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A Characterization of Astral (n_4) Configurations

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Abstract. A conjecture of Branko Grünbaum concerning what astral (n_4) configurations exist is shown to be true.

1. Introduction

An (n_4) *configuration* is a collection of points and straight lines in the Euclidean plane, where each point is on exactly four lines and each line passes through exactly four points. A configuration is *connected* if, starting at an arbitrary point, it is possible to reach any other point in the configuration by travelling only on lines of the configuration and changing lines only at points of the configuration. An (n_4) configuration is *astral* if the subgroup of the isometries of the Euclidean plane which map the configuration onto itself generate exactly two transitivity classes of points and two transitivity classes of lines (see [1] and [2]). In a connected astral (n_4) configuration, the points of the configuration must lie on the vertices of two concentric, convex, regular m-gons, where $m = n/2$, and the lines of the configuration must be common diagonals of the m-gons. If the vertices of an m-gon are consecutively labelled $v_1, \ldots, v_m$, a diagonal has *span a* if it connects vertices v_i and v_{i+a}. The configuration may also be constructed by taking a regular, convex m-gon and all diagonals of span a and c, where the diagonals of the two spans happen to intersect appropriately, and making the intersection points the vertices of the inner m-gon (for an example, see Fig. 1). In [4] it was shown that if four diagonals of a regular polygon intersect, then the m-gon must have m divisible by six, so any configuration whose points lie on the vertices of two concentric, convex, regular m-gons must have m divisible by six as well.

Following the notation in [2], such a configuration will be notated as $m\#a_b c_d$, where m is the number of vertices of the outside m-gon and a and c are the spans of diagonals. Considering only the span a diagonals, one diagonal has intersection points on it from

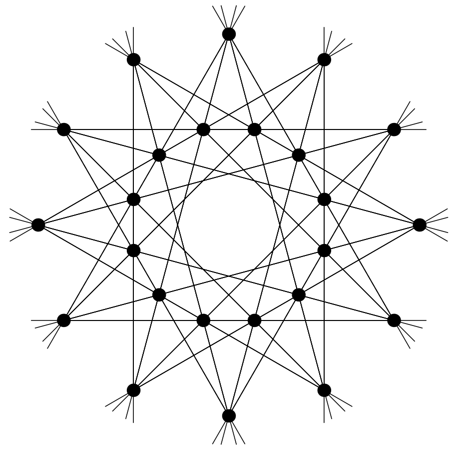

Fig. 1. The astral (n_4) configuration $12\#4_1 5_4$.

the other span a diagonals; b is the number of these intersection points, counted from the midpoint of the diagonal, until the inner m-gon vertex is reached (and similarly for d and the span c diagonals). Thus, a_b and c_d are the same point of the configuration. Additional configurations may be constructed from a given configuration by taking p concentric copies of the configuration, equally spaced, for some choice of p: in this case the final configuration will be said to consist of *p multiples* of the original configuration (see Fig. 2 for an example with $p = 3$). The final type of astral (n_4) configurations are constructed by taking two concentric copies of a configuration whose points lie on the vertices of two concentric, regular, convex polygons and rotating one through an arbitrary angle with respect to the other (see [3]); these are discussed further in Section 4.

Theorem 1. *All astral (n_4) configurations with points that lie on the vertices of two concentric, regular, convex m-gons, where $m = n/2$ and lines which are common diagonals of the m-gons are listed in the following: there are two infinite families, $(6k)\#(3k-j)_{3k-2j}(2k)_j$ for $j = 1, \dots, 2k-1$, $j \neq k$, and $j \neq 3k/2$, and $(6k)\#(3k-2j)_j(3k-j)_{2k}$, for $j = 1, \dots, k-1$. There are 27 connected sporadic configurations, with $m = 30, 42$, and 60, listed in Table 5, where a configuration is sporadic if it is not a member of one of the infinite families. Finally, there are multiples of the sporadic configurations.*

The history of this theorem is somewhat confused. The existence of the infinite families of configurations and of the sporadic configurations was proven in [2]. In that paper Grünbaum conjectured that these were all the astral (n_4) configurations. He emended the conjecture in [3], where he stated that these were all the connected astral configurations, although in [2] he discussed the fact that given s copies of an astral (n_4) configuration $m\#a_b c_d$ one could construct the disconnected astral configuration $sm\#sa_{sb}sc_{sd}$ (see Theorem 2 below). In a private communication he indicated that the proper form of the conjecture is as stated in Theorem 1.

