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Mazur Sets in Normed Spaces*

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Abstract. In this paper we investigate different questions concerning Mazur sets in normed spaces, which point out the close connections between geometric functional analysis and discrete geometry. Motivated by a result of Chen and Lin, we study the relationship between Mazur disks and weak* denting points of the dual unit ball. We prove that the only Mazur sets of the spaces ℓ_1^n are points and closed balls. Finally, a new stability property for the family of all sets which are intersections of closed balls is found.

1. Introduction

Some families of convex sets play a central role in questions related to the geometry of normed spaces, on the one hand, and to Minkowski's convexity theory and discrete geometry, on the other. This is the case of $\mathcal{M}$, the family of all intersections of closed balls, and $\mathcal{P}$, the family of all Mazur sets, introduced in [4]. A closed, convex and bounded set C is called a *Mazur set* provided the following strong separation property is satisfied: for every hyperplane H with $dist(C, H) > 0$, there is a ball D such that $C \subset D$ and $D \cap H = \emptyset$. As a consequence of the separation theorem, $\mathcal{P} \subset \mathcal{M}$ and normed spaces satisfying $\mathcal{P} = \mathcal{M}$ are called *Mazur spaces*. Convex bodies of constant width are probably the most interesting examples of Mazur sets. Recall that a bounded,

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closed and convex set C is said to have *constant width* λ if $\sup f(C) - \inf f(C) = \lambda$ for every norm-one functional f. Throughout this paper X denotes a normed space and B its unit ball. As usual, X^* stands for the dual space of X.

Remark 1.1. Every closed, bounded and convex set of constant width is a Mazur set.

Detail. Consider such a set C with constant width λ. Let f be a norm-one functional, let $\varepsilon > 0$ and let $H = \{x \in X : f(x) = \sup f(C) + \varepsilon\}$. We are looking for a ball containing C but missing H. Since $\sup f(C) - \inf f(C) = \lambda$, there are $x, y \in C$ satisfying $f(x) - f(y) \geq \lambda - \varepsilon/2$. Now consider the ball $y + \lambda B$, which obviously contains C since $\mathrm{diam}(C) = \lambda$. If there is $z \in H \cap (y + \lambda B)$, then

$$f(z) - f(y) = \sup f(C) + \varepsilon - f(y) \geq \lambda + \varepsilon/2,$$

which is a contradiction. Though most books on convexity have classical results about convex bodies of constant width, a rigorous and comprehensive treatment of this topic (in finite-dimensional spaces) can be found in [1]. For the infinite-dimensional case, refer to [6] and [7].

By a *disk* in X we mean the intersection of a hyperplane with a ball centered on the hyperplane. Chen and Lin [2] used the notion of a *semi-denting point* to obtain the following characterization of disks which are intersections of balls: the disk $K_f = B \cap (\ker f) \in \mathcal{M}$ if and only if the norm-one functional f is a semi-denting point of B^*, that is, for every $\varepsilon > 0$, there is a weak* slice $S = S(x, \delta) = \{g \in B^* : g(x) \geq 1 - \delta\}$ where $x \in X$, $\|x\| = 1$ and $\delta > 0$ such that $\mathrm{diam}(\{f\} \cup S) < \varepsilon$. The Chen–Lin characterization suggests the possibility of characterizing Mazur disks in a similar way, namely replacing semi-denting points by a suitable stronger condition of dentability. Section 2 is devoted to showing that a functional which defines a Mazur disk is necessarily a weak* denting point. However, in the opposite direction, even if f is a strongly exposed point of B^* the disk $B \cap (\ker f)$ need not be a Mazur set.

A normed space satisfies the binary intersection property (BIP) if every collection of mutually intersecting closed balls has nonempty intersection. In Section 2 of [4] we proved that when a normed space has the BIP then every nonempty intersection of closed balls $C = \bigcap_i B_i$ satisfies $\bigcap_i B_i + \lambda B = \bigcap_i (B_i + \lambda B)$ for every $\lambda > 0$. However, in the general case, the question of whether $C + \lambda B \in \mathcal{M}$ whenever $C \in \mathcal{M}$ remains open. Since adding a ball λB to the convex set C is, in a sense, the opposite of performing $C \sim \lambda B = \{x \in C : \mathrm{dist}(x, X \setminus C) \geq \lambda\}$, it is natural to ask whether $C \sim \lambda B \in \mathcal{M}$ whenever $C \in \mathcal{M}$ and $C \sim \lambda B$ is nonempty. We prove in Section 3 that, quite surprisingly, this is always the case.

