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Kalai's Squeezed 3-Spheres Are Polytopal

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Abstract. In 1988 Kalai [5] extended a construction of Billera and Lee to produce *many* triangulated $(d-1)$-spheres. In fact, in view of the upper bounds on the number of simplicial d-polytopes by Goodman and Pollack [2], [3], he derived that for every dimension $d \geq 5$, *most* of these $(d-1)$-spheres are not polytopal. However, for $d = 4$, this reasoning fails. We can now show that, as already conjectured by Kalai, all of his 3-spheres are in fact polytopal.

We also give a shorter proof for Hebble and Lee's result [4] that the dual graphs of these 4-polytopes are Hamiltonian.

1. Introduction

This paper is about triangulated spheres and the question whether or not the members of a certain family of them are *realizable*, that is, if they arise as boundary complexes of simplicial polytopes. While for all two-dimensional spheres this is true by Steinitz' theorem, already one dimension higher there exist simplicial spheres that cannot be realized in a convex way. The first example for this—the so-called *Brückner sphere*—was found by Grünbaum and Sreedharan in 1967, who realized that a certain simplicial 3-sphere on eight vertices does not represent the combinatorial type of any 4-polytope, contrary to what Brückner originally thought. (See Chapter 5 of [11] for a more thorough discussion and references.)

In 1988 Kalai extended a construction by Billera and Lee, and showed that starting with $d = 5$, there exist many more simplicial $(d-1)$-spheres than simplicial d-polytopes, and that therefore, in a very strong sense, *most* simplicial spheres are not realizable. In contrast, it is the main goal of this paper to show that all of Kalai's 3-spheres *do* arise as boundary complexes of simplicial 4-polytopes.

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In the remainder of this Introduction, we present the context of these constructions, including the known upper (resp. lower) bounds for the numbers of simplicial polytopes (resp. spheres).

The most important invariant of a $(d-1)$-dimensional simplicial sphere $\mathcal{S}$ is its **f-vector** $f(\mathcal{S}) = (f_{-1}, f_0, f_1, \ldots, f_{d-1})$, where $f_i = f_i(\mathcal{S})$ counts the number of i-dimensional faces of $\mathcal{S}$, and $f_{-1} = 1$. In 1971 McMullen [7] conjectured a characterization of the f-vectors of boundary complexes of simplicial d-polytopes in terms of an encoding of $f(\mathcal{S})$, the so-called *g-vector*. First define the **h-vector** $h(\mathcal{S}) = (h_0, h_1, \ldots, h_d)$ of $\mathcal{S}$ by

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1} \qquad \text{for} \quad k = 0, 1, \ldots, d.$$

The h-vector of any simplicial sphere satisfies the **Dehn–Sommerville equations** $h_k = h_{d-k}$ for $k = 0, 1, \ldots, \lfloor d/2 \rfloor$. Now the **g-vector** of $\mathcal{S}$ is $g(\mathcal{S}) = (g_0, g_1, \ldots, g_{\lfloor d/2 \rfloor})$, where $g_0 := h_0 = 1$ and

$$g_k := h_k - h_{k-1} \qquad \text{for} \quad k = 1, 2, \ldots, \lfloor d/2 \rfloor.$$

We say that $g(\mathcal{S})$ forms an **M-sequence** if $g_0 = 1$ and $g_{k-1} \geq \partial^k(g_k)$ for $k = 1, \ldots, \lfloor d/2 \rfloor$, where

$$\partial^k(g_k) = \binom{a_k - 1}{k-1} + \binom{a_{k-1} - 1}{k-2} + \cdots + \binom{a_2 - 1}{1} + \binom{a_1 - 1}{0},$$

and the integers $a_k > a_{k-1} > \cdots > a_2 > a_1 \geq 0$ are determined by the **binomial expansion**

$$g_k - 1 = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_2}{2} + \binom{a_1}{1}$$

of $g_k - 1$ with respect to k. See Chapter 8 of [11] for more details. We can now state McMullen's conjecture:

Theorem 1 (g-Conjecture/Theorem). *An integer vector $g = (g_0, g_1, \ldots, g_{\lfloor d/2 \rfloor})$ is the g-vector of the boundary complex of a simplicial d-polytope P if and only if it is an M-sequence.*

In the same year, 1979, Stanley [9] proved the necessity and Billera and Lee [1] the sufficiency of McMullen's conditions. Stanley's proof that the g-vector of any simplicial polytope is an M-sequence used the Hard Lefschetz Theorem for the cohomology of projective toric varieties, but in the meantime a simpler proof by McMullen using his *polytope algebra* is available.

Billera and Lee invented an ingenious construction to produce, for every M-sequence g, a simplicial d-polytope with this g-vector. Very briefly, they first find a shellable ball B as a collection of facets of a cyclic polytope C, such that the g-vector of ∂B is the given M-sequence. Then they construct a realization of C and a point z that sees exactly the facets in B, and obtain a realization of ∂B as a simplicial polytope by taking the vertex figure at z of $\operatorname{conv}(\{z\} \cup C)$.