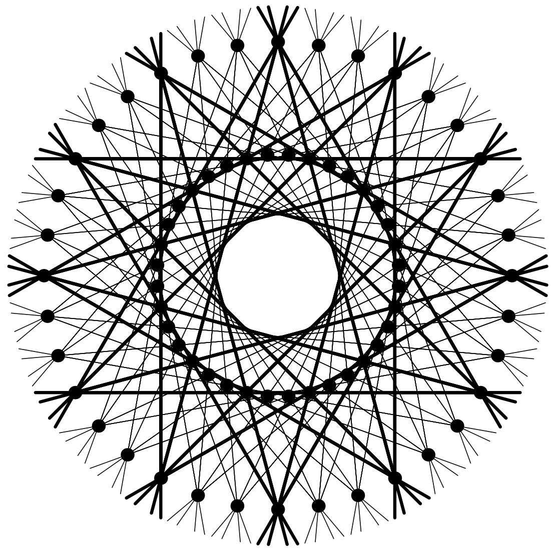

Fig. 2. The astral (n_4) configuration $36\#12_3 15_{12}$, constructed from three copies of $12\#4_1 5_4$ (one copy is shown with thicker lines).

Theorem 2. *A configuration* $6k\#a_bc_d$ *is connected iff* $\mathrm{GCD}(m, a, b, c, d) = 1$. *If* $\mathrm{GCD}(m, a, b, c, d) = q > 1$, *then the configuration* $6k\#a_bc_d$ *is constructed from q concentric copies, equally spaced, of the configuration* $6k/q\#(a/q)_{b/q}(c/q)_{d/q}$.

Proof. Suppose a configuration (the "main configuration") $6k\#a_bc_d$ is made up of q concentric copies, rotated through equal angles, of a smaller, connected configuration $6k'\#a'_{b'}c'_{d'}$ (the "subconfiguration"). Notice that each copy contributes $6k'$ vertices to the total number m of vertices, so $m = 6k'q$. If a line has span a' (respectively, c') when considered as part of the subconfiguration, then it must have span $a'q$ (respectively, $c'q$) considered as part of the main configuration, since there are now $q - 1$ more vertices of the main configuration to be counted between each two vertices of the subconfiguration containing the line. Similarly, there are now $q - 1$ more intersection points per original intersection point to be counted when determining b and d in the main configuration, so $b = b'q$ and $c = c'q$.

Now suppose $6k\#a_bc_d$ is a configuration whose vertices lie on the vertices of two concentric, regular, convex m-gons, and suppose $\mathrm{GCD}(k, a, b, c, d) = q$. Any subconfiguration $m'\#a'_{b'}c'_{d'}$ must have $6|m'$; if $m' = m/q$ and $q \nmid k$ but $q|m$, then $6 \nmid m'$. So if $\mathrm{GCD}(m, a, b, c, d) = q$ and $6|(m/q)$, then $q|k$, so $\mathrm{GCD}(k, a, b, c, d) = q$. Choose a vertex v_0 on the outer polygon of the configuration. Label the other vertices of the outer polygon $v_1, v_2, \ldots, v_{m-1}$ proceeding counterclockwise from v_0. Consider the vertices connected to v_0: using the span a and c lines, v_0 is connected to v_a and v_c. By travelling along the span a line and changing to a different span a line at the bth intersection point (on the inner polygon of the configuration), v_0 is connected to v_b and, similarly, to v_d using the span c line. It follows that v_0 must be connected to all $v_{i_1a+i_2b+i_3c+i_4d}$, where $i_1, \ldots, i_4$

are integers and the subscript is taken modulo m. That is, if $p = \mathrm{GCD}(a, b, c, d)$, v_0 is connected to all v_{jp} (jp taken modulo m), and the subconfiguration induced by the v_{jp} is the connected component of $6k\#a_b c_d$ containing v_0. Finally, the number of components of $6k\#a_b c_d$ equals $\mathrm{GCD}(m, p) = q$, so $6k\#a_b c_d$ consists of precisely q copies of the subconfiguration induced by the v_{jp} (since the choice of v_0 was arbitrary). □

Corollary 1. *The configurations listed in Theorem* 1 *which are connected are those with configuration symbol* $6k\#a_b c_d$, *where* $\mathrm{GCD}(k, a, b, c, d) = 1$.