In every normed space, points and (closed) balls are the easiest examples of Mazur sets and, for this reason, we can call them *trivial* Mazur sets. In Section 3 we are also concerned with the following question: are there normed spaces with only trivial Mazur sets? We prove that, for every $n \geq 3$, this is the case for the spaces ℓ_1^n. We do not know if, in a finite-dimensional setting, the property of having only trivial Mazur sets is actually a characterization for these spaces when $\dim X \geq 3$. For the case of two-dimensional spaces, Theorem 6.5 in [4] implies that every intersection of balls in ℓ_1^2 is a Mazur set.

2. Mazur Disks and Weak* Denting Points

Recall that a disk is a set of the form $\{x \in y + \lambda B\colon\ f(x) = f(y)\}$, where $f \in X^*\backslash\{0\}$, $y \in X$ and $\lambda > 0$. Let $K_f = \{x \in B : f(x) = 0\}$, let $L_f = \{x \in B\colon\ f(x) \geq 0\}$ and let $M_f = \{x \in B\colon\ f(x) \leq 0\}$. The following two geometric results will be of use for the proof of Proposition 2.3.

Lemma 2.1 [5], [4]. *If $f \in X^*\backslash\{0\}$, $y \in X$, $\lambda > 0$ and the ball B' contains the disk $y + \lambda K_f$, then B' also contains one of the two "half-balls" $y + \lambda M_f$ or $y + \lambda L_f$.*

Lemma 2.2 [3]. *The norm-one functional f is a weak* denting point of B^* if and only if for every bounded subset C in X with* $\inf f(C) > 0$, *there is a ball B' containing C such that* $\inf f(B') > 0$

Proposition 2.3. *If $f \in S^*$ and there is a Mazur set $C \subset \ker f$ with nonempty (relative) interior, then f is a weak* denting point of B^*. This applies, in particular, when the disk $K_f = f^{-1}(0) \cap B$ is a Mazur set.*

Proof. From the Chen–Lin characterization of weak* denting points described in Lemma 2.2 above, it suffices to show that given a bounded nonempty subset $A \subset X$ such that $\inf f(A) > 0$, there exists a ball B' containing A such that $\inf f(B') > 0$. Suppose that $M = \sup\{\|x\|\colon x \in A\}$, choose $a \in X$ such that $0 < f(a) < \inf f(A)$ and let $\lambda > M + \|a\| + f(a)$. Finally, let D be a homothetic image of C containing the set $a + \lambda K_f$. Now, D is also a Mazur set, with $\inf f(D) = f(a) > 0$, so there exists a ball $B' \supset D \supset a + \lambda K_f$ such that $\inf f(B') > 0$. By Lemma 2.1, B' must contain either $a + \lambda M_f$ or $a + \lambda L_f$. The former is impossible, since $\inf f(a + \lambda M_f) = f(a) - \lambda < 0$. Moreover, if $x \in A$, then $f(x) > f(a)$ and $\|x - a\| \leq M + \|a\| < \lambda$, so $x \in a + \lambda L_f \subset B'$, which was to be shown. □

Corollary 2.4. *In the dual of a Mazur space, every semi-denting point of the unit ball is a weak* denting point.*

It is clear that every element in the closure of the set of weak*-denting points of B^* is a semi-denting point of B^*. By Corollary 2.4, if X is a Mazur space, then the set of semi-denting points of B^* is precisely the closure of the set of weak*-denting points of B^*. It would be interesting to determine whether this property is a characterization of Mazur spaces.

It is not difficult to construct a ball D in $\mathbb{R}^3$ for which there exist nonextreme points x which are in the closure of the exposed points (consider, for instance, the euclidean unit ball B_2 in the hyperplane $z = 0$ and the c_0 unit ball B_0 in $y = 0$ and define $D = \operatorname{conv}\{B_2 \cup B_0\}$). Any such x is an example of a semi-denting point which is not a denting point. Moreover, notice that the set of semi-denting points of the unit ball is always closed, while this is not the case for the set of denting points.

Example. The converse to the proposition above is not valid. Indeed, let X be the space $\mathbb{R}^3$ with the ℓ_1 norm and let f be the functional in three-dimensional ℓ_∞ defined by the element $(1, 1, 1)$. This vertex is a strongly exposed point of the dual unit ball. The disk $D = B \cap f^{-1}(0)$ obviously contains the points $(-\frac{1}{2}, \frac{1}{2}, 0)$ and $(-\frac{1}{2}, 0, \frac{1}{2})$. Moreover, for each element $u \in D$ we have $u_1 \geq -\frac{1}{2}$. Indeed, if $u_1 < -\frac{1}{2}$, then $\frac{1}{2} < u_2 + u_3$, so $\sum |u_i| \geq -u_1 + u_2 + u_3 > 1$, a contradiction. Let g be the functional defined by the element $(1, 0, 0)$. We have $\inf g(D) = -\frac{1}{2} > -1$. We will show that for any ball $B' = x + \lambda B$ containing D, we have $\inf g(B') \equiv g(x) - \lambda \leq -1$, which implies that D is not a Mazur set. Indeed, if $B' = x + \lambda B \supset D$, then both $(-\frac{1}{2}, 0, \frac{1}{2})$ and $(-\frac{1}{2}, \frac{1}{2}, 0)$ are in B', which is equivalent to

