Fig. 16**

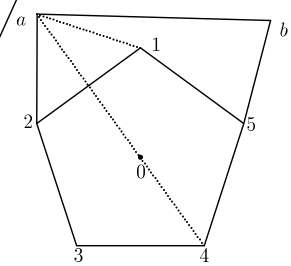



Construction: $0a = 0b, 1a = 1b,$

$ab = a4, 2a = 25$

Property: $ab = b3$

**Fig. 17**

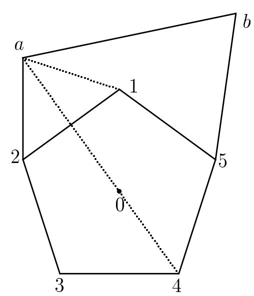



Construction: $1a = 10, ab = a3,$

$ab = b4$

Properties: $a1 = a2, ab = a5$

**Fig. 18**

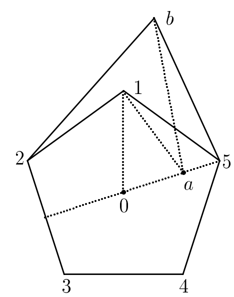



Construction: $1a = 10, ab = a3,$

$4a = 4b$

Properties: $a1 = a2, ab = a5$

**Fig. 19**

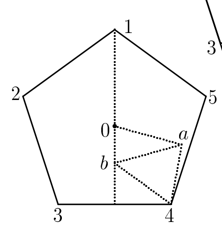



Construction: $1a = 10, ab = a2,$

$ab = b5$

Properties: $ab = a3, a4 = 04, a1 = a4$

**Fig. 20**

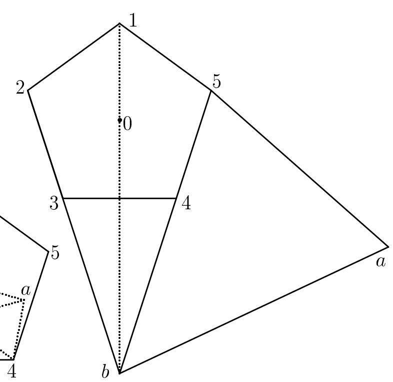



Construction: $1a = 1b, ab = a0,$

$ab = b3, ab = b4$

Property: $b2 = b5$

**Fig. 21**

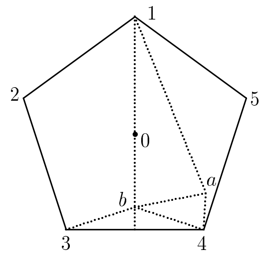



Construction: $1a = 1b, ab = a0,$

$ab = b2, ab = b5$

Property: $b3 = b4$

**Fig. 22**

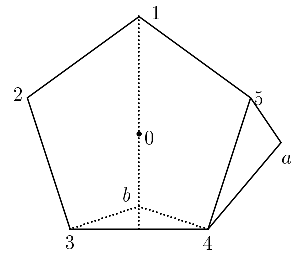



Construction: $b0 = b3, 1a = 1b,$

$ab = b0$

Properties: $ab = b3, ab = b4, b2 = b5$

**Fig. 23**

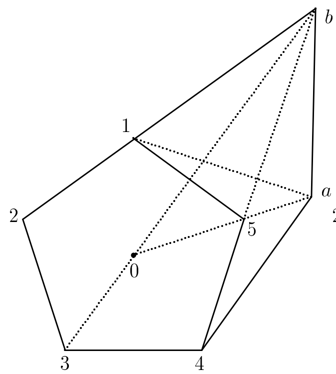



Construction: $b0 = b3, 1a = 1b,$

$ab = b2$

Properties: $ab = b5, b0 = b4, b3 = b4$

**Fig. 24**

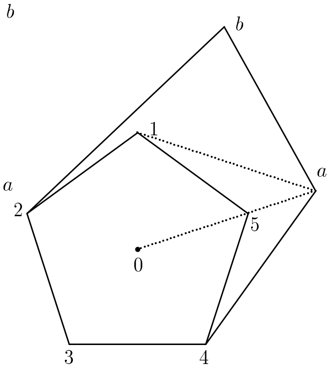



Construction: $1a = a0, ab = a0$

$25 = 5b$

Properties: $a2 = a3, ab = a1, ab = a4$

**Fig. 25**

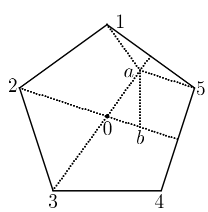



Construction: $1a = a0, ab = a0$

$ab = 5b$