We next discuss Kalai's 1988 extension of their construction, by which he built so many simplicial spheres that *most* of them (in a sense to be made precise below) fail to be polytopal. He achieved this by giving a rule to produce many lists I of $(d+1)$-tuples of vertices, which span pure simplicial complexes $B(I)$. The underlying space of every such complex turns out to be a simplicial, shellable d-ball, which he called a **squeezed ball**, and therefore the boundary $S(I)$ of $B(I)$ is a simplicial $(d-1)$-sphere, a **squeezed sphere**. Lee shows in [6] that Kalai's squeezed spheres are shellable.

Let $s(d,n)$ denote the number of simplicial $(d-1)$-spheres, $sq(d,n)$ the number of squeezed $(d-1)$-spheres, and $c(d,n)$ the number of combinatorial types of simplicial d-polytopes with n labeled vertices. Goodman and Pollack [2], [3] derive the upper bound

$$\log c(d,n) \le d(d+1)n \log n \tag{1}$$

using a theorem of Milnor that bounds the sum of the Betti numbers of real algebraic varieties, while Kalai's squeezed spheres provide the following lower bound for $s(d,n)$:

$$\begin{aligned} \log s(d,n) \ge \log sq(d,n) \ &\ge \ \frac{1}{(n-d)(d+1)}\binom{n-\lfloor (d+2)/2\rfloor}{\lfloor (d+1)/2\rfloor} \\ &= \ \Omega(n^{\lfloor (d+1)/2\rfloor -1}) \qquad \text{for fixed } d. \end{aligned}$$

These bounds reveal that $\lim_{n\to\infty} c(d,n)/sq(d,n) = 0$ for $d \ge 5$, which means that for $d \ge 5$ most of Kalai's spheres are not polytopal—there are simply too many of them. However, we learn nothing for $d \le 4$. We prove in Proposition 1 below that $sq(4,n) \le 2^{n-5}n!$ for $n \ge 5$, which is strictly less than the bound from (1) for all $n \ge 5$.

The rest of the paper is organized as follows: In Section 2 we collect some facts about *cyclic polytopes*, an essential ingredient of our proof. In Section 3 we first present the details of Kalai's construction, and then show how to realize any of his 3-spheres as boundary complexes of simplicial 4-polytopes (Theorem 2). Finally, Section 4 uses the pictures constructed in Section 3 to give a shorter proof of Hebble and Lee's result that the dual graphs of squeezed 3-spheres are Hamiltonian.

2. Some Facts on Cyclic Polytopes

The convex hull of n distinct points on the **moment curve** $\mu_d\colon\ t \mapsto (t, t^2, \dots, t^d)$ in $\mathbb{R}^d$ is called a d-dimensional **cyclic polytope** with n vertices. The combinatorial type of this polytope is independent of the choice of the n points on the moment curve, and so one can talk about **the** cyclic polytope $C_d(n)$. In fact, any d-dimensional order d curve also gives rise to the same combinatorial types of polytopes.

We switch from d and n to $d+1$ and $n+1$, and consider a set $X = \{x_0 = \mu(t_0), \dots, x_n = \mu(t_n)\}$ of $n+1$ distinct points on the moment curve $\mu_{d+1} =: \mu$, ordered by their first coordinates. For any $f \subset \{0, 1, \dots, n\}$, write F_f for the subset of X indexed by f, and $i(F)$ for the indices of a subset F of X. The supporting hyperplane $H(F)$ of

a $(d+1)$-subset $F \subset X$ is given by $H(F) = \{x \in \mathbb{R}^{d+1}\colon \gamma(F) \cdot x = -\gamma_0(F)\}$, where $\gamma(F) = (\gamma_1(F), \ldots, \gamma_{d+1}(F)) \in \mathbb{R}^{d+1}$ and $\gamma_0(F) \in \mathbb{R}$ are defined by

$$0 \;=\; \prod_{i \in i(F)} (t - t_i) = \sum_{j=0}^{d+1} \gamma_j(F) t^j = \gamma_0(F) + \gamma(F) \cdot \mu(t). \tag{2}$$

Observe that $\gamma_{d+1}(F) = 1$; we say that $\gamma(F)$ **points upwards**.

Gale's **evenness criterion** tells us which $(d+1)$-subsets F of X are vertex sets of facets of the cyclic polytope $C = \operatorname{conv}(X)$: for any $i, j \in \{0, 1, \ldots, n\} \backslash i(F)$, the number of elements of $i(F)$ between i and j must be even.

Define the **end set** W_{end} of $F_f \subset X$ to be the rightmost contiguous block $\{r_f + 1, \ldots, \max f\}$ of the indices f of F, where $r_f = \max\{i \in \mathbb{N}\colon i < \max f,\ i \notin f\}$. Let F be a facet of C and take $x_j = \mu(t_j) \in X \backslash F$. If the cardinality of the end set of F is odd, we get $\prod_{i \in i(F)} (t_j - t_i) < 0$ because $j \notin i(F)$, and therefore $\gamma(F) \cdot x_j < -\gamma_0(F)$. Since $\gamma_{d+1}(F) = 1$, we conclude that the whole cyclic polytope C is *below* F, and call F an **upper facet** of C. If $\#W_{\text{end}}$ is even, we analogously call F a **lower facet** of C. Finally, define an **outer normal vector** $\alpha(F)$ of any facet F of C by $\alpha(F) = \gamma(F)$ (resp. $\alpha(F) = -\gamma(F)$) if F is an upper (resp. lower) facet of C, and set $\alpha_0(F) = -\gamma_0(F)$ (resp. $\alpha_0(F) = \gamma_0(F)$). By this, we obtain $C \subset \{x \in \mathbb{R}^{d+1}\colon \alpha(F) \cdot x \le \alpha_0(F)\}$ for all facets F of C.

