Discrete Comput Geom 26:527–541 (2001)
DOI: 10.1007/s00454-001-0035-x

Discrete & Computational
Geometry

Solution of the Illumination Problem for Bodies with md $M = 2$

V. Boltyanski

El Centro de Investigación en Matematicas,
A.P. 402, 36000 Guanajuato, GTO, Mexico
boltian@cimat.mx

Abstract. We solve here the *Gohberg–Markus–Hadwiger Covering Problem* (or, what is the same, the *illumination problem*) for compact, convex bodies $M \subset R^n$ with md $M = 2$. Moreover, we outline an idea for a complete solution, using md M.

1. Introduction

Let $M \subset R^n$ be a be a compact, convex body, let $q \in R^n$ be an arbitrary point, and let k be a real number with $0 < k < 1$. The image of M under the homothety with center q and ratio k is said to be a *diminished copy* of M. The least integer p such that M can be covered by p diminished copies of M is denoted by $b(M)$.

In 1957 Gohberg and Markus proved the following theorem (by singular conditions in the USSR, their article was not published until 1960 [21]).

Theorem 1. *If $M \subset R^2$ is a compact, convex figure distinct from a parallelogram, then $b(M) = 3$. For every parallelogram $b(M) = 4$.*

An equivalent theorem (in another form) was proved by Levi [27].

Observing that $b(M) = 2^n$ for every n-dimensional parallelotope, Gohberg and Markus formulated the following problem:

Prove that $b(M) \le 2^n$ for every compact, convex body $M \subset R^n$, the equality being held only for parallelotopes.

The same problem was formulated by Hadwiger [23], based on [27]. Therefore we call it the *GMH problem*.

Boltyanski [8] formulated the *Illumination Problem*. A boundary point a of a convex body $M \subset R^n$ is *illuminated* by the direction of a nonzero vector $e \in R^n$ if, for $\lambda > 0$ small enough, the point $a + \lambda e$ belongs to the interior of M. Furthermore, we say that the directions of nonzero vectors $e_1, \ldots, e_p$ *illuminate* the boundary of M if every point $x \in \operatorname{bd} M$ is illuminated by at least one of these directions. The least integer p such that there exist nonzero vectors $e_1, \ldots, e_p$, whose directions illuminate the boundary of M, is denoted by $c(M)$.

Theorem 2 [8]. *For every compact, convex body $M \subset R^n$, the equality $b(M) = c(M)$ holds.*

This theorem allows us to give another form of the GMH problem:

Illumination Problem. Prove that $c(M) \le 2^n$ for every compact, convex body $M \subset R^n$, the equality being held only for parallelotopes.

Furthermore, Hadwiger [24] formulated the problem to illuminate the boundary of a compact, convex body $M \subset R^n$ by a minimal number of *point light sources* situated in R^n. For compact, convex bodies in R^n, the Levi problem [27], the GMH problem [21], [23], the illumination problem [8], and the problem of illuminating by point sources [24] are equivalent (see Theorem 34.3 in [17]).

There are some partial results in this direction. If a compact, convex body $M \subset R^n$ is smooth (this means that every boundary point a is *regular*, i.e., there is only one support hyperplane of M through a), then $b(M) = n + 1$. This may be implied from [22]. In [8] Boltyanski proved a more general theorem: *If a compact, convex body $M \subset R^n$ has no more than n nonregular boundary points, then $b(M) = n + 1$.* For $n = 3$, Charazishvili [20] obtained a finer result: *If a compact, convex body $M \subset R^3$ has no more than four nonregular boundary points, then $b(M) = 4$.*

Lassak [26] proved that *if a compact, convex three-dimensional body M is centrally symmetric, then $b(M) \le 8$.* In the same paper [26] Lassak proved that *for every three-dimensional body M of constant width the inequality $b(M) \le 6$ holds.* Furthermore, Bezdek [4] justified the illumination problem for every convex polytope $M \subset R^3$ with affine symmetry, i.e., $c(M) \le 8$ in that case. Using information recently obtained, Dekster proved that $c(M) \le 8$ *for every three-dimensional compact, convex body that is symmetric about a plane.*

Martini [28] established that *for every n-dimensional zonotope distinct from a parallelotope the inequality $c(M) \le 3 \cdot 2^{n-2}$ holds.* We call this inequality *Martini's estimate*. Boltyanski and Soltan proved [19] that Martini's estimate holds for all zonoids and Boltyanski [13] established this estimate for all belt bodies. In [16] Boltyanski and Martini described all belt bodies for which Martini's estimate holds and proved the inequality $c(M) \le 5 \cdot 2^{n-3}$ for other belt bodies.

We note that Bezdek [5] found the dual formulation of the illumination problem (see Lemma 1 in Section 4 below). With the help of that formulation, Bezdek [6] gave dual proofs for several of the above-mentioned results. Finally, in [7] Bezdek and Bisztriczki proved that the illumination problem has a positive solution for all n-dimensional cyclic polytopes.

2. The Functional md

In 1976 Boltyanski introduced the functional md [10]. Let H be a subset of the unit sphere S^{n-1} that is not one-sided, i.e., H is not contained in any closed hemisphere of S^{n-1}. In other words, there is no vector $e \neq 0$ with $\langle a, e\rangle \leq 0$ for every $a \in H$.

Furthermore, let $a_0, a_1, \ldots, a_m$ be vectors in R^n, $1 \leq m \leq n$. We say that the vectors are *minimally dependent* if

(i) they are positively dependent (i.e., there are positive coefficients $\lambda_0, \lambda_1, \ldots, \lambda_m$ with $\lambda_0 a_0 + \lambda_1 a_1 + \cdots + \lambda_m a_m = 0$);
(ii) any m of the vectors $a_0, a_1, \ldots, a_m$ are linearly independent.

In other words, $a_0, a_1, \ldots, a_m$ are minimally dependent if they are the vertices of an m-dimensional simplex that contains the origin in its relative interior.

Finally, by md H we denote the greatest integer m such that H contains m minimally dependent vectors (md $H = 0$ if there is no system of minimally dependent vectors in H). Note that $0 \leq \operatorname{md} H \leq n-1$ if H is one-sided, whereas $1 \leq \operatorname{md} H \leq n$ if H is not one-sided.

The first result obtained with the help of md H was a generalization of the classical Helly theorem. We formulate this result, since it is connected with the illumination problem.

