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Characterization of Planar Pseudo-Self-Similar Tilings*

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Abstract. A pseudo-self-similar tiling is a hierarchical tiling of Euclidean space which obeys a nonexact substitution rule: the substitution for a tile is not geometrically similar to itself. An example is the Penrose tiling drawn with rhombi. We prove that a nonperiodic repetitive tiling of the plane is pseudo-self-similar if and only if it has a finite number of derived Voronoï tilings up to similarity. To establish this characterization, we settle (in the planar case) a conjecture of E. A. Robinson by providing an algorithm which converts any pseudo-self-similar tiling of $\mathbb{R}^2$ into a self-similar tiling of $\mathbb{R}^2$ in such a way that the translation dynamics associated to the two tilings are topologically conjugate.

1. Introduction

Much is known from an ergodic theoretic standpoint regarding **self-similar tilings** (some fundamental references being [15], [17], [18], and [20]). However, there is a closely related class of tilings, the **pseudo-self-similar tilings**, which we would like to claim have the same dynamical properties. Pseudo-self-similar tilings have a hierarchical structure: if the entire tiling is expanded by a well-chosen similarity of $\mathbb{R}^n$, then one can recover the tile in the original tiling at any point $x \in \mathbb{R}^n$ by looking in a finite window around x in the expanded tiling. A famous example of a pseudo-self-similar tiling is the Penrose tiling drawn with "thick" and "thin" rhombi, which are decorated with arrows to specify how they are allowed to fit together. Each expanded rhombus is replaced by configurations of rhombi in the original size, but the smaller tiles do not fit onto it exactly—some parts stick out over the edge, as shown in Fig. 1. However, the Penrose rhombs can be cut

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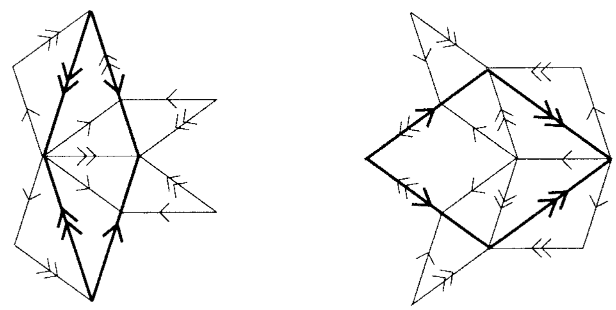

Fig. 1. Expanded thick and thin Penrose rhombi and their replacements.

into triangles which then satisfy the strict self-similarity condition, and this procedure is reversible, see p. 540 of [7]. In this situation we say that the original tiling and the modified version are **mutually locally derivable (MLD)**. The concept of mutual local derivability was originally discussed in [2].

The main motivation for this paper comes from the work of the first author [14], where it was conjectured that *a nonperiodic tiling is pseudo-self-similar if and only if it has finitely many derived Voronoï tilings up to similarity*. The "if" direction was established in [14], where it was also shown that self-similar tilings have a finite derived Voronoï family. It was necessary to show that any pseudo-self-similar tiling of $\mathbb{R}^2$ could be converted to a self-similar tiling in order to prove the "only if" direction. This provides a result which is interesting in its own right. We combined the result with the methods of [14] to confirm the characterization in the planar case.

The use of derived Voronoï tilings is motivated by a similar construct in symbolic dynamical systems theory. Tilings and their translation dynamical systems are continuous, higher-dimensional cousins of infinite symbolic sequences and their shift dynamical systems. One can recode a sequence in terms of some recurrent word W in the sequence by noting the **return words** for W: these are the words in the sequence starting at an occurrence of W and ending at the next occurrence of W. Originally, primitive substitutive sequences were characterized by Durand [6] as those which have a finite number of recodings when one considers the recodings for every initial word in the sequence. Later the condition was expanded to include recodings for arbitrary words [9]. In higher dimensions the return words are more difficult to mimic, as the notion of the "next" copy of a set of tiles in $\mathcal{T}$ is unclear. Voronoï tiles give a convenient geometric solution: two occurrences are "next to" one another if their Voronoï cells are adjacent. Pseudo-self-similar tilings generalize primitive substitutive sequences, so the characterization arrived at in this paper represents a nontrivial generalization of the characterization given in [6].