The remainder of this section and the following two sections deals with the proof of Theorem 1.

To find astral (n_4) configurations whose vertices lie on two concentric, regular, convex m-gons, it suffices to determine when a regular m-gon has two pairs of two diagonals of the same span which intersect in a single point. In [4] Poonen and Rubinstein determined how many intersection points of 2, 3, 4, 5, 6, or 7 diagonals there are in a regular m-gon and provided information about what diagonals are used to form such intersections; they showed that 8 or more diagonals can meet only in the center of an even-sided polygon. In particular, they found 4 one-parameter families of intersecting triples of diagonals, along with 65 sporadic triples (i.e., triples not members of the infinite families), and they found 12 one-parameter families of intersecting quadruples.

Poonen and Rubinstein listed their intersecting triples (respectively, quadruples) as hextuples (respectively, octuples) of numbers summing to 1, listing the fraction of the circumference traversed between successive endpoints of the diagonals in question: call this the arclength. Given a hextuple $\{a, b, c, d, e, f\}$, to form the ith diagonal of the triple, choose a starting position on the polygon and number by 1, 2, ..., 6 the vertices reached by traversing through arclength a, then through arclength b, etc. Vertices i and $i + 3$ are connected to form the ith diagonal; in an octuple, the construction is similar, but the vertices are labelled $1, \ldots, 8$ and vertices i and $i + 4$ are joined (see Fig. 3). The least common denominator of the fractions in the hextuple or octuple is a factor of m, the number of vertices in the polygon. To convert an octuple of arclengths into a configuration, assuming that it consists of two pairs of same-span diagonals, the following algorithm is used: first, multiply each fraction by m, leaving a list of the number

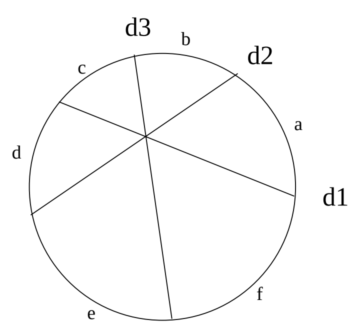

Fig. 3. Converting the hextuple $\{a, b, c, d, e, f\}$ into a triple of diagonals $\{d_1, d_2, d_3\}$.

of arcs between each endpoint of a diagonal; call this list $L = (L_1, \ldots, L_8)$. For each i, $i = 1, \ldots, 8$, make a new list L' which contains the sum from L_i to L_{i+4} in the ith slot (with indices summed mod 8). Let a be the smallest element of L' and let c be the second smallest element. To calculate b, find the two positions in L' which have a in them, say L_j and L_k, and set $b = L_j + \cdots + L_{k-1}$ (again, taking indices mod 8). Similarly, to find d, sum between the positions in L which correspond to the appearance of c in L'. The configuration corresponding to the octuple is $m\#a_b c_d$.

Poonen and Rubinstein show in [4] that the maximum number of diagonals of a regular m-gon which meet in a point other than the center is three if m is not divisible by six. Thus, for any astral (n_4) configuration, if $m = n/2$, $m = 6k$.

2. The Infinite Families of Configurations

Poonen and Rubinstein determined that there are twelve one-parameter families of intersecting quadruples of diagonals, which they listed in a table in [4]. Of these, four contain two pairs of same-span diagonals. I have given them names (i.e., family i) according to the order in which they were listed in [4]. They are listed by arclengths in Table 1.

If $t = j/m$, where $m = 6k$, and the lists are multiplied through by m, the lists in Table 2 are generated, which may be converted into configuration symbols as outlined above.

Define $\hat{k}$ to be the greatest integer strictly less than k. Following the instructions above, the octuples in Table 2 are converted into the configurations in Table 3, with parameters as indicated.

Note that there are several variants of a configuration symbol which correspond to the same configuration. These variants yield the following *configuration identities*, which may be combined in any order:

$$m\#a_b c_d = m\#c_d a_b, \tag{1}$$

$$m\#a_b c_d = m\#a_b c_{m-d}, \tag{2}$$

$$m\#a_b c_d = m\#a_b c_{-d}, \tag{3}$$

$$m\#a_b c_d = m\#(m-a)_b c_d. \tag{4}$$

To show that the families of configurations in Table 3 are the same families of configurations as Grünbaum found, it is necessary to reparametrize and use the configuration identities.