$$|-\tfrac{1}{2} - x_1| + |x_2| + |\tfrac{1}{2} - x_3| \leq \lambda \quad \text{and} \quad |-\tfrac{1}{2} - x_1| + |\tfrac{1}{2} - x_2| + |x_3| \leq \lambda.$$

It follows that

$$\tfrac{1}{2} + x_1 + x_2 + \tfrac{1}{2} - x_3 \leq \lambda \quad \text{and} \quad \tfrac{1}{2} + x_1 + \tfrac{1}{2} - x_2 + x_3 \leq \lambda.$$

Adding these two inequalities, transposing and dividing by 2 yields $g(x) - \lambda \equiv x_1 - \lambda \leq -1$, which completes the proof. Note that, according Proposition 3.9, any hyperplane in the space ℓ_1^n does not contain Mazur sets with nonempty relative interior, while the dual ball $B_{\ell_\infty^n}$ contains weak* denting points.

3. Spaces with Only Trivial Mazur Sets

The purpose of this section is to show that the above example is only a particular case of a more general situation, as the next proposition shows. Three useful lemmas, the first of them stated without proof, are in order before proving our next result. We include the proof of the third lemma for the sake of completeness.

Lemma 3.1. *If $T\colon X \to Y$ is a linear isometry and $y \in Y$ is fixed, then $C \in \mathcal{P}$ in X if and only if $(y + T(C)) \in \mathcal{P}$ in Y.*

In particular, if C is a Mazur set, then so is $\psi(C)$, for every linear onto isometry ψ of X and also, for every $x \in X$ and $\lambda > 0$, the homothetic image $x + \lambda C$ of C is again a Mazur set.

Lemma 3.2. *If $K_f = \ker f \cap B \notin \mathcal{M}$, then, for every $C \in \mathcal{M}$, the (relative) interior of $C_f = \{x \in C : f(x) = \sup f(C)\}$ is empty.*

Proof. By a homothetic transformation, we can assume that $\sup f(C) = 0$ and that the relative interior of C_f contains K_f. By hypothesis, there exists a point x with $f(x) > 0$ such that every ball containing K_f contains x. However, then the same is true of every ball containing C_f. □

Lemma 3.3 [4]. *Given two Mazur sets C and D, the set $C \hat{+} D = \overline{C + D}$ is always a Mazur set.*

Proof. Let C and D be two Mazur subsets of a Banach space X. Consider a functional $f \in X^*$ and $\lambda \in \mathbb{R}$ such that $\sup f(C \hat{+} D) < \lambda$. Denote $\alpha = \sup f(C)$ and $\beta = \sup f(D)$. Clearly, $\sup f(C \hat{+} D) = \sup f(C) + \sup f(D)$ and so $\alpha + \beta < \lambda$. Therefore, there are two real numbers α' and β' satisfying $\alpha < \alpha'$, $\beta < \beta'$ and $\alpha' + \beta' < \lambda$. Now, since C and D are Mazur sets, there are two closed balls B_1 and B_2 such that $C \subset B_1$ and $D \subset B_2$ satisfying $\sup f(B_1) < \alpha'$ and $\sup f(B_2) < \beta'$. The sum of the two balls B_1 and B_2 is again a ball B_3 that obviously contains $C \hat{+} D$ and satisfies

$$\sup f(B_3) = \sup f(B_1) + \sup f(B_2) < \alpha' + \beta' < \lambda. \qquad \square$$

Proposition 3.4. *The only Mazur sets in ℓ_1^3 are points and balls.*

Proof. If $C \subset \ell_1^3$ is a Mazur set, it is also an intersection of balls. Then, according to Proposition 4.1 in [4], we know that $C = \bigcap_{i=1}^4 f_i^{-1}([a_i, b_i])$ where $a_i \leq b_i, i = 1, \ldots, 4$, and

$$f_1(x_1, x_2, x_3) = x_1 + x_2 + x_3, \qquad f_2(x_1, x_2, x_3) = -x_1 + x_2 + x_3,$$
$$f_3(x_1, x_2, x_3) = -x_1 - x_2 + x_3, \qquad f_4(x_1, x_2, x_3) = x_1 - x_2 + x_3.$$

We assume that C is a Mazur set which is neither a point nor a closed ball, which leads us to a contradiction. To that end, we consider three different cases.