To show the MLD between pseudo-self-similar and self-similar tilings of $\mathbb{R}^2$, we present an algorithm which takes any pseudo-self-similar tiling of $\mathbb{R}^2$ and redraws it as a self-similar tiling, using an iterative process. The procedure involves first converting the tiling to a derived Voronoï tiling as in [14]. This tiling is very easy to work with since it is composed entirely of tiles which are convex polygons. Next, we use the pseudo-self-similarity to determine an initial redrawing of the tile boundaries, which produces a tiling which is not yet self-similar, but much closer to being so. We iterate the process, redrawing at each step, until in the limit we have obtained a self-similar tiling. The tiles of this limiting tiling may be nonconvex and have fractal boundaries, but they will be seen to be topological disks. We should note that the method of "redrawing the boundary" has been used before: see, e.g., [3], [5], and [10]. We should especially mention the work of Kenyon [11], since our methods are very similar to his at several points.

The paper is organized as follows. In Section 2 we collect all the basic definitions. The results are stated in Section 3. The main theorem is proved in Sections 4 and 5. The characterization of pseudo-self-similar tilings in terms of derived Voronoï tilings is obtained in Section 6. Section 7 contains concluding remarks.

2. Preliminaries

Here we collect the definitions and set up the terminology. The reader should be warned that there is a great deal of inconsistency in terminology in the literature on this subject. We usually do not attempt to trace the history of the terms we use; some of the early references are [2], [4], and [20]. We closely follow [14] in our definitions.

Generally speaking, a tiling of $\mathbb{R}^2$ is a covering of the plane by finitely many basic shapes (tiles), which overlap only on their boundaries.

We will find that it is convenient to have labelings (visually interpreted as colorings) for our tiles, allowing us to distinguish congruent tile shapes if we choose. So we fix a finite set of labels $\mathcal{A}$. Taking a compact set $A \subset \mathbb{R}^2$ which is the closure of its interior, and a label $l \in \mathcal{A}$, we define a **prototile** t as a pair (A, l). The **support** of t is $\mathrm{supp}(t) = A$; the **label** (or tile type) of t is $l(t) = l$. Throughout the paper **translations** are the only allowable transformations of prototiles, standing in contrast to the work of Radin and others (see [15], [16], and references therein) which allows arbitrary rotation of prototiles.

Definition 2.1. Given a finite prototile set τ, a **tile** T is a pair $(\mathrm{supp}(t) - g, l(t))$ for some $g \in \mathbb{R}^2$ and $t \in \tau$. We let $\mathrm{supp}(T) = \mathrm{supp}(t) - g$ and $l(T) = l(t)$. We say

$$\mathcal{T} = \{T_j = (\mathrm{supp}(t_{i_j} - g_j), l(t_{i_j})) \colon\ j \in \mathbb{N},\ t_{i_j} \in \tau,\ g_j \in \mathbb{R}^2\}$$

is a **tiling** if $\mathbb{R}^2 = \bigcup_j \mathrm{supp}(T_j)$ and $\mathrm{int}(\mathrm{supp}(T_i)) \cap \mathrm{int}(\mathrm{supp}(T_j)) = \emptyset$ for $i \neq j$.

Different tilings that we consider need not have the same prototile set. For convenience of notation we suppress subscripts and refer to any $\mathcal{T}$-tile as $T \in \mathcal{T}$. When considering two tilings, it is natural to identify them if they are equal, up to a one-to-one correspondence of the label sets.

A $\mathcal{T}$-**patch** P is a finite subset of the tiling $\mathcal{T}$. The **support of a patch** P is defined by $\mathrm{supp}(P) = \bigcup\{\mathrm{supp}(T) \colon\ T \in P\}$. The **diameter of a patch** P is $\mathrm{diam}(P) = \mathrm{diam}(\mathrm{supp}(P))$. The **translate** of a tile T by a vector $g \in \mathbb{R}^2$ is $T + g = (\mathrm{supp}(T) + g, l(T))$. The translate of a patch P is $P + g = \{T + g \colon\ T \in P\}$. We say that two patches P_1, P_2 are **translationally equivalent** if $P_2 = P_1 + g$ for some $g \in \mathbb{R}^2$.