Let $H \subset S^{n-1}$ be a subset that is not one-sided. A closed half-space P of R^n is said to be *H-convex* if its outward unit normal belongs to H. Furthermore, a closed set $M \subset R^n$ is said to be *H-convex* [9] if it is representable as the intersection of a family of H-convex half-spaces.

Theorem 3 [10]. *Let $M_1, \ldots, M_q$ be H-convex sets in R^n, $q \geq \operatorname{md} H + 2$. If every* $\operatorname{md} H + 1$ *of the sets has a point in common, then $M_1 \cap \cdots \cap M_q \neq \emptyset$.*

A similar theorem holds for any infinite family of H-convex sets if at least one of them is compact.

Now let $M \subset R^n$ be a compact, convex body. By $H(M) \subset S^{n-1}$ we denote the set of all vectors, each of which is the unit outward normal of M at a regular boundary point. For brevity, the integer md $H(M)$ is denoted by md M. The set $H(M)$ is not one-sided, and therefore $1 \leq \operatorname{md} M \leq n$.

Furthermore, we denote the family of all translates of M by $T(M)$ and its Helly dimension [29] by $\operatorname{him} T(M)$, i.e., the minimal integer p such that for every subfamily $\{M_1, \ldots, M_q\} \subset T(M)$ with $q > p+1$ the following assertion holds: if every $p+1$ of the sets $M_1, \ldots, M_q$ has a point in common, then $M_1 \cap \cdots \cap M_q \neq \emptyset$.

Szökefalvi-Nagy [31] established the following result:

Theorem 4. *Let $M \subset R^n$ be a compact, convex body. The equality* $\operatorname{him} T(M) = 1$ *holds if and only if M is an n-dimensional parallelotope.*

It is easily shown that $\operatorname{md} M = 1$ if and only if M is a parallelotope. Thus the Szökefalvi-Nagy theorem may be formulated in the following form: *The equality* him

$T(M) = 1$ *holds if and only if* $\operatorname{md} M = 1$. In [10] there is a generalization of this assertion for arbitrary $\operatorname{md} M$:

Theorem 5. *For every compact, convex body* $M \subset R^n$, *the equality* $\operatorname{him} T(M) = \operatorname{md} M$ *holds.*

We remark that the Szökefalvi-Nagy theorem contains little more than the particular case $\operatorname{md} M = 1$ of Theorem 5. Indeed, Theorem 4 contains a *geometrical description* of all compact, convex bodies with $\operatorname{him} T(M) = 1$. In this connection the following problem arises:

Szökefalvi-Nagy Problem: *Give a geometrical description of all compact, convex bodies with* $\operatorname{him} T(M) = m$ (*i.e.*, $\operatorname{md} M = m$) *for* $m = 2, \dots, n$.

(Note that the term *Szökefalvi-Nagy problem* was introduced by Boltyanski.) This problem is solved in [15] and [12] for $m = 2$. The following theorem [15] describes all three-dimensional bodies with $\operatorname{md} M = 2$; the *stacks* and the *outcuts* mentioned in its statement are defined below.

Theorem 6. *For a compact, convex body* $M \subset R^3$ *that is distinct from a parallelotope, the equality* $\operatorname{md} M = 2$ *holds in and only in the following three cases*: (a) M *is the direct vector sum of a segment and a two-dimensional figure distinct from a parallelogram*; (b) M *is a stack*; (c) M *is an outcut.*

Now we give descriptions of the stacks and outcuts mentioned in Theorem 6. A compact, convex body $M \subset R^3$ is a *stack* if it is representable (up to a translate) in the following form. Let I_1, I_2, I_3 be three segments with a common endpoint 0 which are not contained in a plane. Denote the parallelogram $I_i \oplus I_j$ by P_{ij}. Let $F_1 \subset P_{13}$ (respectively, $F_2 \subset P_{23}$) be a compact, convex figure that contains the side I_1 (respectively, I_2) of the parallelogram and at least one point of its opposite side, but does not contain this opposite side. Furthermore, let U_1 (respectively, U_2) be the infinite cylinder with the basis F_1 (respectively, F_2) and the generator parallel to I_2 (respectively, I_1). Then $M = U_1 \cap U_2$.

A compact, convex body $M \subset R^3$ is an *outcut* if it is representable (up to a translate) in the following form. Let I_1, I_2, I_3 be as above. For every two indices $i < j = 2, 3$ choose a compact, convex figure $G_{ij} \subset I_i \oplus I_j$ that contains both the segments I_i, I_j, but does not coincide with the parallelogram $I_i \oplus I_j$. Furthermore, let V_{ij} be the infinite cylinder with the basis G_{ij} and the generator parallel to the segment (taken from I_1, I_2, I_3) distinct from I_i, I_j. Then $M = V_{12} \cap V_{13} \cap V_{23}$.

The articles [1]–[3], [11], and [25] contain further results on the Szökefalvi-Nagy problem. Some other applications of $\operatorname{md} M$ are given in Section 44 of [17].

3. An Idea of a Solution

First we discuss the illumination problem for the case $n = 3$. Let $M \subset R^3$ be a compact, convex body. If $\operatorname{md} M = 1$, then $c(M) = 8$.

If $\operatorname{md} M = 2$, then M is one of the bodies described in Theorem 6. It is easily shown that in case (a) the equality $c(M) = 6$ holds. Indeed, let $M = I \oplus F$ where $I = [a, b]$ is a segment and F is a compact, convex two-dimensional figure distinct from a parallelogram. The cylinder M has two bases $a \oplus F$ and $b \oplus F$. It is necessary to have three directions to illuminate all points of the base $a \oplus F$. None of these directions illuminates any point of the opposite base $b \oplus F$. Therefore it is necessary to have three more directions to illuminate all points of the base $b \oplus F$. Thus $c(M) = 6$.