3. Realizing Kalai's 3-Spheres

3.1. *Kalai's Idea*

First define a partial order $\preceq$ on $\binom{\mathbb{N}}{d+1}$ by $\{i_1, i_2, \ldots, i_{d+1}\}_< \preceq \{j_1, j_2, \ldots, j_{d+1}\}_<$ if $i_k \le j_k$ for every $k = 1, \ldots, d+1$. Here the notation $A = \{a_1, \ldots, a_r\}_<$ means that the elements of the set A are listed in increasing order. For the standard poset terminology used in the following, see [10].

For an odd integer $d > 0$ and $n \in \mathbb{N}$, let $\mathcal{F}_d(n)$ be the collection of $(d+1)$-subsets of $[n] := \{1, 2, \ldots, n\}$ of the form $\{i_1, i_1+1\} \cup \{i_2, i_2+1\} \cup \cdots \cup \{i_e, i_e+1\}$, where $e = (d+1)/2$, $i_1 \ge 1$, $i_e < n$, and $i_{j+1} \ge i_j + 2$ for all relevant j. Let I' be an initial set (order ideal) of $\mathcal{F}_d(n)$ with respect to the partial order $\preceq$ on $\binom{\mathbb{N}}{d+1}$. Informally, $f' \preceq g'$ for $f', g' \in \mathcal{F}_d(n)$ if f' arises from g' by pushing some elements in g' to the left.

For even $d > 0$, put $\mathcal{F}_d(n) = \{\{0\} \cup f'\colon\ f' \in \mathcal{F}_{d-1}(n)\} =: 0 * \mathcal{F}_{d-1}(n)$ with the induced partial order, and set $I := 0 * I'$.

Finally, let $B(I)$ be the simplicial complex (the **squeezed** d**-ball**) spanned by I, denote the boundary complex of $B(I)$ by $S(I)$ (the **squeezed** $(d-1)$**-sphere**), and do the same for I'.

3.2. *The Structure of* 3*-Balls*

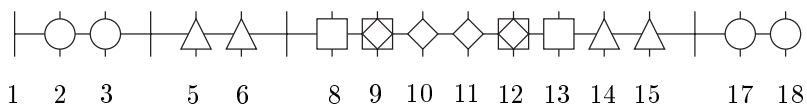
To specialize Kalai's construction to $d = 4$, we first study squeezed 3-balls. Take $n \ge 4$ in $\mathbb{N}$, write (i, j) for an element $\{i, i+1, j, j+1\} \subset [n]$ of $\mathcal{F}_3(n)$, and define the **gap**

of $(i, j) \in \mathcal{F}_3(n)$ to be the number $j - i - 2$ of integers between $i + 1$ and j. From the fact that any two elements of $\mathcal{F}_3(n)$ with the same gap are translates of each other and therefore $\preceq$-comparable, we conclude that any $\preceq$-antichain in $\mathcal{F}_3(n)$ can be linearly ordered by increasing gap, and denote this order by $\sqsubset$. We remark that the difference between the gaps of any two elements in a $\preceq$-antichain must be at least 2, as otherwise the two elements would be $\preceq$-comparable. In particular, the maximal number of elements of a $\preceq$-antichain in $\mathcal{F}_3(n)$ is $\lceil (n-3)/2 \rceil$.

Any order ideal $I' \subset \mathcal{F}_3(n)$ for $n \in \mathbb{N}$ is generated by the set $G' = \{g'_1, g'_2, \ldots, g'_r\}_\sqsubset$ of its maximal elements, for some $r \leq \lceil (n-3)/2 \rceil$. By our discussion, the $g'_k = (i_k, j_k)$ satisfy

(1) $j_k \geq i_k + 2$ for $k = 1, \ldots, r$, and
(2) $i_k > i_{k+1}$ and $j_k < j_{k+1}$ for $k = 1, \ldots, r-1$.

As an example, let I' be the ideal generated by $G' = \{(9, 11), (8, 12), (5, 14), (2, 17)\}_\sqsubset$:



Note that if $g' \sqsubset h' \in G'$, then g' is nested inside h' (possibly with overlap). From Fig. 1 we read off the structure of the 3-ball $B(I')$ generated by G', and its boundary $S(I')$.

Now put $\mathcal{F}_4(n) = 0 * \mathcal{F}_3(n)$ with the induced partial order, and $I = 0 * I'$. The 4-ball $B(I)$ spanned by I is a cone over the 3-ball $B(I')$, whose boundary complex is the squeezed 3-sphere $S(I)$.

Proposition 1. *There are at most* $2^{n-4}(n+1)!$ *squeezed* 3*-spheres with* $n + 1 \geq 5$ *labeled vertices. In particular,* $\log sq(4, n) = \Theta(n \log n)$.