The following definition will be useful in the proof of our main theorem.

Definition 2.2. A **patch with a marked tile** is a pair (P, T) where P is a patch and $T \in P$. Two patches with marked tiles (P_1, T_1) and (P_2, T_2) are said to be translationally equivalent if $P_2 = P_1 + g$ and $T_2 = T_1 + g$ for some $g \in \mathbb{R}^2$.

Definition 2.3. We say that a tiling $\mathcal{T}$ has **finite local complexity** (abbreviated as an **FLC tiling**), if for any $R > 0$, there are finitely many $\mathcal{T}$-patches of diameter less than R, up to translation.

To a subset $F \subset \mathbb{R}^2$ and a tiling $\mathcal{T}$ we can associate a $\mathcal{T}$-patch as follows:

$$[F]^{\mathcal{T}} = \{T \in \mathcal{T}\colon \operatorname{supp}(T) \cap F \neq \emptyset\}.$$

An important patch is the one associated to a point $y \in \mathbb{R}^2$, given by $[y]^{\mathcal{T}} := [\{y\}]^{\mathcal{T}}$.

Notation. We denote by $B_R(y)$ the closed ball of radius R centered at y and write $N_R(F) = \{x \in \mathbb{R}^2\colon \operatorname{dist}(x, F) \leq R\}$.

Definition 2.4. (See [2].) Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two tilings. We say that $\mathcal{T}_2$ is **locally derivable (LD) from** $\mathcal{T}_1$ with a radius $R > 0$ if for all $x, y \in \mathbb{R}^2$,

$$[B_R(x)]^{\mathcal{T}_1} = [B_R(y)]^{\mathcal{T}_1} + (x - y) \quad \Rightarrow \quad [x]^{\mathcal{T}_2} = [y]^{\mathcal{T}_2} + (x - y). \tag{2.1}$$

If $\mathcal{T}_1$ is LD from $\mathcal{T}_2$ and $\mathcal{T}_2$ is LD from $\mathcal{T}_1$, then we say that $\mathcal{T}_1$ and $\mathcal{T}_2$ are **mutually locally derivable (MLD)**.

Remark. Mutual local derivability is a useful equivalence relation. It is natural in the context of a **tiling dynamical system** defined as the translation action on a space of tilings (see [14], [15], and [18] for definitions). In fact, MLD is the tiling analog of a finite block code in symbolic dynamics. If two tilings are MLD, then the associated translation dynamical systems are topologically conjugate [14], hence they have the same dynamical and ergodic-theoretic properties. (We should note, however, that a topological conjugacy between tiling systems does not, in general, imply that the tilings are MLD, see [13] and [16].)

The following lemma is immediate from the definitions.

Lemma 2.5. *Suppose that $\mathcal{T}_1$ and $\mathcal{T}_2$ are two tilings and $\mathcal{T}_2$ is LD from $\mathcal{T}_1$ with the radius R. Then for any $L > R$ and any compact sets $F_1, F_2 \subset \mathbb{R}^2$ with $F_2 = F_1 + g$,*

$$[N_L(F_2)]^{\mathcal{T}_1} = [N_L(F_1)]^{\mathcal{T}_1} + g \quad \Rightarrow \quad [N_{L-R}(F_2)]^{\mathcal{T}_2} = [N_{L-R}(F_1)]^{\mathcal{T}_2} + g.$$

The definition of LD can be extended to include functions.