Furthermore, if we have case (b) of Theorem 6, i.e., M is a stack, then $c(M) = 5$. Indeed, in the above description of the stack, denote by e_1, e_2, e_3 the endpoints of the segments I_1, I_2, I_3 distinct from 0, respectively. We say that $I_1 \oplus I_2$ is the *lower* base of the stack M and denote its *upper* base by P, i.e., $P = M \cap \Gamma$ where Γ is the support plane of M parallel to the lower base. We remark that P is a parallelogram (which may degenerate into a segment or a point) with the sides parallel to ones of the lower base. In addition, every side of P is smaller than the corresponding side of the lower base (since the figure F_1 does not coincide with $I_1 \oplus I_3$ and analogously for F_2). Therefore the stack M is not the direct vector sum as in case (a). Consider the directions of four vectors:

$$\begin{aligned} p_1 &= e_1 + e_2 + \lambda e_3, & p_2 &= e_1 - e_2 + \lambda e_3, \\ p_3 &= -e_1 + e_2 + \lambda e_3, & p_4 &= -e_1 - e_2 + \lambda e_3. \end{aligned}$$

If $\lambda > 0$ is small enough, these four directions illuminate the whole boundary of M except for a small neighborhood of the upper base P. Furthermore, denote by p_5 the vector emanating from the center of the upper base of M and going to the center of the lower base. Since the sides of the upper base are *smaller* than the corresponding sides of the lower base, the direction of the vector p_5 illuminates the upper base P with its neighborhood. Thus the boundary of M is illuminated by five directions, i.e., $c(M) \leq 5$. On the other hand, consider the vertices a_1, a_2, a_3, a_4 of the lower base $I_1 \oplus I_2$ and the center a_5 of the upper base. Every two of these five points is *antipodal*, i.e., is situated in two parallel support planes of M. Consequently, no direction simultaneously illuminates *two* of the points a_1, a_2, a_3, a_4, a_5, and therefore $c(M) \geq 5$. This proves the equality $c(M) = 5$.

At last, if M is an outcut, then again $c(M) = 5$. Indeed, let e_1, e_2, e_3 be as above. Consider the directions of the vectors

$$q_1 = -e_1 + \lambda(e_2 + e_3), \qquad q_2 = -e_2 + \lambda(e_1 + e_3), \quad q_3 = -e_3 + \lambda(e_1 + e_2),$$
$$q_4 = e_1 + e_2 + e_3, \qquad q_5 = -e_1 - e_2 - e_3.$$

If $\lambda > 0$ is small enough, these five directions illuminate the whole boundary of M. To prove this, we set $G = G_{12} \cup G_{13} \cup G_{23}$, $Q = \mathrm{cl}(\mathrm{bd}\, M \setminus G)$ and denote the intersection $Q \cap G_{ij}$ by B_{ij}, $i < j = 2, 3$. Let g_{ij} be a point of the arc B_{ij} not belonging to the relative boundary of the parallelogram $I_i \oplus I_j$, $i < j = 2, 3$. Then (if $\lambda > 0$ is small enough) the direction of q_1 illuminates the part of the arc B_{12} with endpoints b_1, g_{12} and the direction of q_2 illuminates the part of the arc B_{12} with endpoints b_2, g_{12}. Therefore both directions illuminate all points of the arc B_{12}. By similar reasoning, the directions of q_1, q_2, q_3 illuminate all points of the set $B_{12} \cup B_{13} \cup B_{23}$. Furthermore, every point of the set $G \setminus (B_{12} \cup B_{13} \cup B_{23})$ is illuminated by the direction of q_4. Finally, all points of the set $Q \setminus (B_{12} \cup B_{13} \cup B_{23})$ are illuminated by the direction of q_5. Thus the boundary of M is illuminated by five directions, i.e., $c(M) \leq 5$. On the other hand, consider the five points $0, e_1, e_2, e_3, b$ where $b \in \operatorname{bd} V_{12} \cap \operatorname{bd} V_{13} \cap \operatorname{bd} V_{23}$ is distinct from 0. Every two of these five points is antipodal, and therefore $c(M) \geq 5$. This proves the equality $c(M) = 5$.

Combining cases (a)–(c) considered in Theorem 6, we obtain that if $\operatorname{md} M = 2$, then $c(M) \leq 6$, i.e., $c(M) \leq 2^3 - 2^1$. Consequently, *to solve the illumination problem for three-dimensional bodies, it is enough to establish that if* $\operatorname{md} M = 3$, *then* $c(M) \leq 7$, *i.e.*, $c(M) \leq 2^3 - 2^0$. Nevertheless, the exotic nature of the bodies $M \subset R^3$ with $\operatorname{md} M = 3$ remains outside the framework of this article.

The above discussion leads us to the following:

Hypothesis. *If $M \subset R^n$ is a compact, convex body with* $\operatorname{md} M = m \geq 2$, *then* $c(M) \leq 2^n - 2^{n-m}$.

In this article we justify this hypothesis for $m = 2$, i.e., we prove the following assertion:

Main Theorem. *Let $M \subset R^n$ be a compact, convex body with* $\operatorname{md} M = 2$. *Then* $c(M) \leq 2^n - 2^{n-2}$, *i.e., Martini's estimate* $c(M) \leq \frac{3}{4} \cdot 2^n$ *holds.*

To prove the Main Theorem, first we give the polar descriptions for the illumination problem and the functional md.

4. Polar Description of the Problem

In what follows we suppose that R^n is an n-dimensional, vectorial Euclidean space that is self-adjoint (i.e., the scalar product is introduced in R^n).

For every point $x \in R^n$ distinct from the origin 0, we denote its *polar hyperplane* by x^*, i.e., $x^* = \{y : \langle x, y \rangle = 1\}$. Similarly, for every hyperplane $\Gamma \subset R^n$ not containing 0, we denote its *polar point* (i.e., the point for which the polar hyperplane coincides with Γ) by Γ^*. Furthermore, for every compact, convex body containing 0 in its interior, we denote its *polar body* by M^*:

$$M^* = \{y : \langle y, a \rangle \leq 1 \text{ for all } a \in M\}.$$

Let $M \subset R^n$ be a compact, convex body containing 0 in its interior, and let M^* be its polar body. Let $P_1, \dots, P_k$ be half-spaces of R^n with $0 \in \operatorname{bd} P_i$, $i = 1, \dots, k$. We say that the system of half-spaces $\{P_1, \dots, P_k\}$ *co-illuminates* the body M^* if, for every proper face F of M^* (i.e., $F \neq M^*$), there is an index $i \in \{1, \dots, k\}$ such that $F \subset \operatorname{int} P_i$. By $c^*(M^*)$ we denote the least integer k for which there exists a co-illuminating system $\{P_1, \dots, P_k\}$ for the body M^*.

Lemma 1. *For every compact, convex body $M \subset R^n$ containing the origin in its interior, the equality $c(M) = c^*(M^*)$ holds.*

This lemma is already known. For example, Soltan and Soltan [30] applied this lemma in their solution of the X-raying problem for three-dimensional polytopes. Moreover, there is a stronger form of this lemma in [4]. Nevertheless, for completeness, we give a proof of this lemma.

