Discrete Comput Geom 28:107–114 (2002)
DOI: 10.1007/s00454-001-0080-5

Discrete & Computational
Geometry

Improved Linear Programming Bounds for Antipodal Spherical Codes

Kurt M. Anstreicher

Department of Management Sciences, University of Iowa,
Iowa City, IA 52242, USA
kurt-anstreicher@uiowa.edu

Abstract. Let $S \subset [-1, 1)$. A finite set $\mathcal{C} = \{x_i\}_{i=1}^M \subset \Re^n$ is called a *spherical S-code* if $\|x_i\| = 1$ for each i, and $x_i^T x_j \in S$, $i \neq j$. For $S = [-1, 0.5]$ maximizing $M = |\mathcal{C}|$ is commonly referred to as the *kissing number* problem. A well-known technique based on harmonic analysis and linear programming can be used to bound M. We consider a modification of the bounding procedure that is applicable to *antipodal* codes; that is, codes where $x \in \mathcal{C} \Rightarrow -x \in \mathcal{C}$. Such codes correspond to packings of lines in the unit sphere, and include all codes obtained as the collection of minimal vectors in a lattice. We obtain improvements in upper bounds for kissing numbers attainable by antipodal codes in dimensions $16 \leq n \leq 23$. We also show that for $n = 4$, 6 and 7 the antipodal codes with maximal kissing numbers are essentially unique, and correspond to the minimal vectors in the laminated lattices Λ_n.

1. Introduction

Let $S \subset [-1, 1)$. A finite set $\mathcal{C} = \{x_i\}_{i=1}^M \subset \Re^n$ is called a *spherical S-code* if $\|x_i\| = 1$ for each i, and $x_i^T x_j \in S$, $i \neq j$. When $S = [-1, \cos\theta]$ the points of $\mathcal{C}$ are the centers of nonoverlapping *spherical caps* of angular diameter θ, and if $\theta = \pi/3$ the points of $\mathcal{C}$ are the centers of nonoverlapping spheres of radius $\frac{1}{2}$, all of which touch the sphere of radius $\frac{1}{2}$ centered at the origin. Maximizing the number of such spheres is commonly referred to as the *kissing number* problem in $\Re^n$.

There is a very large literature concerning spherical codes, and the related *Tammes problem*: find a $[-1, \cos\theta]$-code of given cardinality M that maximizes θ (see Chapters 1 and 3 of [5] and references therein). In addition to their purely geometrical interest these problems have a number of significant applications, for example to the construction of constant-energy codes for a Gaussian communication channel [5, Section 3.1]. A fundamental problem connected with spherical codes is to bound $M = |\mathcal{C}|$ for a given S.

An approach based on harmonic analysis and linear programming [9], [12] allows for the computation of explicit bounds on M for fixed n, and also asymptotic bounds on the sizes of spherical codes and related sphere packings for large n. In Chapters 13 and 14 of [5] this approach is applied with $S = [-1, 0.5]$ to obtain bounds on M for $n = 3, \ldots, 24$, and to give precise characterizations of spherical codes that solve the kissing number problem in dimensions 8 and 24. For recent results concerning spherical codes, the Tammes problem and the linear programming bounds see pp. xxiii–xxv of [5].

In this paper we consider a modification of the linear programming bound that is applicable when $\mathcal{C}$ is *antipodal*; that is, $x \in \mathcal{C} \Rightarrow -x \in \mathcal{C}$. Antipodal codes include all codes obtained as the set of minimal vectors in a lattice, so the antipodal bound applies to the size of any such *lattice code* in $\Re^n$. An antipodal code can also be viewed as a packing of lines in the unit sphere, which is the lowest-dimensional case of the packings of subspaces, or *Grassmannian packings*, considered in [4]. Bounds for antipodal codes, or packings of lines, have been previously considered in [3], [7], and [11].

In the next section we describe the linear programming bound of [9], and a variant that is valid for antipodal codes. In Section 3 the antipodal bound is applied in the case of $S = [-1, 0.5]$ to obtain bounds on the kissing number attainable by antipodal codes in dimensions $n = 3, \ldots, 24$. (For all such n *except* 13, 14 and 15 the highest known kissing number corresponds to an antipodal code.) We obtain improvements in the best known upper bound on M in dimensions $16 \le n \le 23$. In Section 4 we use the solutions of the linear programming problems to obtain additional results for certain dimensions. In particular we prove that for $n = 4$, 6 and 7 the antipodal codes that attain the maximal kissing number are essentially unique, and correspond to the minimal vectors in the laminated lattices Λ_n.