Case 1: *C is a three-dimensional set*. As a first step, we prove that for each of the coordinate functionals $g_j(x) = x_j$, $j = 1, 2, 3$, each of the sets $C_j^+ = \{x \in C : g_j(x) = \sup g_j(C)\}$ and $C_j^- = \{x \in C : g_j(x) = \inf g_j(C)\}$ consists of a single point. We write the argument for C_3^+, since other cases are analogous. Lemma 3.2 ensures that C_3^+ cannot have dimension two. Indeed, the disk $\ker g_3 \cap B$ is not an intersection of balls since g_3 is not a semi-denting point of the unit ball of $(\mathbb{R}^3, \|\cdot\|_\infty)$ (actually, g_3 is not even an extreme point). Consequently, C_3^+ is either a point or a segment. Supposing it were the latter, consider the set $D = \psi(C)$, where ψ is a $\pi/2$ rotation with respect to the z axis. By Lemma 3.1, we know that $D \in \mathcal{P}$ and, by Lemma 3.3, also that $C + D \in \mathcal{P}$. However, the set $\{x \in C + D\colon \sup g_3(x) = \sup g_3(C + D)\}$ contains the set $C_3^+ + D_3^+$, which has nonempty (relative) interior. Again, Lemma 3.2 implies that $C + D$ is not an intersection of balls, which is a contradiction, so C_3^+ must be a point.

The second step is to prove that $C_3^+ = \bigcap_{i=1}^4 f_i^{-1}(b_i)$, $C_1^+ = f_1^{-1}(b_1) \cap f_2^{-1}(a_2) \cap f_3^{-1}(a_3) \cap f_4^{-1}(b_4)$, $C_2^+ = f_1^{-1}(b_1) \cap f_2^{-1}(b_2) \cap f_3^{-1}(a_3) \cap f_4^{-1}(a_4)$ and $C_3^- = \bigcap_{i=1}^4 f_i^{-1}(a_i)$. Since all the cases can be proved in a similar way, we only prove the first of them. First, since C_3^+ is an extreme point of $\bigcap_{i=1}^4 f_i^{-1}([a_i, b_i])$, it is the intersection of at least three hyperplanes from $\{f_i^{-1}(a_i), f_i^{-1}(b_i), i = 1, \ldots, 4\}$ (and at most four of them).

Claim 1. *It is not possible that $C_3^+ \in f_1^{-1}(a_1) \cap f_3^{-1}(a_3)$ nor that $C_3^+ \in f_2^{-1}(a_2) \cap f_4^{-1}(a_4)$.*

Indeed, assume that $C_3^+ \in f_1^{-1}(a_1) \cap f_3^{-1}(a_3)$. Since $x_1 + x_2 + x_3 \geq a_1$ and $-x_1 - x_2 + x_3 \geq a_3$ for every $x = (x_1, x_2, x_3) \in C$, adding these two inequalities, we obtain

$2x_3 \geq a_1 + a_3$, with equality if $\{x\} = C_3^+$. Thus

$$g_3(C_3^+) \equiv \sup g_3(C) \geq \inf g_3(C) \geq (a_1 + a_3)/2 = g_3(C_3^+),$$

which implies that C is not a three-dimensional set, since it would be contained in the hyperplane $g_3(x) = g_3(C_3^+)$. The proof that $C_3^+ \notin f_2^{-1}(a_2) \cap f_4^{-1}(a_4)$ is analogous.

Claim 2. *C_3^+ is the intersection of three hyperplanes from the set $\{f_i^{-1}(b_i), i = 1, \ldots, 4\}$.*

To prove the claim, we just need to show that $f_i(C_3^+) > a_i$, for every $i = 1, \ldots, 4$. Suppose that this is not so and, for instance, $f_1(C_3^+) = a_1$. The following argument can be easily followed by sketching the plane H defined by $\{x\colon\ g_3(x) = g_3(C_3^+)\}$. Let $L_i^+ = H \cap f_i^{-1}(b_i)$ and $L_i^- = H \cap f_i^{-1}(a_i)$. Since C is a three-dimensional set, $a_i < b_i$ for every $i = 1, \ldots, 4$ and thus L_1^-, L_1^+, L_4^- and L_4^+ define a rectangle R with C_3^+ contained in one of its sides (precisely, in the side of R which is contained in L_1^-). Consider now the set $R' = R \cap f_3^{-1}([a_3, b_3])$. Claim 1 stated that $f_3(C_3^+) > a_3$ and this implies that R' is again a rectangle. Indeed, by looking at the definition of the f_i's, one sees that the line L_3^-—which is the intersection of H and $f_3^{-1}(a_3)$—and the line L_3^+ are parallel to the lines L_1^- and L_1^+. Finally, $R'' = R' \cap f_2^{-1}([a_2, b_2])$ is either a rectangle or a segment but it cannot be a single point, which is a contradiction.