Proof. Denote the integer $c(M)$ by k, and let $p_1, \dots, p_k$ be unit vectors whose directions illuminate the boundary of M. Consider the half-spaces $P_1, \dots, P_k$ with outward normals $p_1, \dots, p_k$, respectively, which contain the origin in their boundary hyperplanes:

$$P_i = \{x : \langle p_i, x\rangle \le 0\}, \qquad i = 1, \dots, k.$$

Let F be a proper face of the body M^* and let Γ be a support hyperplane of M^* that contains F. Then $x = \Gamma^*$ is a boundary point of M. For every point $z \in F$ the equality $\langle z, x\rangle = 1$ holds. Let $j \in \{1, \dots, k\}$ be an index such that the direction of the vector p_j illuminates the point $x \in \operatorname{bd} M$, i.e., $x + \lambda p_j \in \operatorname{int} M$ for $\lambda > 0$ small enough. We fix such a number λ. For each point x' contained in a small neighborhood of $x + \lambda p_j$ the inequality $\langle z, x'\rangle \le 1$ holds, and therefore $\langle z, x + \lambda p_j\rangle < 1$. Since $\langle z, x\rangle = 1$, we have $\langle z, \lambda p_j\rangle < 0$, and consequently $\langle z, p_j\rangle < 0$. This means that $z \in \operatorname{int} P_j$. This is true for every point $z \in F$, and hence $F \subset \operatorname{int} P_j$. Thus the system $\{P_1, \dots, P_k\}$ co-illuminates the body M^*, i.e., $c^*(M^*) \le c(M)$.

We now prove the inverse inequality. Denote the integer $c^*(M^*)$ by l, and let $Q_1, \dots$, Q_l be half-spaces with the boundaries through 0 which co-illuminate the body M^*. Denote by $q_1, \dots, q_l$ the unit outward normals of the half-spaces $Q_1, \dots, Q_l$, respectively. We show that the directions of the vectors $q_1, \dots, q_l$ illuminate the boundary of M.

Indeed, let $u \in \operatorname{bd} M$. Then u^* is a support hyperplane of the body M^*. The intersection $G = u^* \cap M^*$ is a proper face of the body M^*. Let $j \in \{1, \dots, l\}$ be an index for which $G \subset \operatorname{int} Q_j$, i.e., $\langle v, q_j\rangle < 0$ for every point $v \in G$. Since the set G is compact, there exists a number $\mu > 0$ such that $\langle v, q_j\rangle < -\mu$ for every $v \in G$. Hence there exists a neighborhood W of G such that $\langle v, q_j\rangle < 0$ for every point $v \in W$. Moreover, since $\langle v, u\rangle \le 1$ when $v \in M^*$, we have

$$\langle v, u + \lambda q_j\rangle = \langle v, u\rangle + \langle v, \lambda q_j\rangle \le 1 + \lambda\langle v, q_j\rangle < 1$$

for every point $v \in M^* \cap W$ and every $\lambda > 0$.

Furthermore, since $u^* \cap (M^* \backslash W) = \emptyset$, for every point $v \in M^* \backslash W$ the inequality $\langle v, u\rangle < 1$ holds. By compactness of the set $M^* \backslash W$, there exists $\varepsilon > 0$ such that $\langle v, u\rangle < 1 - \varepsilon$ for every $v \in M^* \backslash W$. This implies

$$\langle v, u + \lambda q_j\rangle < 1 \qquad \text{for} \quad v \in M^* \backslash W$$

if $\lambda > 0$ is small enough. We fix such a number λ. Combining both the cases $v \in M^* \cap W$ and $v \in M^* \backslash W$, we conclude that for every point $v \in M^*$ the inequality $\langle v, u + \lambda q_j\rangle < 1$ holds. This means that $u + \lambda q_j$ is an *interior* point of the body M, i.e., the point u is illuminated by the direction of the vector q_j. Thus the directions of the vectors $q_1, \dots, q_l$ illuminate the boundary of M, i.e., $c(M) \le c^*(M^*)$. □

Recall that a boundary point b of a convex body $N \subset R^n$ is *exposed* if there exists a support hyperplane Γ of N such that $\Gamma \cap N = \{b\}$. The set of all exposed boundary points of N is denoted by $\exp N$.

Lemma 2. *Let $M \subset R^n$ be a compact, convex body with the origin* 0 *in its interior and let M^* be its polar body. Let, furthermore, $\Gamma \subset R^n$ be a support hyperplane of M^* that has only one point b in common with M^*. Then Γ^* is a regular boundary point of M and b^* is the only support hyperplane of M through Γ^*. Conversely, if Γ^* is a regular boundary point of M and b^* is the only support hyperplane of M through Γ^*, then b is an exposed boundary point of M^* and Γ is a support hyperplane of M^* which has only the point b in common with M^*. Moreover, in both the cases, b is the outward normal of M at the point Γ^*.*

This lemma is already known (see, for example, Theorem 3.2 in [17]).

Now we can give a polar description of the functional md. Let $M \subset R^n$ be a compact, convex body containing the origin 0 in its interior, and let M^* be its polar body. By $\mathrm{md}^*(M^*)$ we denote the largest integer k for which there are points $b_0, b_1, \dots, b_k \in \exp(M^*)$ such that $\mathrm{conv}\{b_0, b_1, \dots, b_k\}$ is a k-dimensional simplex containing 0 in its relative interior.

Lemma 3. *Let $M \subset R^n$ be a compact, convex body containing the origin* 0 *in its interior, and let M^* be its polar body. Then* $\mathrm{md}^*(M^*) = \mathrm{md}\, M$.