Proof. By Proposition 3.3 of [5], distinct 4-balls $B(I)$ whose vertices are labeled according to their construction give rise to distinct 3-spheres $S(I)$ labeled in this way, and distinct initial sets $I \subset \mathcal{F}_4(n)$ obviously induce distinct such 4-balls. Every initial set I is of the form $0 * I'$ for a unique order ideal $I' \subset \mathcal{F}_3(n)$. Therefore, by relabeling vertices, $sq(4, n+1)$ is at most $(n+1)!$ times the number of distinct order ideals in $\mathcal{F}_3(n)$, depending on the combinatorial symmetries of $S(I)$. By Fig. 1, every such order ideal can be represented by a lattice path of length $n-4$ taking steps only in the positive i- or negative j-directions, and starting at $(i, j) = (1, n-1)$. There are 2^{n-4} of these, and they all give rise to distinct ideals. □

3.3. *A Bird's-Eye View of the Realization Construction*

Observe that by Gale's Evenness Criterion, every $f \in I$ corresponds to a lower facet F_f of a cyclic polytope. By adapting the ideas of Billera and Lee, we will now realize any $S(I)$ as the boundary complex of a 4-polytope P by appropriately realizing a cyclic 5-polytope C, and choosing a viewpoint v close to the negative e_5-axis that sees exactly

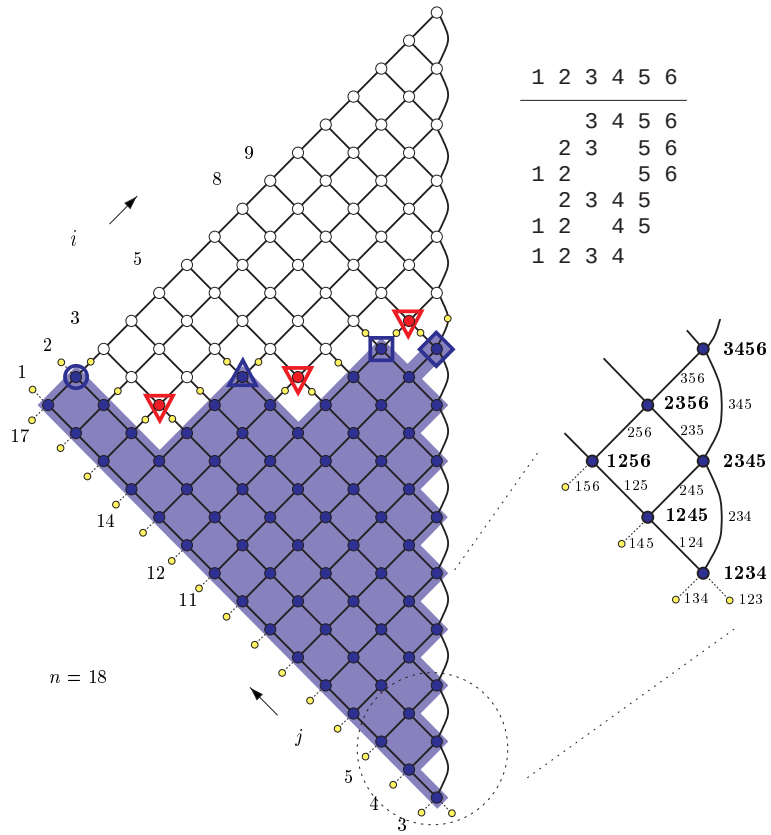


Fig. 1. The Kalai poset $\mathcal{F}_3(18)$. The shaded circles are the facets of the 3-ball $B(I')$ with generators $G' = \{(9,11),(8,12),(5,14),(2,17)\}_{\sqsubset}$. The minimal elements H' of $\mathcal{F}_3(18)\backslash I'$ are marked by ∇'s, and $\sqsubset$ orders the elements of $E' = G' \cup H'$ from left to right (see Observation 1). Straight lines between facets correspond to $\prec$-covering relations between elements of I', and straight and curved lines together to inner ridges of $B(I')$. The small circles are the facets of $S(I') = \partial B(I')$. The set of facets of the Kalai sphere $S(I)$ is the union of $B(I')$ and $0 * S(I')$.

the facets of C in $B(I)$. The convex 4-polytope P is then the vertex figure at v of $\operatorname{conv}(C \cup \{v\})$, and $S(I)$ its boundary.

Specifically, let $\mu = \mu_5\colon \mathbb{R} \to \mathbb{R}^5$, $t \mapsto (t, t^2, \ldots, t^5)$ be the moment curve in dimension 5. Given an order ideal $I = 0 * I'$ in $\mathcal{F}_4(n)$ where $n = \max \bigcup I$, we execute the following steps:

1. Choose $N' > 0$ and place $0 = t_0 < t_1 < \cdots < t_n \in \mathbb{R}_{\geq 0}$ such that

$$\begin{aligned} \prod_{i \in f \backslash \{0\}} t_i &< N' \qquad \text{for all} \quad f \in I, \quad \text{and} \\ \prod_{i \in f \backslash \{0\}} t_i &> N' \qquad \text{for all} \quad f \in \mathcal{F}_4(n) \backslash I. \end{aligned} \tag{S1}$$

Solutions for (S1) exist with $t_1 > 0$ arbitrarily small. We will find a solution for

this system of inequalities by processing the elements of $E' = G' \cup H'$ in $\sqsubset$-order, where G' is the set of $\preceq$-maximal elements of I', and H' is the set of $\preceq$-minimal elements of $\mathcal{F}_3(n) \backslash I'$.