2. Linear Programming Bounds

Let $\mathcal{C} = \{x_i\}_{i=1}^M$ be a spherical S-code in $\Re^n$, $n \ge 3$. In this section we describe a well-known linear programming bound for the size M of such a code. The *distance distribution* of the code is the function $\alpha(\cdot)\colon [-1, 1] \to \Re$ defined as

$$\alpha(s) = \frac{|\{(i, j)\colon x_i^T x_j = s\}|}{M}.$$

It is then easy to see that

$$\alpha(1) = 1, \tag{1a}$$

$$\sum_{s \in S} \alpha(s) = M - 1, \tag{1b}$$

$$\alpha(s) \ge 0, \qquad \alpha \in S. \tag{1c}$$

Let $\Phi_k(\cdot)$, $k = 0, 1, \ldots$, denote the Gegenbauer, or ultraspherical, polynomials

$$\Phi_k(t) = \frac{P_k^{(\beta,\beta)}(t)}{\binom{k+\beta}{k}}, \tag{2}$$

where $P_k^{(\beta,\beta)}$ is the Jacobi polynomial with $\beta = (n-3)/2$ [1]. The normalization in (2) is chosen so that $\Phi_k(1) = 1$ for all k. Using techniques from harmonic analysis it can be shown [9], [5, Chapters 9 and 13] that

$$1 + \sum_{s \in S} \alpha(s)\Phi_k(s) \geq 0, \qquad k = 1, 2, \ldots. \tag{3}$$

By combining (2) and (3) a bound on M can be obtained via the linear programming problem

$$\begin{aligned} \text{(LP)} \quad & \max \sum_{s \in S} \alpha(s) \\ & \text{s.t.} \ \sum_{s \in S} \alpha(s)\Phi_k(s) \geq -1, \qquad k = 1, 2, \ldots, \\ & \alpha(s) \geq 0, \qquad s \in S. \end{aligned}$$

Note that (LP) has both an infinite number of variables and constraints. In practice a bound on M can be obtained by working with a finite number of constraints $k = 1, \ldots, K$, and using a feasible solution to the dual problem to bound the optimal value of (LP).

Our interest here is in modifying the problem (LP) to obtain an improved bound when $\mathcal{C}$ is antipodal. In this case it is obvious that the distance distribution satisfies $\alpha(s) = \alpha(-s)$, $s \in [-1, 1]$. Since the polynomials $\Phi_k(\cdot)$ are odd for k odd, it follows immediately that the constraints (3) are satisfied with equality for all odd k. Let $S_+ = S \cap [0, 1]$ and $S_{++} = S \cap (0, 1]$. Since $\Phi_k(\cdot)$ are even for k even, the constraints (3) for even k can be written as

$$2 + \alpha(0)\Phi_{2k}(0) + 2 \sum_{s \in S_{++}} \alpha(s)\Phi_{2k}(s) \geq 0, \qquad k = 1, 2, \ldots.$$

A bound for M, the size of the code, can then be based on the linear programming problem

$$\begin{aligned} \text{(LP+)} \quad & \max \alpha(0) + 2 \sum_{s \in S_{++}} \alpha(s) \\ & \text{s.t.} \ \alpha(0)\Phi_{2k}(0) + 2 \sum_{s \in S_{++}} \alpha(s)\Phi_{2k}(s) \geq -2, \qquad k = 1, 2, \ldots, K, \\ & \alpha(s) \geq 0, \qquad s \in S_+. \end{aligned}$$

(If $0 \notin S$ the variable $\alpha(0)$ is omitted in (LP+).) The dual of (LP+) is the problem

$$\begin{aligned} \text{(LD+)} \quad & \min 2 \sum_{k=1}^{K} f_{2k} \\ & \text{s.t.} \ \sum_{k=1}^{K} f_{2k}\Phi_{2k}(s) \leq -1, \qquad s \in S_+, \\ & f_{2k} \geq 0, \qquad k = 1, \ldots, K. \end{aligned}$$

In practice it may be impossible to solve (LD+) exactly due to the infinite number of constraints. However, by solving an approximation of (LD+) using a finite set of points $s_1, s_2, \ldots, s_N$ we can obtain values f_{2k}, $k = 1, \ldots, K$, so that

$$1 + \sum_{k=1}^{K} f_{2k}\Phi_{2k}(s) \leq \varepsilon, \qquad s \in S_+, \tag{4}$$

where $0 \leq \varepsilon < 1$. A bound on the size of the code is then given as follows.