Proof. Denote the integer $\mathrm{md}^*(M^*)$ by k. Let $b_0, b_1, \dots, b_k$ be the exposed boundary points of M^* such that $\mathrm{conv}\{b_0, b_1, \dots, b_k\}$ is a k-dimensional simplex containing 0 in its relative interior. Let $\Gamma_0, \Gamma_1, \dots, \Gamma_k$ be support hyperplanes of M^* with $\Gamma_i \cap M^* = \{b_i\}$, $i = 0, 1, \dots, k$. By virtue of Lemma 2, $\Gamma_i{}^*$ is a regular boundary point of M, and $b_i{}^*$ is the support hyperplane of M through $\Gamma_i{}^*$, $i = 0, 1, \dots, k$. Moreover, b_i is an outward normal of the body M at the point $\Gamma_i{}^*$, i.e., $b_i = \mu_i p_i$ where $\mu_i > 0$ and p_i is the unit outward normal of M at the point $\Gamma_i{}^*$, $i = 0, 1, \dots, k$. Since 0 is a relative interior point of the simplex $\mathrm{conv}\{b_0, b_1, \dots, b_k\}$, there are positive numbers $\lambda_0, \lambda_1, \dots, \lambda_k$ such that $\lambda_0 b_0 + \lambda_1 b_1 + \cdots + \lambda_k b_k = 0$, i.e.,

$$\lambda_0 \mu_0 p_0 + \lambda_1 \mu_1 p_1 + \cdots + \lambda_k \mu_k p_k = 0,$$

and moreover every k of the vectors $b_0, b_1, \dots, b_k$ (i.e., every k of the vectors $p_0, p_1, \dots, p_k$) are linearly independent. This means that the vectors $p_0, p_1, \dots, p_k$ are minimally dependent. Hence, $\mathrm{md}\, M \geq k$, i.e., $\mathrm{md}\, M \geq \mathrm{md}^*(M^*)$.

Now we prove the opposite inequality. Denote the integer $\mathrm{md}(M)$ by l. Let $\Gamma_0^*, \Gamma_1^*, \dots, \Gamma_l^*$ be regular boundary points of M such that the unit outward normals $p_0, p_1, \dots, p_l$ of the body M at these points are minimally dependent. Denote the support hyperplanes of M at these points by $b_0^*, b_1^*, \dots, b_l^*$, respectively. By virtue of Lemma 2, b_i is an exposed boundary point of M^* and Γ_i is a support hyperplane of M^* which has only the point b_i in common with M^*, $i = 0, 1, \dots, l$. Since the vectors $p_0, p_1, \dots, p_l$ are minimally dependent, every l of them is linearly independent and there are positive numbers $\nu_0, \nu_1, \dots, \nu_l$ such that $\nu_0 p_0 + \nu_1 p_1 + \cdots + \nu_k p_k = 0$. Furthermore, $b_i = \mu_i p_i$, $i = 0, 1, \dots, l$, where $\mu_0, \mu_1, \dots, \mu_l$ are positive numbers. Therefore we have the dependence

$$\frac{\nu_0}{\mu_0} \quad b_0 + \frac{\nu_1}{\mu_1} \quad b_1 + \cdots + \frac{\nu_l}{\mu_l} \quad b_l = 0$$

with positive coefficients. This means that $b_0, b_1, \ldots, b_l$ are the vertices of an l-dimensional simplex containing 0 in its relative interior. Hence, $\operatorname{md}^*(M^*) \geq l$, i.e., $\operatorname{md}^*(M^*) \geq \operatorname{md} M$. □

5. Convex Bodies with md $M = 2$

In this section we formulate the solution of the Szökefalvi-Nagy problem for the case $\operatorname{md} M = 2$ given in [12]. We explain this solution here, since it is used below in the proof of the Main Theorem.

Let M be a compact, convex body with $\operatorname{md} M = 2$ and let $M = M_1 \oplus \cdots \oplus M_q$ be its decomposition into the direct vector sum of indecomposable convex sets. Since $\operatorname{md} M$ is the maximal of the integers $\operatorname{md} M_1, \ldots, \operatorname{md} M_q$ (see Theorem 25.4 in [17]), every integer $\operatorname{md} M_i$ is not greater than 2 ($i = 1, \ldots, q$), and for at least one index i the equality $\operatorname{md} M_i = 2$ holds. Without lost of generality, we may suppose that for an integer p, $1 \leq p \leq q$, we have $\operatorname{md} M_1 = \cdots = \operatorname{md} M_p = 2$ and (if $p < q$) $\operatorname{md} M_{p+1} = \cdots = \operatorname{md} M_q = 1$, i.e., the summands $M_{p+1}, \ldots, M_q$ are *segments* (since they are indecomposable).

Therefore to solve the Szökefalvi-Nagy problem for $\operatorname{md} M = 2$, it remains to give a geometrical description of all compact, convex, indecomposable bodies with $\operatorname{md} M = 2$. This is made in Theorem 7 below. First we give some necessary definitions.

Definition 1. A compact, convex, indecomposable body $M \subset R^n$ is said to be an n-dimensional *stack* if it is representable (up to a translate) in the following form. Let $I_1, \ldots, I_n$ be n segments with a common endpoint 0 which are not contained in a hyperplane. Furthermore, let $F_j \subset I_j \oplus I_n$ be a compact, convex figure that contains the side I_j of the parallelogram $I_j \oplus I_n$ and at least one point of its opposite side, $j = 1, \ldots, n-1$. Finally, let $U_j = F_j \oplus L_{jn}$ where L_{jn} is the $(n-2)$-dimensional subspace containing all segments $I_1, \ldots, I_n$ except I_j and I_n. Then $M = U_1 \cap \cdots \cap U_{n-1}$.

Definition 2. A compact, convex, indecomposable body $M \subset R^n$ is said to be an n-dimensional *outcut* if it is representable (up to a translate) in the following form. Let $I_1, \ldots, I_n$ be as above. For every two indices $i < j \leq n$ choose a compact, convex figure $G_{ij} \subset I_i \oplus I_j$ that contains both the segments I_i, I_j, but does not coincide with the parallelogram $I_i \oplus I_j$. Furthermore, let $V_{ij} = G_{ij} \oplus L_{ij}$ where L_{ij} is the $(n-2)$-dimensional subspace containing all segments $I_1, \ldots, I_n$ except I_i, I_j. Then $M = \bigcap_{i<j} V_{ij}$.

Definition 3. A compact, convex, indecomposable body $M \subset R^n$ is said to be an n-dimensional *stack-outcut* if it is representable (up to a translate) in the following form. Let $I_1, \ldots, I_n$ be as above. Let k be an integer, $2 \leq k < n$, and let $\varphi\colon \{k+1, \ldots, n\} \to \{1, \ldots, k\}$ be a map that is onto in the case $k = 2$. For every two indices $i < j \leq k$ choose a compact, convex figure $G_{ij} \subset I_i \oplus I_j$ that contains both the segments I_i, I_j, but does not coincide with the parallelogram $I_i \oplus I_j$. Furthermore, for every index $j \geq k+1$ choose a compact, convex figure $F_j \subset I_j \oplus I_{\varphi(j)}$ that contains the side I_j of the parallelogram

$I_j \oplus I_{\varphi(j)}$ and at least one point of its opposite side. Finally, let $V_{ij} = G_{ij} \oplus L_{ij}$ and $U_j = F_j \oplus L_{j\varphi(j)}$. Then $M = (\bigcap_{i<j\le k} V_{ij}) \cap (\bigcap_{j>k} U_j)$.