Table 1. The infinite families with pairs of same-span diagonals.

Family									Range; $t \in \mathbb{Q}$
1	t	t	t	$\frac{1}{6} - 2t$	$\frac{1}{6}$	$\frac{1}{3} + t$	$\frac{1}{6}$	$\frac{1}{6} - 2t$	$0 < t < \frac{1}{12}$
2	t	$\frac{1}{6} - t$	$\frac{1}{6} - t$	$\frac{1}{6} - t$	t	$\frac{1}{6}$	$\frac{1}{6} + t$	$\frac{1}{6}$	$0 < t < \frac{1}{6}$
4	$2t$	$\frac{1}{2} - t$	$2t$	$\frac{1}{6} - 2t$	t	$\frac{1}{6} - t$	t	$\frac{1}{6} - 2t$	$0 < t < \frac{1}{12}$
12	$2t$	$\frac{1}{6} - t$	t	$\frac{1}{6} - t$	t	$\frac{1}{6} - t$	$2t$	$\frac{1}{2} - 3t$	$0 < t < \frac{1}{6}$

Table 2

Family									Range; $j \in \mathbb{N}$
1	j	j	j	$-2j+k$	k	$j+2k$	k	$-2j+k$	$0 < j < k/2$
2	j	$-j+k$	$-j+k$	$-j+k$	j	k	$j+k$	k	$0 < j < k$
4	$2j$	$-j+3k$	$2j$	$-2j+k$	j	$-j+k$	j	$-2j+k$	$0 < j < k/2$
12	$2j$	$-j+k$	j	$-j+k$	j	$-j+k$	$2j$	$-3j+3k$	$0 < j < k$

Table 3

Family	Configuration	Parameter range
1	$6k\#(j+k)_{(2j+5k)}(2k)_{(j+4k)}$	$j = 1, \ldots, \hat{k}/2$
2	$6k\#(3k-2j)_j(3k-j)_{(4k)}$	$j = 1, \ldots, k-1$
4	$6k\#(2k-j)_{(k-2j)}(2k)_{(k+j)}$	$j = 1, \ldots, \hat{k}/2$
12	$6k\#(2k)_{(k-j)}(j+2k)_{(2j+k)}$	$j = 1, \ldots, k-1$

Table 4

Family	Reparametrization	New configuration	New parameter range
1	$j \mapsto k-i$	$6k\#(3k-i)_{(3k-2i)}(2k)_{-i}$	$i = \widehat{3k}/2+1, \ldots, 2k-1$
4	$j \mapsto i-k$	$6k\#(3k-i)_{(3k-2i)}(2k)_i$	$i = k+1, \ldots, \widehat{3k}/2$
12	$j \mapsto 2k-i$	$6k\#(2k)_i(3k-i)_{3k-2i}$	$i = 1, \ldots, k-1$

First, consider Family 2. Since $6k\#(3k-2j)_j(3k-j)_{(4k)}$ is equivalent to $6k\#(3k-2j)_j(3k-j)_{(2k)}$ using identity (2), this is Grünbaum's second family, with $j = 1, \ldots, k-1$.

Families 1, 4, and 12 together form Grünbaum's first family of configurations. To see this, first reparametrize as indicated in Table 4.

Applying identity (3) to family 1, identity (1) to family 12, and replacing i with j, the three families combine to form Grünbaum's first family.

3. The Sporadic Configurations

In finding the infinite families of quadruples, Poonen and Rubinstein "merged" the infinite families of triples they had found, by developing a system of linear conditions to determine when two triples overlapped to form a quadruple. Mostly, the one-parameter families of triples combined to form one-parameter families of quadruples; however, for small values of m, they found a finite number of quadruples which were formed from two infinite-family triples but were not themselves members of the infinite families of quadruples [4]. It can be verified that these particular quadruples, which all have $m = 12, 18, 24$, correspond to configurations which are members of the infinite families.