Lemma 1. *Let f_{2k}, $k = 1, \ldots, K$, be nonnegative numbers satisfying* (4). *If $\mathcal{C}$ is an antipodal spherical code, then $M = |\mathcal{C}| \leq 2 + 2(\sum_{k=1}^{K} f_{2k})/(1 - \varepsilon)$.*

Proof. Since $\mathcal{C}$ is antipodal, the identities (1a) and (1b) imply that $M \leq v(\text{LP+}) + 2$, where $v(\text{LP+})$ denotes the solution objective value in (LP+). By weak duality [6] $v(\text{LP+}) \leq 2(\sum_{k=1}^{K} f_{2k})$, where f_{2k}, $k = 1, \ldots, K$, is feasible in (LD+). However, if f_{2k}, $k = 1, \ldots, K$, are nonnegative and satisfy (4), then $f_{2k}/(1 - \varepsilon)$, $k = 1, \ldots, K$, are feasible in (LD+). □

3. Bounds on Kissing Numbers

We now consider the bound of Lemma 1 applied to the case of $S = [-1, 0.5]$, often referred to as the kissing number problem. In this case $S_+ = [0, 0.5]$. Bounds for this case based on explicit feasible solutions to (LD+) have previously been described [3], but to our knowlege there has been no attempt to solve (LD+) numerically. For $n = 3, 4, \ldots, 24$ we solve the approximation of (LD+) obtained using $K = 6$, and the constraints generated by $\{s_j\}_{j=1}^{2001}$, $s_j = 0.00025(j - 1)$. Let f_{2k}, $k = 1, \ldots, K$, be the solution of this linear programming problem, and let $\Phi(s) = 1 + \sum_{k=1}^{K} f_{2k}\Phi_{2k}(s)$. To obtain the value ε required for the bound in Lemma 1 we use the following simple technique. Let j be such that $\Phi(a_j) \approx 0$, $\Phi'(s_j) > 0$, $\Phi'(s_{j+1}) < 0$. Let $d_2 = \max\{\Phi''(s_j),\ \Phi''(s_{j+1})\}$. Then $\Phi''(s) \leq d_2 < 0$, $s \in [s_j, s_{j+1}]$, assuming that $\Phi''(\cdot)$ is negative and monotonic on this interval, which is easily checked. It follows that for $0 \leq \delta \leq 0.00025$,

$$\Phi(s_j + \delta) \leq \Phi(s_j) + \delta\Phi'(s_j) + \frac{\delta^2}{2}d_2,$$

from which we obtain an upper bound of the form

$$\varepsilon = \Phi(s_j) - \frac{\Phi'(s_j)}{2d_2}.$$

In Table 1 we give bounds on the kissing number attainable by antipodal codes, as well as the original linear programming bounds and highest known kissing numbers, from [5]. The antipodal bounds are rounded down to the next even integer, since M must be even for an antipodal code. For $4 \leq n \leq 15$ the antipodal bounds computed here agree with bounds given by explicit polynomials in [3] and [11]. For $16 \leq n \leq 23$, however, our bounds are better than those of [3] and [11]. For $n = 5$, 10 and 14 the

Table 1. Best known kissing numbers and linear programming bounds.

n	Best known	Lattice?	Original bound	Antipodal bound
3	12	yes	13	12
4	24	yes	25	24
5	40	yes	46	40‡
6	72	yes	82	72
7	126	yes	140	126
8	240	yes	240	240
9	306	no†	380	366
10	500	no†	595	548‡
11	582	no†	915	820
12	840	no†	1,416	1,228
13	1,130	no	2,233	1,866
14	1,582	no	3,492	2,938‡
15	2,564	no	5,431	4,962
16	4,320	yes	8,313	8,158‡
17	5,346	yes	12,215	11,478
18	7,398	yes	17,877	16,122
19	10,668	yes	25,901	22,724
20	17,400	yes	37,974	32,340
21	27,720	yes	56,852	46,878
22	49,896	yes	86,537	70,164
23	93,150	yes	128,096	111,126
24	196,560	yes	196,560	196,560

†Antipodal.
‡See text.

linear programming bounds are integral, and it is shown in [3] that codes attaining these bounds cannot exist. Consequently the bounds may be reduced by 2, and these reduced values are reported in Table 1. We use a similar technique to improve the bound for $n = 16$ in the next section. As described above $K = 6$ was used in the formulation of the problem used to obtain these bounds, but $f_{10} = f_{12} = 0$ in the solution for all n except for $n = 3$.