2. Make sure that the viewpoint to be defined will not see any upper facets of $C = C_5(n+1) = \operatorname{conv}\{0, \mu(t_1), \mu(t_2), \ldots, \mu(t_n)\}$ that contain 0, by choosing $t_1 > 0$ so small that

$$t_1 t_{n-2} t_{n-1} t_n < N'. \tag{S2}$$

3. Choose ε, with $0 < \varepsilon < t_1$, so small that for all $e, f \in \mathcal{F}_4(n)$,

$$e \prec f \quad \Longrightarrow \quad \gamma(F_e) \cdot \mu(\varepsilon) < \gamma(F_f) \cdot \mu(\varepsilon). \tag{S3}$$

4. Choose $\varepsilon > 0$ even smaller, if necessary, such that the viewpoint $v := \mu(\varepsilon) - \varepsilon N' e_5$ satisfies

$$\begin{aligned} &\alpha(F) \cdot v > \alpha_0(F) && \text{for} \quad f_F \in I, \\ &\alpha(F) \cdot v < \alpha_0(F) && \text{for all lower facets } F \text{ of } C \text{ such that} \quad f_F \notin I, \\ &\alpha(F) \cdot v < \alpha_0(F) && \text{for all upper facets } F \text{ of } C, \end{aligned} \tag{S4}$$

where $\alpha(F)$ is the outer normal vector of F we defined at the end of Section 2.

We conclude that v sees exactly the facets of C in $B(I)$, and obtain $S(I)$ as above.

3.4. *How to Realize Kalai's 3-Spheres*

We now give the details of the construction and prove the following theorem.

Theorem 2. *Every squeezed 3-sphere $S(I)$ given by an order ideal I in the poset $(\mathcal{F}_4(n), \preceq)$ with $n \geq \max \bigcup I$ can be realized as the boundary complex of a simplicial, convex 4-polytope.*

Remark 1. The construction shows the stronger result that every squeezed 4-ball $B(I)$ can be realized as a regular triangulation of a convex 4-polytope.

Proof of Theorem 2. Given an ideal $I \subset \mathcal{F}_4(n)$, we may assume that $n = \max \bigcup I$ since $\mathcal{F}_4(n) \subseteq \mathcal{F}_4(n')$ for $n \leq n'$. By definition, every order ideal $I \subset \mathcal{F}_4(n)$ has the form $I = 0 * I'$, where $I' = \langle G' \rangle \subset \mathcal{F}_3(n)$ is generated by its maximal elements $G' = \{g'_1, g'_2, \ldots, g'_r\}$ with $g'_k = (i_k, j_k)$. Choose $N' > 0$, introduce n variable points $0 < t_1 < t_2 < \cdots < t_n$ in $\mathbb{R}_{>0}$, and consider the set H' of $\preceq$-minimal elements of $\mathcal{F}_3(n) \backslash I'$.

Observation 1. *Consider any two consecutive elements $e' = (i, j) \sqsubset\!\cdot\ f' = (k, \ell)$ of a $\sqsubset$-ordered $\preceq$-antichain G' of $\mathcal{F}_3(n)$. Then the unique $\prec$-minimal element m' in $\mathcal{F}_3(n) \backslash \langle G' \rangle$ with $\operatorname{gap}(e') < \operatorname{gap}(m') < \operatorname{gap}(f')$ exists and is $m' = (k+1, j+1)$. In particular, the number of $\prec$-minimal elements in $\mathcal{F}_3(n) \backslash \langle G' \rangle$ is no greater than $\lfloor (n-3)/2 \rfloor$.*

Sketch of Proof. The first statement follows by inspection of Fig. 1. For the second assertion, note that the set H' has maximal cardinality if $G' = \{(i, n-i)\colon i = 1, 2, \ldots, \lceil (n-3)/2 \rceil\}$. $\square$

Using Observation 1, we linearly order $E' = G' \cup H'$ by $\sqsubset$, see Fig. 1. To carry out Step 1 of our program, first choose some small $\delta > 0$. Our goal is to place the t's in $\mathbb{R}_{>0}$ such that

$$\prod_{i \in g'} t_i = N' - \delta \quad \text{for} \quad g' \in G' \qquad \text{and} \qquad \prod_{i \in h'} t_i = N' + \delta \quad \text{for} \quad h' \in H'. \tag{S1$'$}$$

Observation 2. *The cardinality of $E' = G' \cup H'$ is at most $n-3$. In particular, there are fewer equalities in* (S1$'$) *than there are variables.*

Proof. Because $n = \max \bigcup I$, the largest element of $(E', \sqsubset)$ is in G'. Using Observation 1 again,

$$\#E' = \#G' + \#H' \le \left\lceil \frac{n-3}{2} \right\rceil + \left\lfloor \frac{n-3}{2} \right\rfloor = n-3,$$

which proves Observation 2. $\square$

We now begin the construction by placing the t's corresponding to the $\sqsubset$-smallest element of E' in such a way in $\mathbb{R}_{>0}$ that (S1$'$) is satisfied. This is clearly possible. The general step of constructing a solution to (S1$'$) is based on the following lemma.