We assume now that the three hyperplanes whose intersection is proved in Claim 2 to be the point C_3^+ are $\{f_i^{-1}(b_i), i = 1, 2, 3\}$. We just need to show that $f_4(C_3^+) = b_4$. To do that, we continue in much the same way as in Claim 2. The intersection $S = H \cap f_1^{-1}([a_1, b_1]) \cap f_2^{-1}([a_2, b_2]) \cap f_3^{-1}([a_3, b_3])$ is a segment (otherwise, $H \cap f_i^{-1}([a_i, b_i])$ cannot be a point). Hence $S \cap f_4^{-1}([a_4, b_4])$ reduces to a point only if $f_4(C_3^+) = b_4$.

In the third step, by a translation, we may assume that the point C_3^+ is $(0, 0, 1)$. As a consequence, $b_i = 1$ for every $i = 1, \ldots, 4$. Moreover, by a suitable homothety relative to the point $(0, 0, 1)$, we may assume that $C_1^+ = f_1^{-1}(1) \cap f_2^{-1}(a_2) \cap f_3^{-1}(a_3) \cap f_4^{-1}(1) = (1, 0, 0)$. This implies that $a_2 = a_3 = -1$. Now $C_2^+ = f_1^{-1}(1) \cap f_2^{-1}(1) \cap f_3^{-1}(a_3) \cap f_4^{-1}(a_4) = f_1^{-1}(1) \cap f_2^{-1}(1) \cap f_3^{-1}(-1) = (0, 1, 0)$ and so $a_4 = -1$. Reasoning with C_3^- in the same form, we obtain that $a_4 = -1$. In other words, C is the unit ball, which is a contradiction.

Case 2: *C is a two-dimensional set*. Let f be a norm-one functional and let α be a real number such that $C \subset f^{-1}(\alpha)$. By Proposition 5.1 of [4], if C is a Mazur set, then the vector sum of C with the unit ball must be a three-dimensional Mazur set. If we prove that $C + B$ is not a ball, then we have a contradiction. Clearly, $\inf f(C + B) = \alpha - 1$ and $\sup f(C + B) = \alpha + 1$. This implies that the only possibility is that $C + B$ is a ball of radius 1. However, since C is a two-dimensional set, there is a norm-one functional g satisfying $\inf g(C) < \sup g(C)$. Then

$$\sup g(C + B) - \inf g(C + B) = (\sup g(C) + 1) - (\inf g(C) - 1) > 2,$$

which implies that $C + B$ cannot have radius 1. Consequently, $C + B$ is not a ball and we have found a contradiction.

Case 3: *C is a one-dimensional set*. Since C is an interval, by a homothetic transformation we may assume that it has the form $C = [-x, x]$ for some x of norm one. Writing $x = (x_1, x_2, x_3)$, at least one of the components—say x_1—is nonzero. Define $y = (-x_1, x_2, x_3)$ and let $D = [-y, y]$. Since D is an isometric copy of C, it is a Mazur set and hence, by Lemma 3.3, $C + D$ is a Mazur set. However, since C and D are linearly independent, their sum is two-dimensional, contradicting Case 2. □

An argument somewhat similar to the one employed in Case 2 can be used to obtain a more general result in this direction.

Proposition 3.5. *Let X be a Banach space in which every Mazur set which has nonempty interior is a ball. Then the only Mazur sets in X are points and balls.*

Proof. The idea of the proof is fairly simple: if C is a Mazur set which is neither a point nor a ball, then $C + B$ is a Mazur set with nonempty interior which is not a ball, contradicting the hypothesis. The only thing to be explained is why $C + B$ is not a ball. In Case 2, since C had dimension two, we knew of the existence of two functionals f and g satisfying $\sup f(C) - \inf f(C) = 0$ and $\sup g(C) - \inf g(C) > 0$ which led to a contradiction with the fact that $C + B$ was a ball. However, in the general case, how do we ensure the existence of such a pair of functionals? We simply cannot: if D is a convex set of constant width λ, then $\sup f(D) - \inf f(D) = \lambda$ for every norm-one functional f. We try to avoid this difficulty by using the equivalence of the following two facts:

(i) The set C is a ball or a point.
(ii) There is $\lambda > 0$ such that $C + \lambda B$ is a ball.