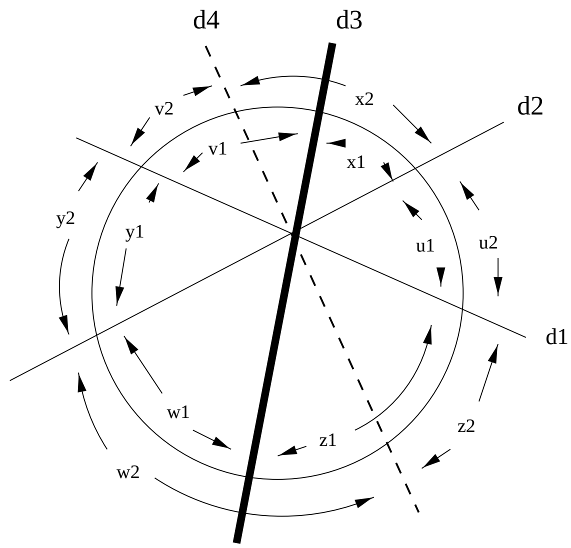

Fig. 4. Relationships between arclengths of intersecting triples.

Therefore, any sporadic astral configurations must contain a sporadic triple, since all the configurations which are formed from two overlapping infinite-family triples are in the infinite families of configurations.

Given four diagonals which intersect in a single point, the quadruple may be decomposed as two sets of three diagonals, each of which intersect in the same point: i.e, if d_1, d_2, d_3, and d_4 are the four diagonals, then $\{d_1, d_2, d_3\}$ form one triple and $\{d_1, d_2, d_4\}$ form the other triple. Following the notation in [4], the triple $\{d_1, d_2, d_3\}$ decomposes the circle into the six arclengths $\{u_1, x_1, v_1, y_1, w_1, z_1\}$, and the triple $\{d_1, d_2, d_4\}$ decomposes the circle into the six arclengths $\{u_2, x_2, v_2, y_2, w_2, z_2\}$. If two distinct triples overlap to form a quadruple, where the first two diagonals are the same, then the following relationships must hold (see Fig. 4):

$$\begin{aligned} u_1 &= u_2, \\ y_1 &= y_2, \\ v_1 + x_1 &= v_2 + x_2, \\ w_1 + z_1 &= w_2 + z_2, \\ x_1 &\neq x_2 \qquad \text{(so that the triples are distinct).} \end{aligned}$$

If the pair of triples satisfies the above equations, they are called *mergeable*. In this case the quadruple they form has arclengths $\{u_1, x_1, |x_2-x_1|, v_2, y_1, w_1, |w_2-w_1|, z_2\}$. I call the process of comparing pairs of triples to see if they satisfy the necessary equations *merging* the triples.

To determine the sporadic configurations, the sporadic triples must be merged with each other and with the infinite-family triples.

Poonen and Rubinstein proved the following lemma (note that here "configuration" simply refers to a certain set of diagonals, and "denominator" refers to the least common denominator of the arclengths in the set of diagonals, i.e., m):

"**Lemma 5.1.** *If a configuration of $k \geq 2$ diagonals meeting at an interior point other than the center has denominator dividing d, then any configuration of diagonals meeting at that point has denominator dividing* $\mathrm{LCM}(2d, 3)$*"* [4, *p*. 146].

Thus, it suffices to check whether the sporadic triples of a given denominator are mergeable only with triples with denominators as indicated by the lemma. That is,

Sporadic triples	Possibly mergeable triples
$m =$ 30	$m =$ 30, 60
42	42, 84
60	30, 60, 120
84	42, 84, 168
90	30, 60, 90, 180
120	30, 60, 120
210	30, 42, 60, 84, 210, 420

Using *Mathematica*, the sporadic triples were merged with triples of the appropriate denominator, and the resulting octuples were converted into configurations. The sporadic configurations obtained are listed in Table 5.

Merging the sporadic triples of the other denominators either resulted in octuples from the merge which did not correspond to configurations ($m = 84, 120$; the octuples did not contain two pairs of diagonals of the same span) or in no results from the merge ($m = 90, 210$).