Definition 2.6. Let $\Phi\colon X \to \mathbb{R}^2$ be a function for some $X \subseteq \mathbb{R}^2$. We say that Φ is LD from a tiling $\mathcal{T}$ with a radius $R > 0$ if $\forall x, y \in X$,

$$[B_R(x)]^{\mathcal{T}} = [B_R(y)]^{\mathcal{T}} + (x - y) \quad \Rightarrow \quad \Phi(x) = \Phi(y) + (x - y). \tag{2.2}$$

Definition 2.7. A tiling $\mathcal{T}$ is called **repetitive** if for any patch $P \subset \mathcal{T}$ there is a real number $R > 0$ such that for any $x \in \mathbb{R}^2$ there is a $\mathcal{T}$-patch P' such that $\operatorname{supp}(P') \subset B_R(x)$ and P' is a translate of P. The minimal such R, denoted $R(P)$, is called the **repetitivity radius** of P. A repetitive tiling is called **linearly repetitive** if there exists a constant $C > 0$ such that

$$R(P) \leq C \operatorname{diam}(P) \qquad \text{for any} \quad P \subset \mathcal{T}.$$

We should note that repetitive tilings are called "almost periodic" in [14] and "locally isomorphic" in [18].

In everything that follows, we let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an expanding, orientation-preserving similitude; that is, φ is a linear map such that for some $\lambda > 1$ we have $|\varphi g| = \lambda |g|$ for all $g \in \mathbb{R}^2$. We call such a φ an **expansion map** and refer to λ as the **size** of the expansion. Clearly, $\lambda = \|\varphi\| = \sqrt{\det \varphi}$. Because it is orientation-preserving, there can be no reflections, so if we view $\mathbb{R}^2$ as the complex plane, the expansion φ can be identified with the multiplication by a complex number ζ, $|\zeta| = \lambda$. We can define the expansion $\varphi\mathcal{T}$ of a tiling $\mathcal{T}$: it is the tiling given by $\cup_{T \in \mathcal{T}} (\varphi(\operatorname{supp}(T)), l(T))$.

Definition 2.8. Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an expansion map. A repetitive FLC tiling $\mathcal{T}$ is said to be **pseudo-self-similar with expansion** φ if $\mathcal{T}$ is LD from $\varphi\mathcal{T}$. A repetitive FLC tiling $\mathcal{T}$ is called **self-similar with expansion** φ if

(i) for any tile $T = (A, l) \in \mathcal{T}$, there is a $\mathcal{T}$-patch whose support is $\varphi(A)$, formally $[\operatorname{int}(\varphi(A))]^{\mathcal{T}}$, and

(ii) if $T' = T - g$ for two $\mathcal{T}$-tiles $T = (A, l)$, $T' = (A', l)$, and $g \in \mathbb{R}^2$, then

$$[\operatorname{int}(\varphi(A'))]^{\mathcal{T}} = [\operatorname{int}(\varphi(A))]^{\mathcal{T}} - \varphi(g).$$

Definition 2.9. A tiling $\mathcal{T}$ is said to be **nonperiodic** if $\mathcal{T} - g \neq \mathcal{T}$ for any nonzero $g \in \mathbb{R}^2$.

Remark. It is easy to see that a self-similar tiling is pseudo-self-similar. In [14] it is required that a pseudo-self-similar tiling $\mathcal{T}$ is MLD with $\varphi\mathcal{T}$. We will show that this property holds whenever $\mathcal{T}$ is nonperiodic. It is known (see Chapter 10 of [7]) that nonperiodicity is necessary for $\varphi\mathcal{T}$ to be LD from $\mathcal{T}$. We should note that the property "$\mathcal{T}$ and $\varphi\mathcal{T}$ are MLD" was already considered by Baake and Schlottmann [1] under the name "inflation–deflation symmetry."

For the characterization of pseudo-self-similar tilings, we recall the construction of the derived Voronoï family from [14].

Definition 2.10. Suppose that $\mathcal{T}$ is a repetitive tiling of $\mathbb{R}^2$. Let $r > 0$, $P_r = [B_r(0)]^{\mathcal{T}}$ and create the **locator set**

$$\mathcal{L}_r = \{q \in \mathbb{R}^2 \text{ such that there exists } P \subset \mathcal{T} \text{ with } P_r = P - q\}.$$

Let $R(r)$ be the repetitivity radius of P_r so that every ball of radius $R(r)$ in $\mathcal{T}$ contains a translate of P_r. The **derived Voronoï tiling** $\mathcal{T}_r$ has a tile t_q for each $q \in \mathcal{L}_r$ with support

$$\operatorname{supp}(t_q) = \{x \in \mathbb{R}^2\colon |q - x| \leq |q' - x| \text{ for all } q' \in \mathcal{L}_r\};$$

t_q is labeled by the translational equivalence class of the patch $[B_{2R(r)}(q)]^{\mathcal{T}}$. The **derived Voronoï family** is defined by $\mathcal{F}(\mathcal{T}) = \{\mathcal{T}_r \colon r > 0\}$.