Obviously, we only need to prove that (ii) implies (i). To this end, given any set D and any $\lambda > 0$, denote $D \sim \lambda B = \{x \in D : \text{dist}(x, X \setminus D) \geq \lambda\}$. On the one hand, if D is a ball of radius μ and $\lambda \geq \mu$, then $D \sim \lambda B$ is a ball of radius $\mu - \lambda$, if $\lambda > \mu$, and a single point when $\lambda = \mu$. On the other hand, $(C + \lambda B) \sim \lambda B = C$. Indeed, it is clear that $C \subset (C + \lambda B) \sim \lambda B$. To prove the reverse inclusion suppose, on the contrary, that there is a point $x \in (C + \lambda B) \sim \lambda B$ which is not in C. Consider a norm-one functional f such that $\sup f(C) < f(x)$. Then

$$\sup f(C + \lambda B) = \sup f(C) + \lambda < f(x) + \lambda \leq \sup f(C + \lambda B)$$

(the last inequality due to the fact $x + \lambda B \subset (C + \lambda B)$), which is a contradiction. □

In Section 2 of [4], we proved that when a normed space has the binary intersection property then every nonempty intersection of closed balls $C = \bigcap_i B_i$ satisfies $\bigcap_i B_i + \lambda B = \bigcap_i (B_i + \lambda B)$ for every $\lambda > 0$. However, in the general case, the question of whether $C + \lambda B \in \mathcal{M}$ whenever $C \in \mathcal{M}$ remains open. Since adding a ball λB to the convex set C is, in a sense, the opposite of performing the "subtraction" $C \sim \lambda B$, it is natural to ask whether $C \sim \lambda B \in \mathcal{M}$ whenever $C \in \mathcal{M}$ and $C \sim \lambda B$ is nonempty. The next proposition, which is a bit surprising in view of the above-mentioned result from [4], shows that this is indeed always the case.

Proposition 3.6. *In every normed space, every intersection of closed balls $C = \bigcap B_i$ with diameter $d > 0$ satisfies $\bigcap_i B_i \sim \lambda B = \bigcap_i (B_i \sim \lambda B)$ for every $0 < \lambda < d/2$. Consequently, $C \sim \lambda B$ is also in $\mathcal{M}$.*

Proof. For each i, let $r_i > 0$ be the radius of the ball B_i. Our hypothesis about λ and d guarantees that each $B_i \sim \lambda B$ is a ball with the same center as B_i and radius $r_i - \lambda$. To show that $\bigcap_i B_i \sim \lambda B \subset \bigcap_i (B_i \sim \lambda B)$, suppose that $x \in B_i$ for each i and $\operatorname{dist}(x, X \backslash C) \geq \lambda$. Since $X \backslash B_i \subset X \backslash C$, we have $\operatorname{dist}(x, X \backslash B_i) \geq \lambda$. Thus, $x \in B_i \sim \lambda B$ for all i. For the reverse inclusion, if $x \in \bigcap_i (B_i \sim \lambda B)$, then $x \in B_i$ for all i, so $x \in C$; moreover, for any $y \in X \backslash C$, we have $y \in X \backslash B_i$ for some i, hence $\|x - y\| \geq \lambda$ and therefore $x \in C \sim \lambda B$. □

Lemma 3.7. *Let $X = (\ell_1(I), \|\cdot\|_1)$, and let $\{g_i\}$ be the usual coordinate functionals. If there is $i \in I$ such that the ball B' contains the disk $y + \lambda K_{g_i}$, then B' also contains the ball $y + \lambda B$. More generally, this same assertion is valid for any set which is an intersection of balls.*

Proof. Since the statement is invariant under a homothetic transformation, we may assume that B' is the unit ball B. Consequently, we want to prove that if, for some fixed i_0 and $y \in \ell_1(I)$ and $\lambda > 0$, we know that $\|y + \lambda x\| \leq 1$ whenever $\|x\| \leq 1$ and $x_{i_0} = 0$, it then follows that $\|y + \lambda u\| \leq 1$ whenever $\|u\| \leq 1$. Fix $i_1 \neq i_0$. Given u with $\|u\| \leq 1$, there is nothing to prove if $u_{i_0} = 0$ so suppose first that $u_{i_0} > 0$ and consider the two cases: (I) $y_{i_1} + \lambda u_{i_1} \geq 0$ and (II) $y_{i_1} + \lambda u_{i_1} < 0$. In case (I), define $x \in \ell_1(I)$ by $x_i = u_i$, $i \neq i_0, i_1$, while $x_{i_0} = 0$ and $x_{i_1} = u_{i_1} + u_{i_0}$. Then $\|x\| \leq 1$ and $x_{i_0} = 0$, hence