Lemma 1. *Let $e' = (i, j) \sqsubset f' = (k, \ell)$ be two consecutive elements of E'.*

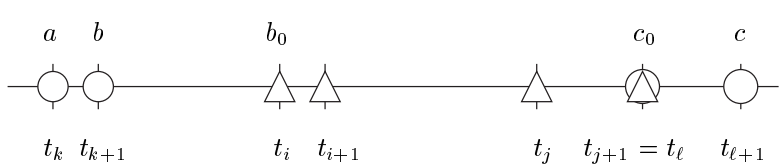
(a) *If $e' \in G'$ and $f' \in H'$, then $0 < k \le i$ and $\ell = j + 1$. If $e' \in H'$ and $f' \in G'$, then $k = i - 1$ and $j \le \ell < n$.* (*See Fig.* 1.)

(b) *Suppose that the $\{t_i\}_{i \in e'}$ have been placed already, but not all $\{t_j\}_{j \in f'}$. Then these latter t's may be placed in such a way in $\mathbb{R}_{>0}$ that $0 < t_k < t_{k+1} < t_\ell < t_{\ell+1}$, and the equality*

$$t_k t_{k+1} t_\ell t_{\ell+1} = M \tag{3}$$

is satisfied, where $M := N' - \delta$ if $f' \in G'$ and $M := N' + \delta$ if $f' \in H'$.

Sketch of Proof for (b). Suppose that $e' \in G'$ and $f' \in H'$. We then have the following situation:



It is straightforward to verify that for any $0 < k \le i$, the points a, b, c may be placed in such a way that $0 < a < b < b_0 < c_0 < c$ and $abc_0c = N' + \delta$. Similarly, if $e' \in H'$

and $f' \in G'$,

for any $j \leq \ell < n$ we may place a, b, c such that $0 < a < a_0 < b_0 < b < c$ and $aa_0bc = N' - \delta$. □

We now complete Step 1 by applying Lemma 1 to all members of E' in $\sqsubset$-order. The definition of $\preceq$ tells us that because the $f' \in E'$ satisfy (S1′), in fact all $f \in \mathcal{F}_4(n)$ satisfy the system (S1).

If in Step 1 we encountered some $e' \in E'$ with $1 \in e'$, then necessarily $e' = \{1, 2, n-1, n\} \in G'$, which imposed the inequality $t_1t_2t_{n-1}t_n < N'$. This inequality in turn remains satisfied if we choose t_1 even small enough to verify (S2). If $1 \notin e'$ for all $e' \in E'$, we are free to do the same. We have completed Step 2, and place any remaining unassigned t's such that $0 = t_0 < t_1 < \cdots < t_n$.

Observation 3.

(a) $\gamma_0(F_f) = 0$ *for any* 5*-element subset* $f \subset \{0, 1, \ldots, n\}$ *that contains* 0.
(b) *For all choices of* $t_1 < \cdots < t_n$, *one can find* $\varepsilon > 0$ *small enough such that the implication* (S3) *holds for all* $f, g \in \mathcal{F}_4(n)$.

Proof of (b). The definition (2) of the γ's implies that for $f = \{0, s_1, \ldots, s_4\}$,

$$\gamma(F_f) \cdot \mu(\varepsilon) = \varepsilon(\varepsilon - s_1) \cdots (\varepsilon - s_4) = \varepsilon s_1 s_2 s_3 s_4 \pm o(\varepsilon). \tag{4}$$

This means that $\gamma(F_f) \cdot \mu(\varepsilon) < \gamma(F_g) \cdot \mu(\varepsilon)$ by definition of $\prec$, for ε small enough. □

Take $0 < \varepsilon < t_1$ as in Observation 3(b), tentatively set $z := \mu(\varepsilon)$, and let $f \in \mathcal{F}_4(n)$. If $f \in I$, there exists some $g \in G := 0 * G'$ with $f \preceq g$, and by (4) we have

$$\gamma(F_f) \cdot z \leq \gamma(F_g) \cdot z = \varepsilon \prod_{i \in g \setminus \{0\}} t_i + O(\varepsilon^2) = \varepsilon(N' - \delta) \pm o(\varepsilon).$$

If $f \notin I$, then there is some $h \in H := 0 * H'$ with $f \succeq h$, and we obtain in a similar way that

$$\gamma(F_f) \cdot z \geq \varepsilon(N' + \delta) \pm o(\varepsilon).$$

Thus, we finally choose $0 < \varepsilon < t_1$ so small that with $z := \mu(\varepsilon)$ and $N := \varepsilon N'$, we have $\gamma(F_f) \cdot z < N$ for $f \in I$, and $\gamma(F_f) \cdot z > N$ for $f \notin I$. Step 3 is now complete.