Given an expanding similitude $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$, we say a family of tilings $\mathcal{F}$ is φ-**finite** if there exist $\mathcal{S}_1, \ldots, \mathcal{S}_M$ in $\mathcal{F}$ so that for any $\mathcal{T} \in \mathcal{F}$, there is an $i \in \{1, \ldots, M\}$ and a $j \in \mathbb{Z}^+$ with $\mathcal{T} = \varphi^j \mathcal{S}_i$. (Recall that we identify two tilings if they are equal up to a one-to-one correspondence between the label sets.)

3. Results

Recall that all tilings considered in this paper are assumed to be repetitive.

Theorem 3.1. *Let $\mathcal{T}$ be a pseudo-self-similar tiling of the plane with expansion φ. Then for any $k \in \mathbb{N}$ sufficiently large there exists a tiling $\mathcal{T}'$ which is self-similar with expansion φ^k, such that $\mathcal{T}$ is MLD with $\mathcal{T}'$. Moreover, all the tiles of $\mathcal{T}'$ are topological disks*.

This settles a conjecture of E. A. Robinson (personal communication). Using the methods of [14], we deduce from Theorem 3.1 the following result which was conjectured in [14].

Theorem 3.2. *A nonperiodic, repetitive tiling of $\mathbb{R}^2$ is pseudo-self-similar if and only if its derived Voronoï family is ψ-finite for an expanding, orientation-preserving similitude ψ.*

Theorem 3.1 also allows us to extend many of the results available for self-similar tilings to the case of pseudo-self-similar tilings.

Corollary 3.3. *Suppose that $\mathcal{T}$ is a pseudo-self-similar tiling of $\mathbb{R}^2$ with expansion φ. Then*

(i) *$\mathcal{T}$ is linearly repetitive*;
(ii) *if $\mathcal{T}$ is nonperiodic, then it is MLD with $\varphi\mathcal{T}$.*

This corollary follows from Theorem 3.1 above and Lemma 2.3 and Theorem 1.1 of [19].

A complex number ζ is called a **complex Perron number** if it is an algebraic integer whose Galois conjugates, other than $\overline{\zeta}$, are all less than $|\zeta|$ in modulus.

Corollary 3.4. *Suppose that $\mathcal{T}$ is a pseudo-self-similar tiling of the plane with expansion φ corresponding to the multiplication by a complex number ζ. Then ζ is a complex Perron number.*

Proof. By Theorem 3.1 and Thurston's characterization of expansion constants for self-similar tilings [20, Theorem 10.1], ζ^k is Perron for all k sufficiently large. However, this implies that ζ itself is Perron, see Lemma 5 of [12]. □

Combining Theorem 3.1 with Corollary 2.2 of [14] and the results of [18], one obtains a wealth of information about translation dynamical systems associated with pseudo-self-similar tilings. For instance, every such tiling dynamical system is uniquely ergodic but not strongly mixing. If the complex expansion constant ζ is a complex Pisot number (i.e., all its Galois conjugates, other than $\overline{\zeta}$, are less than one in modulus), then the associated tiling dynamical system is not weakly mixing.

4. Proof of Theorem 3.1

Any sufficiently well-behaved tiling $\mathcal{S}$ has a well-defined **boundary graph** $\partial\mathcal{S}$. For convenience, we assume in the definition of this graph that the supports of all $\mathcal{S}$-tiles are convex polytopes. The boundary graph $\partial\mathcal{S}$ has **vertices** given by the points in $\mathbb{R}^2$ which intersect supports of three or more tiles and **edges** given by the connected sets in $\mathbb{R}^2$ which intersect supports of exactly two tiles. The graph $\partial\mathcal{S}$ is connected, and we denote the vertex and edge sets by $V(\partial\mathcal{S})$ and $E(\partial\mathcal{S})$, respectively.