$$1 \geq \|y + \lambda x\| = |y_{i_1} + \lambda(u_{i_1} + u_{i_0})| + \sum_{i \neq i_1, i_0} |y_i + \lambda u_i| + |y_{i_0}| \geq \|y + \lambda u\|,$$

which was to be shown. In case (II), define $x \in \ell_1(I)$ by $x_i = u_i$, $i \neq i_1, i_0$, while $x_{i_0} = 0$ and $x_{i_1} = u_{i_1} - u_{i_0}$. Then $\|x\| \leq \|u\| \leq 1$, while $x_{i_0} = 0$, hence

$$\begin{aligned} 1 \geq \|y + \lambda x\| &= \sum_{i \neq i_1, i_0} |y_i + \lambda u_i| + |y_{i_0}| + |y_{i_1} + \lambda(u_{i_1} - u_{i_0})| \\ &= \sum_{i \neq i_1, i_0} |y_i + \lambda u_i| + |y_{i_1} + \lambda u_{i_1}| + \lambda u_{i_0} + |y_{i_0}| \\ &\geq \|y + \lambda u\|. \end{aligned}$$

An analogous argument can be used in case $u_{i_0} < 0$.

Finally, if C contains the disk $y + \lambda K_{g_i}$ and C is an intersection of balls, then by the foregoing result, each of these balls contains the ball $y + \lambda B$, hence so does C. □

Lemma 3.8. *Let C be a closed, convex and bounded subset of $\ell_1(I)$. Suppose that there is a coordinate functional $g_i(x) = x_i$, $i \in I$, such that either the set $C_i^+ = \{x \in C : g_i(x) = \sup g_i(C)\}$ or C_i^- contains a segment. Then C is not a Mazur set.*

Proof. Let S be a segment of length λ which is contained in C_i^+ and let H be the hyperplane $\{x \in \ell_1(I) : g_i(x) = \sup g_i(C) \equiv \alpha\}$. If D is a ball with center y and radius

$\mu > 0$ which contains S, then $D \cap H$ is a disk in H with center $z = y + (\alpha - y_i)e_i$ (where e_i is the ith basis vector) and radius $r = \mu - |\alpha - y_i|$, which necessarily satisfies $r \geq \lambda/2$. From Lemma 3.7 it follows that D contains the ball with center $z \in H$ and radius r. This implies that D always has nonempty intersection with the hyperplane $H' = \{x \in \ell_1(I) : g_i(x) = \sup g_i(C) + \lambda/2\}$ and, therefore, that C is not a Mazur set. □

Proposition 3.9. *The only Mazur sets in ℓ_1^n for every $n \geq 3$ are points and balls.*

Proof. The case $n = 3$ was already proved in Proposition 3.4 and the general case will be proved by induction. Suppose that $n > 3$ and let $\{e_1, e_2, \ldots, e_n\}$ be the canonical basis, let $\{g_1, g_2, \ldots, g_n\}$ be the coordinate functionals ($g_k(x) = x_k$) and, finally, for any $\alpha \in \mathbb{R}$ let $X_k(\alpha) = g_k^{-1}(\alpha)$. By Proposition 3.5, we only need to prove that *if C is a Mazur set with nonempty interior, then C is a ball.*

Step 1: *For every $k = 1, \ldots, n$ and every $\alpha \in \mathbb{R}$, $C \cap X_k(\alpha)$ is a Mazur set in $X_k(\alpha)$, provided it is nonempty.*

If $\alpha = \inf g_k(C)$ or $\alpha = \sup g_k(C)$, then from Lemma 3.8 we conclude that $C \cap X_k(\alpha)$ is a point. (Also, $C \cap X_k(\alpha)$ is empty when $\alpha \notin [\inf g_k(C), \sup g_k(C)]$.) Therefore, we can consider the case $\alpha \in (\inf g_k(C), \sup g_k(C))$, where the relative interior of $C \cap X_k(\alpha)$ is nonempty. Moreover, we lose no generality in assuming that $k = 1$.

Since the statement of Step 1 is invariant under a translation, we prove it for the case $X_1(0) \equiv X_1$, assuming that the relative interior of $X_1 \cap C$ is nonempty. Let $\beta \in \mathbb{R}$ and let $h \in X_1^*$ be a functional satisfying $\inf h(X_1 \cap C) > \beta$. We want to find an X_1-ball D containing $C \cap X_1$ such that $\inf h(D) > \beta$. The Hahn–Banach theorem ensures the existence of an extension $\widehat{h} \in (\ell_1^n)^*$ satisfying $\inf \widehat{h}(C) = \inf h(X_1 \cap C) > \beta$. Since C is a Mazur set, there is a ball B' containing C such that $\widehat{h}(B') > \beta$. Since X_1 is a coordinate hyperplane, $B' \cap X_1$ is the $(n-1)$-dimensional ℓ_1-ball we were looking for.