Definition 4. A polytope $M \subset R^4$ containing the origin in its interior (and any its translate) is said to be a *particular four-dimensional polytope* if there is a basis e_1, e_2, e_3, e_4 of R^4 such that

$$M^* = \operatorname{conv}\{e_1, e_2, e_3, e_4, -e_1 - e_2, -e_1 - e_3, -e_2 - e_4, -e_3 - e_4, e_1 + e_2 + e_3 + e_4\}.$$

Theorem 7. *Let $M \subset R^n$ be a compact, convex, indecomposable body. The equality* md $M = 2$ *holds in and only in the following five cases*:

(i) $n = 2$ *and M is a compact, convex, two-dimensional figure distinct from a parallelogram*.
(ii) $n \ge 3$ *and M is an n-dimensional stack*.
(iii) $n \ge 3$ *and M is an n-dimensional outcut*.
(iv) $n \ge 4$ *and M is an n-dimensional stack-outcut*.
(v) $n = 4$ *and M is a particular four-dimensional polytope*.

The proof is given in [12].

6. Proof of the Main Theorem

First we give estimates of the number $c(M)$ for the bodies described in Definitions 1–4.

Lemma 4. *Let $M \subset R^n$ be an n-dimensional stack. Then $c(M) < \frac{3}{4} \cdot 2^n$.*

Proof. We use the description of the stack given in Definition 1. Let $e_1, \dots, e_n$ be the endpoints of the segments $I_1, \dots, I_n$ distinct from 0, respectively. Denote by $S \subset R^n$ the $(n-1)$-dimensional subspace containing $I_1, \dots, I_{n-1}$. For every $t \in [0, 1]$ denote by $S^{(t)}$ the hyperplane of R^n parallel to S and passing through the point te_n. Since for every $j = 1, \dots, n-1$ the subspace L_{jn} is contained in S, the intersection $S^{(t)} \cap U_j$ coincides with $(S^{(t)} \cap F_j) \oplus L_{jn}$. Moreover, the intersection $I_j^{(t)} = S^{(t)} \cap F_j$ is a segment parallel to I_j, the length of $I_j^{(t)}$ being less than the length of I_j if t is close enough to 1 (since M is indecomposable and hence F_j does not coincide with the parallelogram $I_j \oplus I_n$). This implies

$$\begin{aligned} S^{(t)} \cap M &= S^{(t)} \cap U_1 \cap \cdots \cap U_{n-1} \\ &= \bigcap_{j=1}^{n-1} ((S^{(t)} \cap F_j) \oplus L_{jn}) = \bigcap_{j=1}^{n-1} (I_j^{(t)} \oplus L_{jn}), \end{aligned}$$

i.e., $P^{(t)} = S^{(t)} \cap M$ is an $(n-1)$-dimensional parallelotope with edges parallel to the segments $I_1, \dots, I_{n-1}$.

For $t = 0$ the parallelotope $P^{(0)}$ is the *lower* base of the stack M. For $t = 1$ the parallelotope $P^{(1)}$ is the *upper* base, i.e., $P^{(1)} = M \cap \Gamma$ where $\Gamma \subset R^n$ is the support hyperplane of M parallel to the lower base. We note that $P^{(1)}$ is an $(n-1)$-dimensional parallelotope which may degenerate into a parallelotope of a smaller dimension or into a point, and the edges of $P^{(1)}$ are parallel to ones of the lower base. Denote by p the vector emanating from the center of the upper base of M and going to the center of the lower base. Since the sides of the upper base are *smaller* than the corresponding sides of the lower base, the direction of the vector p illuminates the upper base $P^{(1)}$ and its neighborhood. In other words, there is a number τ, $0 < \tau < 1$, such that the part of bd M above $S^{(\tau)}$ is illuminated by p.

Consider now 2^{n-1} vectors $\pm e_1 \pm \cdots \pm e_{n-1}$. These vectors illuminate all vertices (and hence all relative boundary points) of the $(n-1)$-dimensional parallelotope $P^{(\tau)}$. Consequently there is a positive number λ such that the directions of 2^{n-1} vectors $\pm e_1 \pm \cdots \pm e_{n-1} + \lambda e_n$ also illuminate the relative boundary of $P^{(\tau)}$. Hence, these 2^{n-1} vectors illuminate the relative boundary of $P^{(t)}$ for $0 < t < \tau$. Moreover, the 2^{n-1} vectors considered illuminate all points of the lower base $P^{(0)}$. Thus $2^{n-1}+1$ vectors $p, \pm e_1 \pm \cdots \pm e_{n-1} + \lambda e_n$ illuminate the whole boundary of M, i.e., $c(M) \le 2^{n-1} + 1 < \frac{3}{4} \cdot 2^n$. □

Remark. In fact, the obtained estimate $c(M) \le 2^{n-1}+1$ is exact. Indeed, consider all vertices of the lower base $P^{(0)}$ and the center of the upper base. Every two of these $2^{n-1}+1$ points is *antipodal*, and therefore $c(M) \ge 2^{n-1}+1$. This proves the equality $c(M) = 2^{n-1}+1$.

Lemma 5. *Let $M \subset R^n$ be an n-dimensional outcut. Then $c(M) < \frac{3}{4} \cdot 2^n$.*

Proof. We use the description of the outcut given in Definition 2. Let $e_1, \dots, e_n$ be the endpoints of the segments $I_1, \dots, I_n$ distinct from 0, respectively. Consider all vectors of the kind $-v + \lambda(e_1 + \cdots + e_n)$ where λ is a fixed positive number and v is the sum of an *odd* number of the summands $e_1, \dots, e_n$. The number of the vectors of this kind is equal to

$$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1}.$$

Supplementing one more vector $e_1 + \cdots + e_n$, we obtain $2^{n-1}+1$ vectors. We show that for $\lambda > 0$ small enough the directions of these vectors illuminate the whole boundary of the outcut M.