Properties: $a2 = a3, ab = a1, ab = a4$

**Fig. 26**



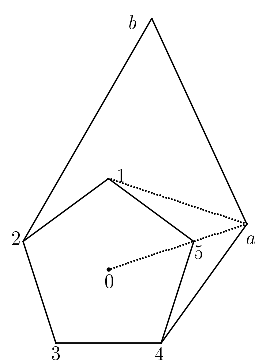

Construction: $1a = a0, ab = a0$

$23 = 3b$

Properties: $ab = a5, a2 = a4,$

$34 = 3b$

**Fig. 27**



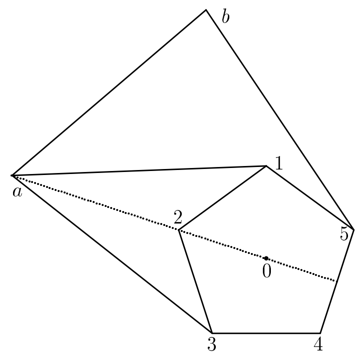

Construction: $1a = a0, ab = a3$

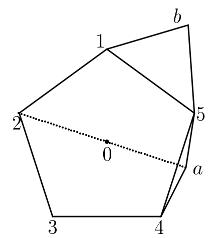
$ab = b5$

Properties: $a0 = a4, ab = a2,$

$a1 = a4$

**Fig. 28**

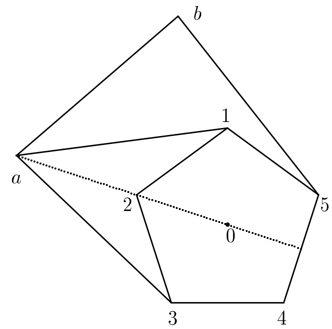


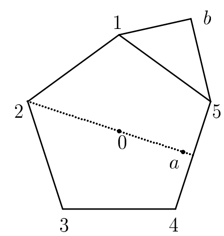




Construction: $25 = 2a, a1 = ab, ab = b0$

Properties: $ab = a3, a4 = a5, 24 = 2a$

**Fig. 29** **Fig. 30**

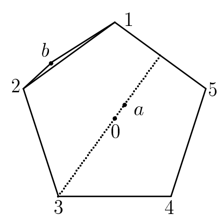


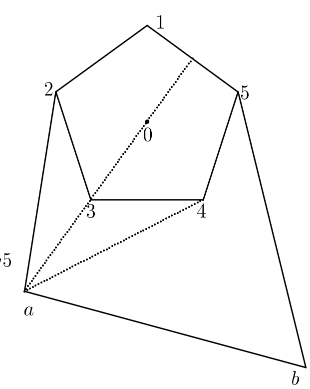




Construction: $a1 = ab, ab = a3, ab = b0, 25 = 2b$

Properties: $24 = 2b, a4 = a5$

**Fig. 31** **Fig. 32**

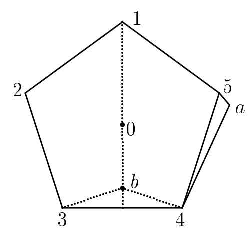



Construction: $23 = 3a, a1 = ab, ab = b0$

Properties: $ab = a5, a2 = a4, 34 = 3a$

**Fig. 33** **Fig. 34**




Construction: $b0 = b3, ab = b2, ab = a1$

Properties: $ab = b5, b0 = b4, b3 = b4$

**Fig. 35**

## References

1. P. Fishburn, Isosceles planar subsets, *Discrete Comput. Geom.*, **19** (1998), 391–398.
2. Changqing Xu and Ren Ding, About 4-isosceles planar sets, *Discrete Comput. Geom.*, **27** (2002), 287–290.

*Received December* 21, 2002, *and in revised form March* 4, 2003. *Online publication September* 29, 2003.