As can be seen in Table 1 the antipodal bounds are tight for dimensions 3–8 and 24. The tight bounds for dimensions 8 and 24 are to be expected since the original linear programming bounds are tight, and the maximal kissing numbers are attained by lattice codes [5, Chapter 13]. The tight bounds for $3 \le n \le 8$ provide an alternative proof for the known result [13], [14] that the laminated lattices Λ_n have the highest possible kissing numbers for lattices in these dimensions, and also imply that higher kissing numbers, if they exist, can only come from codes that are *not* antipodal. It is known that the maximal kissing number for $n = 3$ is 12 [10].

For some n the solution of (LD+) is particularly well structured, allowing for additional analysis that can either demonstrate that the code attaining the bound is essentially unique, or in fact cannot exist. We pursue this topic in detail in the next section for $n = 4, 6, 7$ and 16.

4. Uniqueness or Nonexistence of Certain Antipodal Codes

In this section we show that:

- For $n = 4$, 6 and 7 the *only* antipodal codes that attain the maximal possible kissing number correspond to orthogonal transformations of the set of minimal vectors of the laminated lattices Λ_n.
- For $n = 16$ there is *no* antipodal code that attains the bound 8160 from (LD+), and therefore this bound can be reduced by 2.

In all cases the analysis uses explicit rational coefficients f_{2k} suggested by the solution of (LD+). For $n \leq 8$ it is known that Λ_n is the unique lattice with maximal density [5, Section 1.5], and, for $4 \leq n \leq 9$, Λ_n is the unique lattice with the highest kissing number [14]. Our method of proving the uniqueness of these codes in dimensions 4, 6 and 7 is similar to that used to prove that for $n = 8$ the minimal vectors from E_8 are the essentially unique code with kissing number 240 [2]; see also Theorem 7 in Section 14.2 of [5]. The fact that in dimensions 4, 6 and 7 the distance distribution for an antipodal code achieving the bound in Table 1 is uniquely determined was previously noted in [3].

For a code $\mathcal{C} = \{x_i\}_{i=1}^M$, let $\alpha_i(s) = |\{j\colon\ x_i^T x_j = s\}|$. A code is called *distance invariant* if $\alpha_i(s)$ is independent of i for all s, and in this case $\alpha_i(s) = \alpha(s)$ for all i and s.

Lemma 2. *Suppose that an antipodal spherical code $\mathcal{C}$ for $n = 4$ and $S = [-1, 0.5]$ has $M = |\mathcal{C}| = 24$. Then $\mathcal{C}$ is distance invariant, and the distance distribution of $\mathcal{C}$ has $\alpha(0) = 6$, $\alpha(\frac{1}{2}) = \alpha(-\frac{1}{2}) = 8$, $\alpha(1) = \alpha(-1) = 1$, $\alpha(s) = 0$, $s \notin \{0, \pm\frac{1}{2}, \pm 1\}$.*

Proof. For $n = 4$ we obtain $f_2 = 6$, $f_4 = 5$ and a bound $2 + 2(f_2 + f_4) = 24$. Let

$$\Phi(s) = 1 + f_2\Phi_2(s) + f_4\Phi_4(s) = 16s^2(s^2 - \tfrac{1}{4}).$$

Then $\Phi(s) \leq 0$ for $s \in S_+$, with roots at 0 and $\frac{1}{2}$. It follows from the complementary slackness property [6] that if $\mathcal{C}$ is an antipodal code with $M = 24$, then the distance distribution for $\mathcal{C}$ must satisfy $\alpha(s) = 0$, $s \notin \{0, \pm\frac{1}{2}, \pm 1\}$, and in addition

$$\begin{aligned} \alpha(0)\Phi_2(0) + 2\alpha(\tfrac{1}{2})\Phi_2(\tfrac{1}{2}) &= -2, \\ \alpha(0)\Phi_4(0) + 2\alpha(\tfrac{1}{2})\Phi_4(\tfrac{1}{2}) &= -2. \end{aligned} \tag{5}$$