Indeed, consider the figure G_{ij}. Every point $x \in (\operatorname{rint} G_{ij}) \cup I_i \cup I_j$ is illuminated by the direction of the vector $e_1 + \cdots + e_n$, since the point $x + \lambda(e_1 + \cdots + e_n)$ belongs to the interior of M.

Now consider a point $y \in G_{ij}$ that does not belong to $(\operatorname{rint} G_{ij}) \cup I_i \cup I_j$. Denote the coordinates of y in the basis $e_1, \dots, e_n$ by $y_1, \dots, y_n$. Both the numbers y_i, y_j are positive and at least one of them is less than 1 (since G_{ij} is distinct from the parallelogram $I_i \oplus I_j$). Let, for definiteness, $0 < y_i < 1$. Then the direction of the vector $e_j + \lambda(e_1 + \cdots + e_n)$ illuminates the point y. This shows that all points of the set $\bigcup_{i,j} G_{ij}$ are illuminated by the $2^{n-1}+1$ vectors considered. In other words, if $x \in \operatorname{bd} M$ belongs to a two-dimensional subspace of R^n spanned by two vectors taken from $e_1, \dots, e_n$, then y is illuminated.

Now consider a point $y \in \operatorname{bd} M$ that belongs to a four-dimensional subspace of R^n spanned by four vectors taken from $e_1, \ldots, e_n$, say, y belongs to the subspace spanned by e_1, e_2, e_3, e_4. We may suppose that y does not belong to a two-dimensional subspace as above, i.e., at least *three* of its coordinates y_1, y_2, y_3, y_4 are distinct from zero. Moreover, no more than one of the coordinates is equal to 1 (since any figure G_{ij} is distinct from the parallelogram $I_i \oplus I_j$). Let, for definiteness, $0 \le y_1 < 1$, $0 < y_4 \le 1$, and $0 < y_2 \le y_3 < 1$. Then the direction of the vector $-(e_2 + e_3 + e_4) + \lambda(e_1 + \cdots + e_n)$ illuminates the point y (as in the cases $y \in S$ and $y \notin S$ where S is the three-dimensional subspace spanned by e_2, e_3, e_4). Thus if $y \in \operatorname{bd} M$ belongs to a four-dimensional subspace of R^n spanned by four vectors taken from $e_1, \ldots, e_n$, then y is illuminated by the direction of at least one of the $2^{n-1} + 1$ vectors considered.

Similarly, if $y \in \operatorname{bd} M$ belongs to a six-dimensional subspace of R^n spanned by six vectors taken from $e_1, \ldots, e_n$, then y is illuminated by at least one of the $2^{n-1} + 1$ directions considered, etc. □

Remark. In fact, the obtained estimate $c(M) \le 2^{n-1} + 1$ is exact. We show this for a particular outcut (a similar reasoning can be realized for any outcut). Namely, let $e_1, \ldots, e_n$ be a basis in R^n. Furthermore, let α be a number with $0 < \alpha < 1$. For every pair of indices $i < j$ taken from the set $\{1, \ldots, n\}$, we denote the set $\operatorname{conv}\{0, e_i, e_j, \alpha e_i + \alpha e_j\}$ by G_{ij} and construct the outcut $M' \subset R^n$ with these sets G_{ij}, $1 \le i < j \le n$. Denote by F the set contained in $\operatorname{bd} M'$ that includes the points $0, e_1, \ldots, e_n$ and all points $x_1 e_1 + \cdots + x_n e_n$ such that an *odd* number (not lesser than three) of coordinates $x_1, \ldots, x_n$ take the value α, other coordinates being equal to zero. Every two points of the set F are *antipodal* boundary points of the outcut M'. For example, consider the points

$$a = (\alpha, \alpha, \alpha, 0, \ldots, 0), \qquad b = (\alpha, \alpha, \alpha, \alpha, \alpha, 0, \ldots, 0).$$

The point b belongs to the support hyperplane $x_4 + x_5 = 2\alpha$, and the point a belongs to the parallel support hyperplane $x_4 + x_5 = 0$, i.e., the points a and b are antipodal (similarly for any two points of the set F). Since the number of the points of the set F is equal to $2^{n-1}+1$, we have $c(M') \ge 2^{n-1}+1$. This proves the equality $c(M') = 2^{n-1}+1$.

Lemma 6. *Let $M \subset R^n$ be an n-dimensional stack-outcut. Then the inequality $c(M) < \frac{3}{4} \cdot 2^n$ holds.*

Proof. The stack-outcut is a "combination" of the stack and the outcut, and the proof of Lemma 6 is obtained as a union of the proofs of Lemmas 4 and 5. We only indicate the vectors whose directions illuminate the boundary of the stack-outcut, and do not repeat details of the previous reasonings.

We use the description of the stack-outcut given in Definition 3. Let $e_1, \ldots, e_n$ be the endpoints of the segments $I_1, \ldots, I_n$ distinct from 0, respectively. Consider all vectors of the kind $-v \pm e_{j+1} \pm \cdots \pm e_n + \lambda(e_1 + \cdots + e_n)$ where λ is a fixed positive number and v is the sum of an *odd* number of the vectors $e_1, \ldots, e_k$. The number of these vectors is equal to 2^{n-1}. We add the vector $e_1 + \cdots + e_n$. The directions of these $2^{n-1}+1$ vectors illuminate the whole boundary of M. □

Lemma 7. *Let $M \subset R^4$ be a four-dimensional particular polytope. Then $c(M) < \frac{3}{4} \cdot 2^4$.*

Proof. The polar polytope M^* has *nine* vertices:

$$e_1,\ e_2,\ e_3,\ e_4,\ -e_1 - e_2,\ -e_1 - e_3,\ -e_2 - e_4,\ -e_3 - e_4,\ e_1 + e_2 + e_3 + e_4$$

(by Definition 4). In [18] is shown that M^* has *eighteen* edges:

$$\begin{gathered}
[e_1, e_1 + e_2 + e_3 + e_4],\ [e_2, e_1 + e_2 + e_3 + e_4],\ [e_3, e_1 + e_2 + e_3 + e_4],\\
[e_4, e_1 + e_2 + e_3 + e_4],\ [-e_1 - e_2, -e_1 - e_3],\ [-e_1 - e_2, -e_2 - e_4],\\
[-e_1 - e_3, -e_3 - e_4],\ [-e_2 - e_4, -e_3 - e_4],\ [e_1, -e_2 - e_4],\ [e_1, -e_3 - e_4],\\
[e_2, -e_3 - e_4],\ [e_3, -e_2 - e_4],\ [e_1, e_4],\ [e_2, -e_1 - e_3],\\
[e_3, -e_1 - e_2],\ [e_4, -e_1 - e_2],\ [e_4, -e_1 - e_3],\ [e_2, e_3].
\end{gathered}$$