We proceed to verify that $v := \mu(\varepsilon) - \varepsilon N' e_5 = z - N e_5$ satisfies the inequalities (S4). For this, recall that all F_f with $f \in \mathcal{F}_4(n)$ satisfy Gale's Evenness Criterion, which means that $\mathcal{F}_4(n)$ is exactly the set of lower facets of the cyclic polytope $C = \mathrm{conv}(X)$ that contain $x_0 = 0$. However, *any* $F \subset X$ of odd cardinality satisfying Gale's Evenness Criterion with even end set must contain 0, and we conclude that $\mathcal{F}_4(n)$ is in fact the set of *all* lower facets of C.

Recall from Section 2 that $\alpha(F) = \gamma(F)$ and $\alpha_0(F) = -\gamma_0(F)$ if F is an upper facet of C, and that $\alpha(F) = -\gamma(F)$ and $\alpha_0(F) = \gamma_0(F)$ if F is a lower facet of C. We discuss all facets F_f of C in turn:

Lower facets of C:

- If $f \in I \subset \mathcal{F}_4(n)$, then by construction $\gamma(F_f)\cdot z < N$, and this implies $\gamma(F_f)\cdot v < 0$ (remember that $\gamma_5(F) = 1$ for all F) and $\alpha(F_f)\cdot v > 0 = \alpha_0(F_f)$, which means that F_f is visible from v.
- If $f \in \mathcal{F}_4(n)\backslash I$, we conclude from $\gamma(F_f)\cdot z > N$ that $\alpha(F_f)\cdot v < 0 = \alpha_0(F_f)$, which says that F_f is not visible from v.

Upper facets of C:

- If $0 \notin f = \{s_1, \ldots, s_5\}$, then (2) and $\varepsilon < t_1$ imply $\gamma(F_f)\cdot z + \gamma_0(F_f) = \prod_{i=1}^{5}(\varepsilon - s_i) < 0$, and

$$\alpha(F_f)\cdot v = \gamma(F_f)\cdot v = \gamma(F_f)\cdot z - N < -\gamma_0(F_f) - N < -\gamma_0(F_f) = \alpha_0(F_f).$$

- If $0 \in f$, then $\gamma_0(F_f) = 0$ and $f = \{0, 1\} \cup \{i, i+1\} \cup \{n\}$ with $2 \le i \le n-2$. By inequality (S2) and the definition of $\prec$, we conclude that necessarily $\gamma(F_f)\cdot z < N$ and

$$\alpha(F_f)\cdot v = \gamma(F_f)\cdot z - N < 0 = \alpha_0(F_f).$$

We have verified the inequalities (S4) and completed the proof of Theorem 2. □

Remark 2. A referee has suggested extending this construction to boundaries of more general even-dimensional squeezed balls. However, so far we have only been able to realize odd-dimensional squeezed spheres directly modeled on the three-dimensional ones, and leave this as an open problem.

4. A Shorter Proof that Squeezed 3-Spheres Are Hamiltonian

In 1973 Barnette [8] conjectured that all simple 4-polytopes admit a Hamiltonian circuit. In [4] Hebble and Lee prove that squeezed 3-spheres are (dual) Hamiltonian by explicitly constructing a Hamiltonian circuit in the dual graph; however, their proof goes through extensive case analysis. A referee has suggested that it might be possible to obtain a simpler proof of this result. In this section we follow the referee's suggestion and obtain a "proof by picture" with fewer case distinctions, which moreover only depend on parity conditions.

Theorem 3 [4]. *The dual graph of any Kalai* 4*-polytope $S(I)$ admits a Hamiltonian circuit. In particular, the polars of these* 4*-polytopes satisfy Barnette's conjecture.*

Proof. Recall from Section 3.2 that the set of facets of $S(I)$ is $B(I') \cup (0 * S(I'))$. We continue to write $(i, j) = \{i, i+1, j, j+1\}$ for facets of $S(I)$ in $B(I')$, and introduce the

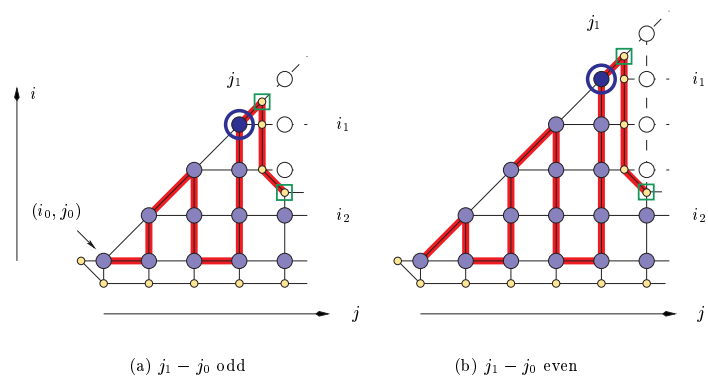


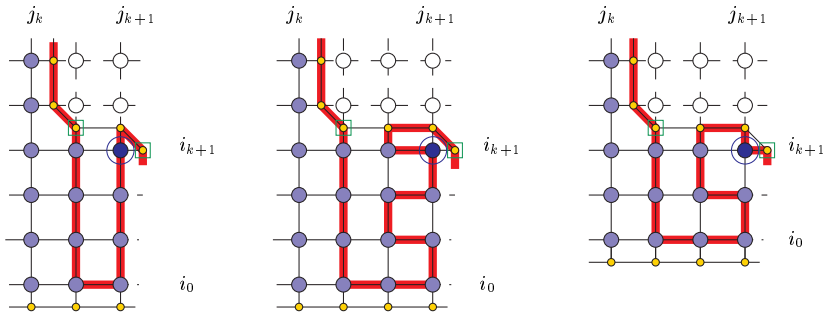
Fig. 2. Steps *Over the top* and *Down*. The circled facet is (i_1, j_1), the upper $\square$ represents $(i_1 + \frac{1}{2}, j_1)$, and the lower $\square$ is $(i_2 + \frac{1}{2}, j_1 + 1)$.