The unique solution of (5) is $\alpha(0) = 6$, $\alpha(\frac{1}{2}) = 8$. From (5) and the fact that the code is antipodal, $\mathcal{C}$ is a 5-design in $\Re^4$ [8]. Since $\mathcal{C}$ is also an S-code with $|S| = 4$, Theorem 7.4 of [8] implies that $\mathcal{C}$ is distance invariant. □

Theorem 3. *Suppose that an antipodal spherical code $\mathcal{C}$ for $n = 4$ and $S = [-1, 0.5]$ has $M = |\mathcal{C}| = 24$. Then there is an orthogonal transformation that maps the elements of $\mathcal{C}$ onto the minimal vectors of the lattice $\Lambda_4 = D_4$.*

Proof. Let $\{x_i\}_{i=1}^{24}$ be the elements of $\mathcal{C}$, and define the lattice L consisting of points of the form

$$\sum_{i=1}^{24} \sqrt{2}\, a_i x_i, \qquad a_i \in \mathbf{Z}, \quad i = 1, \ldots, 24.$$

It is then easy to show that L is an even integral lattice. Since L is generated by vectors of squared-norm 2, Witt's theorem [5, Section 4.3] implies that L is a direct sum of root lattices that are isometric with either A_n, $n \geq 1$, or D_n, $n \geq 4$. The only lattice of this form with at least 24 minimal vectors is D_4. □

In dimensions 6 and 7 very similar analysis obtains the following results.

Lemma 4. *Suppose that an antipodal spherical code $\mathcal{C}$ for $n = 6$ and $S = [-1, 0.5]$ has $M = |\mathcal{C}| = 72$. Then $\mathcal{C}$ is distance invariant, and the distance distribution of $\mathcal{C}$ has* $\alpha(0) = 30$, $\alpha(\frac{1}{2}) = \alpha(-\frac{1}{2}) = 20$, $\alpha(1) = \alpha(-1) = 1$, $\alpha(s) = 0$, $s \notin \{0, \pm\frac{1}{2}, \pm 1\}$.

Proof. Similar to the proof of Lemma 2, using $f_2 = 14$, $f_4 = 21$. □

Theorem 5. *Suppose that an antipodal spherical code $\mathcal{C}$ for $n = 6$ and $S = [-1, 0.5]$ has $M = |\mathcal{C}| = 72$. Then there is an orthogonal transformation that maps the elements of $\mathcal{C}$ onto the minimal vectors of the lattice $\Lambda_6 = E_6$.*

Proof. Similar to the proof of Theorem 3, but with the additional root lattice E_6. □

Lemma 6. *Suppose that an antipodal spherical code $\mathcal{C}$ for $n = 7$ and $S = [-1, 0.5]$ has $M = |\mathcal{C}| = 126$. Then $\mathcal{C}$ is distance invariant, and the distance distribution of $\mathcal{C}$ has* $\alpha(0) = 60$, $\alpha(\frac{1}{2}) = \alpha(-\frac{1}{2}) = 32$, $\alpha(1) = \alpha(-1) = 1$, $\alpha(s) = 0$, $s \notin \{0, \pm\frac{1}{2}, \pm 1\}$.

Proof. Similar to the proof of Lemma 2, using $f_2 = \frac{234}{11}$, $f_4 = \frac{448}{11}$. □

Theorem 7. *Suppose that an antipodal spherical code $\mathcal{C}$ for $n = 7$ and $S = [-1, 0.5]$ has $M = |\mathcal{C}| = 126$. Then there is an orthogonal transformation that maps the elements of $\mathcal{C}$ onto the minimal vectors of the lattice $\Lambda_7 = E_7$.*

Proof. Similar to the proof of Theorem 3, but with the additional root lattices E_6 and E_7. □

It is worthwhile to note that the distance distributions characterized in Lemmas 2, 4 and 6 attain the "special bound" for antipodal codes described in Example 8.4 of [8].