Furthermore, M^* has *fifteen* two-dimensional faces: nine parallelograms,

$$\begin{gathered}
[e_1, -e_3 - e_4, -e_1 - e_3, e_4],\ [e_1, -e_2 - e_4, -e_1 - e_2, e_4],\\
[e_3, -e_1 - e_2, e_4, e_1 + e_2 + e_3 + e_4],\ [e_2, -e_1 - e_3, e_4, e_1 + e_2 + e_3 + e_4],\\
[e_3, -e_2 - e_4, e_1, e_1 + e_2 + e_3 + e_4],\ [e_2, -e_4 - e_3, e_1, e_1 + e_2 + e_3 + e_4],\\
[e_2, -e_4 - e_3, -e_4 - e_2, e_3],\ [e_2, -e_1 - e_3, -e_1 - e_2, e_3],\\
[-e_1 - e_2, -e_1 - e_3, -e_4 - e_3, -e_4 - e_2],
\end{gathered}$$

and six triangles,

$$\begin{gathered}
[e_1, e_4, e_1 + e_3 + e_3 + e_4],\ [e_2, e_3, e_1 + e_3 + e_3 + e_4],\ [e_1, -e_2 - e_4, -e_3 - e_4],\\
[e_4, -e_1 - e_2, -e_1 - e_3],\ [e_2, -e_1 - e_3, -e_3 - e_4],\ [e_3, -e_1 - e_2, -e_2 - e_4].
\end{gathered}$$

Note that six two-dimensional faces (four parallelograms and two triangles) adjoin every vertex of M^*. Three two-dimensional faces (two parallelograms and one triangle) adjoin every edge.

By Euler's theorem, $c_0 - c_1 + c_2 - c_3 = 0$ where c_i is the number of i-dimensional faces of the polytope M^*, $i = 1, 2, 3, 4$. Since $c_0 = 9$, $c_1 = 18$, $c_2 = 15$, we obtain $c_3 = 6$, i.e., M^* has *six* three-dimensional faces $W_1, \dots, W_6$. Let $\Pi_1, \dots, \Pi_6$ be open half-spaces with the boundaries through 0 such that $\Pi_i \supset W_i$, $i = 1, \dots, 6$. Then *every* face of M^* is contained in at least one of these open half-spaces (since every face is contained in a three-dimensional one). This means that $c^*(M^*) \le 6$. By Lemma 1, we have $c(M) = 6 < \frac{3}{4} \cdot 2^4$. □

Remark. In fact, $c^*(M^*) = 6$ (and hence $c(M) = 6$), since no pair of three-dimensional faces of M^* is situated in one open half-space (with the boundary hyperplane through 0).

Proof of the Main Theorem. Let $M \subset R^n$ be a compact, convex body with md $M = 2$. Consider its decomposition $M = M_1 \oplus \dots \oplus M_q$ into the direct vector sum of compact, convex, indecomposable sets. We may suppose that for an integer p, $1 \le p \le q$, we have md $M_1 = \cdots =$ md $M_p = 2$ and (if $p < q$) md $M_{p+1} = \cdots =$ md $M_q = 1$, i.e., the summands $M_{p+1}, \ldots, M_q$ are segments. Denote the dimension of the set M_k by n_k, $k = 1, \ldots, p$. If we verify that for the sets $M_1, \ldots, M_p$ Martini's estimate holds, i.e., $c(M_k) \leq \frac{3}{4} \cdot 2^{n_k}$ for $k = 1, \ldots, p$, then we obtain

$$\begin{aligned} c(M) &= c(M_1) \cdot \ \cdots \ \cdot c(M_p) \cdot 2^{q-p} \\ &\leq (\tfrac{3}{4} \cdot 2^{n_1}) \cdot \ \cdots \ \cdot (\tfrac{3}{4} \cdot 2^{n_p}) \cdot 2^{q-p} = (\tfrac{3}{4})^p \cdot 2^n \leq \tfrac{3}{4} \cdot 2^n \end{aligned}$$

(since $n = n_1 + \cdots + n_p + q - p$), which proves the Main Theorem. Thus it remains to verify that the equalities $c(M_k) \leq \frac{3}{4} \cdot 2^{n_k}$, $k = 1, \ldots, p$, hold. To this end, we consider separately cases (i)–(v) of Theorem 7.

(i) $n_k = 2$ and M_k is a two-dimensional figure distinct from a parallelogram. In this case $c(M_k) = 3 = \frac{3}{4} \cdot 2^{n_k}$, i.e., Martini's estimate is exact.
(ii) $n_k \geq 3$ and M_k is an n_k-dimensional stack. In this case, by Lemma 4, $c(M_k) < \frac{3}{4} \cdot 2^{n_k}$, i.e., Martini's estimate is strict.
(iii) $n_k \geq 3$ and M_k is an n_k-dimensional outcut. In this case, by Lemma 5, $c(M_k) < \frac{3}{4} \cdot 2^{n_k}$, i.e., Martini's estimate is strict.
(iv) $n_k \geq 4$ and M_k is an n_k-dimensional stack-outcut. In this case, by Lemma 6, we also have $c(M_k) < \frac{3}{4} \cdot 2^{n_k}$, i.e., Martini's estimate is strict.
(v) $n_k = 4$ and M_k is a four-dimensional particular polytope. In this case, by Lemma 7, $c(M_k) < \frac{3}{4} \cdot 2^4 = \frac{3}{4} \cdot 2^{n_k}$, i.e., Martini's estimate is strict. □

Remark. The above reasoning shows that for a compact, convex body $M \subset R^n$ with md $M = 2$, the equality $c(M) = \frac{3}{4} \cdot 2^n$ holds if and only if $M = Q \oplus I_1 \oplus \cdots \oplus I_{n-2}$ where Q is a two-dimensional figure distinct from a parallelogram and all I_j are segments. In all other cases $c(M) < \frac{3}{4} \cdot 2^n$.

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Received August 23, 2000, *and in revised form January* 22, 2001. *Online publication August* 9, 2001.