Table 5

	$m = 30$	
$30\#4_1 7_6$	$30\#6_1 7_4$	$30\#6_1 11_{10}$
$30\#6_2 8_6$	$30\#7_2 12_{11}$	$30\#8_1 13_{12}$*
$30\#10_1 11_6$	$30\#10_6 12_{10}$	$30\#10_7 13_{12}$
$30\#11_2 12_7$	$30\#11_6 14_{13}$	$30\#12_1 13_8$
$30\#12_4 14_{12}$	$30\#12_7 13_{10}$	$30\#13_6 14_{11}$
	$m = 42$	
$42\#6_1 13_{12}$	$42\#11_6 18_{17}$	$42\#12_1 13_6$
$42\#12_5 19_{18}$	$42\#17_6 18_{11}$	$42\#18_5 19_{12}$†
	$m = 60$	
$60\#9_2 22_{21}$	$60\#12_5 25_{24}$	$60\#14_3 27_{26}$
$60\#21_2 22_9$	$60\#24_5 25_{12}$	$60\#26_3 27_{14}$

* Erroneously listed in [2] as $30\#8_1 13_2$.
† Erroneously listed in [2] as $42\#17_2 19_{14}$.

These are all possible connected sporadic astral (n_4) configurations (note this list corrects a few typos in [2]).

Combining the results of Sections 2 and 3 with Corollary 1 completes the proof of Theorem 1.

4. Other Astral (n_4) Configurations

Theorem 1 completely characterizes astral (n_4) configurations whose vertices lie on the vertices of two concentric regular m-gons. However, there are some astral (n_4) configurations where the vertices of the configuration do not lie on two regular m-gons; see Fig. 5.

Theorem 3. *All astral* (n_4) *configurations whose vertices do not lie on the vertices of two concentric regular* m*-gons may be constructed by taking two concentric copies of one of the astral configurations indicated in Theorem* 1, *where one copy is rotated through an arbitrary angle* (*i.e., other than* π/m) *with respect to the other.*

Proof. First, consider a configuration (the "main configuration") which is constructed from two copies of an astral configuration listed in Theorem 1. Such a configuration is astral, since if the configuration is made of two copies of a configuration $m\#a_b c_d$ rotated as indicated, then using a combination of rotation through $2\pi/m$ and reflection through the lines of symmetry in the main configuration, it is clear that the span a lines

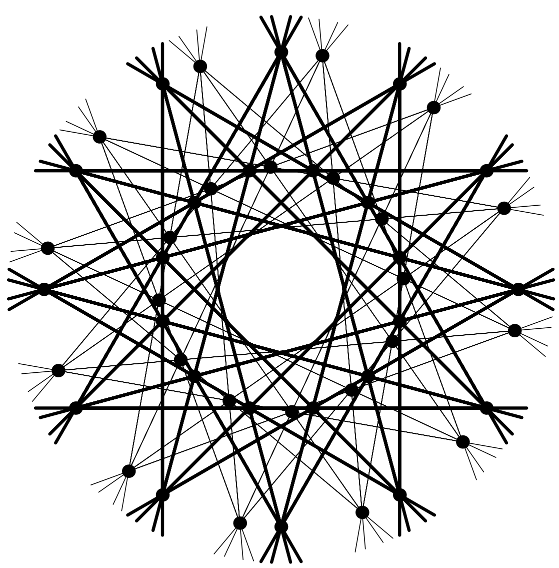

Fig. 5. An astral configuration (48_4), whose vertices do not lie on the vertices of two concentric m-gons; it is constructed from two copies of $12\#4_1 5_4$, with one copy shown with thicker lines.

of both subconfigurations form one symmetry class of lines and the span c lines of both subconfigurations form the other, and that the vertices form two symmetry classes, one on the outer circle and one on the inner circle.

Given an astral (n_4) configuration whose vertices do not lie on the vertices of a regular polygon, notice that due to symmetry considerations, the connected component of a vertex v must be one of the configurations listed in Theorem 1. A configuration constructed from more than two concentric copies of a Theorem 1 configuration rotated through arbitrary angles with respect to one of them results in more than two symmetry classes of points and lines, so the (n_4) configuration is not astral. □

5. Remarks

1. Given the characterization of an astral (n_4) configuration whose vertices lie on the vertices of two concentric regular polygons as one which is formed from two pairs of same-span diagonals of a regular $m = (n/2)$-gon which intersect in a common point, it may seem that it should be straightforward to determine all such astral (n_4) configurations. However, there turned out to be significant subtleties.

2. All computations were done in *Mathematica*. Annotated source code is available at http://www.math.washington.edu/˜berman.

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