Next we give the nonexistence result for $n = 16$. From the solution of (LD+), the polynomial $\Phi(s) = 1 + f_2\Phi_2(s) + f_4\Phi_4(s) + f_6\Phi_6(s) + f_8\Phi_8(s)$ appears to be of the form

$$\Phi(s) = \gamma s^2 (s^2 - \theta)^2 (s^2 - \tfrac{1}{4}),$$

where γ is a scalar and $\sqrt{\theta} \approx 0.185$. Using the fact that the bound from (LD+) is 8160 and $f_6 = 1824$, we compute $\gamma = 495616/85$, $\theta = 3/88$, $f_8 = 20064/13$, $f_4 = 41848/65$, $f_2 = 339/5$. Using the same argument as in the proof of Lemma 2, the solution of (LP+) must satisfy

$$\begin{aligned}
\alpha(0)\Phi_2(0) + 2\alpha(\tfrac{1}{2})\Phi_2(\tfrac{1}{2}) + 2\alpha(\sqrt{\theta})\Phi_2(\sqrt{\theta}) &= -2,\\
\alpha(0)\Phi_4(0) + 2\alpha(\tfrac{1}{2})\Phi_4(\tfrac{1}{2}) + 2\alpha(\sqrt{\theta})\Phi_4(\sqrt{\theta}) &= -2,\\
\alpha(0)\Phi_6(0) + 2\alpha(\tfrac{1}{2})\Phi_6(\tfrac{1}{2}) + 2\alpha(\sqrt{\theta})\Phi_6(\sqrt{\theta}) &= -2,\\
\alpha(0)\Phi_8(0) + 2\alpha(\tfrac{1}{2})\Phi_8(\tfrac{1}{2}) + 2\alpha(\sqrt{\theta})\Phi_8(\sqrt{\theta}) &= -2,
\end{aligned} \tag{6}$$

which has a unique solution $\alpha(0) = 2890/3$, $\alpha(\frac{1}{2}) = 11560/19$, $\alpha(\sqrt{\theta}) = 170368/57$. From (6) and the fact that $\mathcal{C}$ is antipodal, $\mathcal{C}$ is a 9-design in $\Re^{16}$, and is also an S-code with $|S| = 6$. From Theorem 7.4 of [8] $\mathcal{C}$ is distance invariant, so $\alpha(s)$ must be integral for all s. Therefore no antipodal code with $M = 8160$ can exist, and the bound can be reduced to 8158.

Acknowledgement

I am grateful to P. Boyvalenkov, A. Neumaier, and N.J.A. Sloane for helpful comments.

References

1. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*. National Bureau of Standards Applied Mathematics Series, 55. U.S. Department of Commerce, Washington, DC, 1972.
2. E. Bannai and N. J. A. Sloane, Uniqueness of certain spherical codes. *Canad. J. Math.* **33** (1981), 437–449.
3. P. Boyvalenkov, Nonexistence of certain symmetric spherical codes. *Designs Codes Cryptography* **3** (1993), 69–74.
4. J. H. Conway, R. H. Hardin, and N. J. A. Sloane, Packing lines, planes, etc.: packings in Grassmannian space. *Experimental Math.* **5** (1996), 139–159. See also `www.research.att.com/~njas/grass/`.
5. J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, third edition. Springer-Verlag, New York, 1999.
6. G. B. Dantzig, *Linear Programming and Extensions*. Princeton University Press, Princeton, NJ, 1963.
7. P. Delsarte, J.-M. Goethals, and J. J. Seidel, Bounds for systems of lines, and Jacobi polynomials. *Philips Res. Rep.* **30** (1975), 91–105.
8. P. Delsarte, J.-M. Goethals, and J. J. Seidel, Spherical codes and designs. *Geom. Dedicata* **6** (1977), 363–388.
9. G. A. Kabatiansky and V. I. Levenshtein, Bounds for packings on a sphere and in space. *Problems Inform. Transmission* **14** (1978), 1–17.
10. J. Leech, The problem of the thirteen spheres. *Math. Gaz.* **40** (1956), 22–23.
11. V. I. Levenshtein, Bounds for packings in metric spaces and certain applications. *Problemi Kibernet.* **40** (1983), 44–110.
12. V. I. Levenshtein, Universal bounds for codes and designs. In *Handbook of Coding Theory*, Vol. 1 (V. S. Pless and W. C. Huffman, eds.). North-Holland, Amsterdam, 1998, pp. 499–648.
13. C. Musès, The dimensional family approach in (hyper)sphere packing: a typological study of new patterns, structures, and interdimensional functions. *Appl. Math. Comput.* **88** (1997), 1–26.
14. G. L. Watson, The number of minimum points of positive quadratic forms. *Dissertationes Math.* **84** (1971), 1–43.

Received June 25, 2001, *and in revised form August* 18, 2001, *and November* 13, 2001.
Online publication March 27, 2002.