notation $(i + \frac{1}{2}, j) := \{0, i+1, j, j+1\}$ and $(i, j + \frac{1}{2}) := \{0, i, i+1, j+1\}$ for facets of $S(I)$ in $0 * S(I')$. Also, recall from Section 3.2 the definition of the order relations $\preceq$ and $\sqsubset$, and number the set G' of $\preceq$-maximal elements (i_k, j_k) of $B(I')$ in ascending $\sqsubset$-order, starting with $k = 1$.

We start our Hamiltonian circuit in the dual graph of $S(I)$ at the facet $(i_0, j_0) = (1, 3) = \{1, 2, 3, 4\} \in B(I')$. While walking through the other facets of $B(I')$, we also pick up the facets of the form $(i + \frac{1}{2}, j)$ and $(i, j + \frac{1}{2})$ with $i, j \geq 1$ of $S(I')$, and then return to $(1, 3)$ via the set of facets $\{(0, j)\colon 2 \leq j \leq n-1\}$. We also use the difference operators $\Delta j_k = j_{k+1} - j_k$ and $\Delta i_k = i_{k+1} - i_k$. In our circuit we repeatedly go through certain steps, and in the figures we mark the end of one step and the beginning of the next by a square. In all steps, if all facets in G' are processed, go to step *Down* (and then to *Finish*).

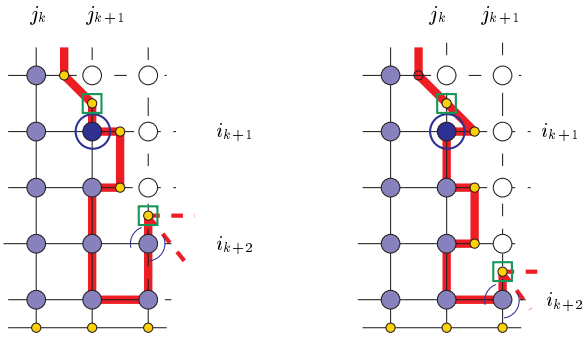
1. *Over the top*: Start at $(i_0, j_0) = (1, 3)$. If $j_1 - j_0$ is odd, continue as in Fig. 2(a). If $j_1 - j_0$ is even, proceed as in Fig. 2(b). In both cases, go on until $(i_1 + \frac{1}{2}, j_1)$. Set $k = 1$, and go to step *Down*.
2. *Down*: If there are no more generators to be processed, go down along the facets $\{(i_\ell, j_k + \frac{1}{2})\colon \ell = k, k-1, \ldots, 1\}$ and continue with step *Finish*. Otherwise, if $\Delta i_k > 0$, continue downwards as in Fig. 2 until $(i_{k+1} + \frac{1}{2}, j_k + 1)$. If $i_{k+1} = i_k$, do nothing. In both cases, increment k by 1, and continue to step *Across*.
3. *Across*: If Δj_k is even, continue as in Fig. 3(a). If Δj_k is odd and not 1 and $i_{k+1} - i_0$ is even, continue as in Fig. 3(b); if $\Delta j_k \neq 1$ and $i_{k+1} - i_0$ are both odd, continue as in Fig. 3(c). If $\Delta j_k = 1$ and Δi_{k+1} is even, proceed as in Fig. 4(a), if Δi_{k+1} is odd, proceed as in Fig. 4(b). In any case, increment k by one, and repeat from step *Down* or *Across* as necessary, depending on whether the facet surrounded by a dashed circle in Fig. 4 is in G or not.
4. *Finish*: Now the only thing left to do is to return to $(1, 3)$ via the set of facets $\{(0, j) : n - 1 \geq j \geq 2\}$, as in Fig. 5.

This completes the proof of Theorem 3. $\square$



(a) Δj_k even (b) Δj_k odd, $i_{k+1} - i_0$ odd (a) Δj_k odd, $i_{k+1} - i_0$ even

Fig. 3. Step *Across* in case Δj_k is even. The circled facet is (i_{k+1}, j_{k+1}).



(a) Δi_{k+1} even (b) Δi_{k+1} odd

Fig. 4. Step *Across* in case $\Delta j_k = 1$. The circled facet is (i_{k+1}, j_{k+1}). Depending on whether the facet surrounded by a dashed circle is in G or not, the next step will be *Down* or *Across*, respectively.

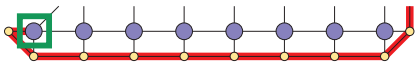


Fig. 5. Step *Finish*.

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