Step 2: *If $n > 3$ and ℓ_1^{n-1} satisfies the statement of the proposition, then so does ℓ_1^n.*

We know that for each $\alpha \in [\inf g_1(C), \sup g_1(C)]$, the set $B_\alpha = C \cap X_1(\alpha)$ is a Mazur set and, by our hypothesis, it is either a ball or a point. We may assume, for instance, that $B_0 = C \cap X_1$ is the ball of the family $\{B_\alpha\}_\alpha$ which has the greatest radius. We assume further that B_0 is the unit ball in X_1. (Recall that the problem is invariant under homothetic transformations and translations). By the second assertion in Lemma 3.7, we can assume that C contains the unit ball of ℓ_1^n. The proof will be accomplished by showing that C actually equals this unit ball. We divide the argument into three steps.

Step 2.1: *For every $k = 2, \ldots, n$, the set $C \cap X_k(0)$ is the unit ball in $X_k(0)$ and it is the ball of the family $\{C \cap X_k(\alpha)\}_\alpha$ which has the greatest radius.* First, note that from Lemma 3.7, we know that if $C \cap X_i(\alpha)$ has radius r, then C contains an ℓ_1^n-ball of radius $\geq r$. This implies that if there are $i \in \{1, \ldots, n\}$ and α such that $C \cap X_i(\alpha)$ has radius r, then, for every $j \in \{1, \ldots, n\}$, there is α_j such that $C \cap X_j(\alpha_j)$ has radius $\geq r$. Finally, since C contains the unit ball, each of the sets $C \cap X_k(0)$ contains an $(n-1)$-dimensional ball of radius 1. Suppose that for some k, α, the slice $C \cap X_k(\alpha)$ contains a ball of radius $r > 1$. Using Lemma 3.7 again, C must contain an ℓ_1^n-ball of radius r, contradicting the choice of $B_0 = C \cap X_1(0)$ as having maximum radius 1.

Step 2.2: *For every* $i = 1, \ldots, n$ *we have* $[\inf g_i(C), \sup g_i(C)] = [-1, 1]$. To prove this fact, it is enough to show that $X_i(1) \cap C = \{e_i\}$ and $X_i(-1) \cap C = \{-e_i\}$ for every $i = 1, \ldots, n$. We see, for instance, that $X_1(1) \cap C = \{e_1\}$. It is clear, on the one hand, that $e_i \in X_i(1) \cap C$. On the other hand, $X_1(1) \cap C$ is a ball in $X_1(1)$ whose intersection with each coordinate hyperplane $X_i(0)$, $i = 2, \ldots, n$, reduces to the point e_1. To see that $X_1(1) \cap C = \{e_1\}$, we use the fact that $D \equiv X_1(1) \cap C$ is an $\ell_1^{(n-1)}$-ball. If it were not equal to $\{e_1\}$, it would have radius $r > 0$ and center $x = e_1 + x_2 e_2 + \cdots + x_n e_n$ in $X_1(1)$. We claim that at least one of the intersections $D \cap X_k(0)$ would contain more than the point e_1. This would obviously be the case if all the x_k were 0. If some $x_k \neq 0$, then the point $e_1 + x_k e_k \in D \cap X_j(0)$ for every $j \notin \{1, k\}$ and is not equal to e_1.

Step 2.3. Suppose that there is $x = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n \in C$ with $\|x\|_1 > 1$. Choose an index $i \in \{1, \ldots, n\}$ such that $\alpha_i \neq 0$. We may assume, for instance, that $\alpha_1 > 0$ since the argument for the other cases is entirely similar. Consider the ball $B_{\alpha_1} = C \cap X_1(\alpha_1)$ which has center y and radius r. Since C contains the unit ball, then B_{α_1} contains the ℓ_1^{n-1}-ball with center $\alpha_1 e_1$ and radius $1 - \alpha_1$. Now, $x \in X_1(\alpha_1)$ and the estimate

$$\|x - \alpha_1 e_1\|_1 = |\alpha_2| + \cdots + |\alpha_n| = \|x\|_1 - \alpha_1 > 1 - \alpha_1$$

implies that $r > 1 - \alpha_1$. Now, Lemma 3.7 ensures that C actually contains the ℓ_1^n-ball with center y and radius r. As a consequence,

$$\sup g_1(C) \geq g_1(y) + r = \alpha_1 + r > 1,$$

which is a contradiction. □

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