Lemma 4.1. *It is enough to prove Theorem* 3.1 *for tilings with the following additional properties*:

$$\textit{the supports of all } \mathcal{T}\textit{-tiles are convex polygons}, \tag{4.1}$$

$$\textit{every vertex of the boundary graph } \partial\mathcal{T} \textit{ has degree } 3. \tag{4.2}$$

Proof. Fix $r > 0$ and consider the derived Voronoï tiling $\mathcal{T}_r$, as in Definition 2.10. It is proved in 4.2 of [14] that $\mathcal{T}_r$ is MLD with $\mathcal{T}$, so we also have that $\varphi\mathcal{T}$ is MLD with $\varphi\mathcal{T}_r$. Since $\mathcal{T}$ is LD from $\varphi\mathcal{T}$, and LD is a transitive relation, we obtain that $\mathcal{T}_r$ is LD from $\varphi\mathcal{T}_r$.

It is well known that the Voronoï tiling has tiles that are convex polytopes intersecting along whole faces (sides in our case). Some vertices may have degree larger than 3, so for every vertex v with more than three edges going out, we create a new tile. We find $a > 0$ less than one-third of the shortest edge of $\mathcal{T}_r$ and put new vertices at the distance a of v on every edge going out of v. We then connect them to get a closed convex curve and delete everything inside (the point v and the line segments going out of it). We obtain a new tile with convex support. We enlarge the label set by adding *new* labels that are in one-to-one correspondence with the translational equivalence classes of the patches $[v]^{\mathcal{T}_r}$.

It is clear that now every vertex has degree 3 and the supports of other tiles remain convex. Denote the new tiling by $\widetilde{\mathcal{T}}$. Observe that $\widetilde{\mathcal{T}}$ is MLD with $\mathcal{T}_r$ since this procedure is reversible. Similarly, $\varphi\widetilde{\mathcal{T}}$ is MLD with $\varphi\mathcal{T}_r$. Thus, $\widetilde{\mathcal{T}}$ is a pseudo-self-similar tiling with all the desired properties, and the lemma follows. $\square$

The key step in the proof of Theorem 3.1 is the construction of the function Ψ described in the next proposition. This map takes the graph of $\mathcal{T}$ and "redraws" the edges by approximating them with paths in the graph of $\varphi^{-k}\,\partial\mathcal{T}$.

Denote by $|\cdot|$ the arc length measure on $\partial\mathcal{T}$; this will be well defined for tilings satisfying (4.1). Let $d_M = d_M(\mathcal{T}) = \sup\{\mathrm{diam}(\mathrm{supp}(T))\colon T \in \mathcal{T}\}$, which is finite by FLC.

Proposition 4.2. *Let $\mathcal{T}$ be a pseudo-self-similar tiling with expansion map φ satisfying* (4.1) *and* (4.2), *and let $\varepsilon > 0$. Then for all $k \in \mathbb{N}$ sufficiently large there exists a continuous injective map $\Psi\colon \partial\mathcal{T} \to \partial(\varphi^{-k}\mathcal{T}) = \varphi^{-k}\,\partial\mathcal{T}$ which takes edges in $\partial\mathcal{T}$ onto unions of edges of $\varphi^{-k}\,\partial\mathcal{T}$ and which satisfies*:

(i) *For all $x \in \partial\mathcal{T}$, $|x - \Psi(x)| < \varepsilon$.*
(ii) *There is $\rho \in (0, 1)$ such that for any $e \in E(\partial\mathcal{T})$ and $\ell \in \Psi(e) \cap E(\partial(\varphi^{-k}\mathcal{T}))$, the restriction $\Psi|_{\Psi^{-1}(\ell)}$ is linear and $|\Psi^{-1}(\ell)| \le \rho|e|$.*
(iii) *There is $R > 0$ such that Ψ is LD from $\mathcal{T}$ with a radius R.*

Next we deduce Theorem 3.1, postponing the proof of the proposition to the next section. Fix k, with $\lambda^k > 2$, and Ψ, given by the proposition. Let

$$\Psi^{(n)} = \varphi^{-nk}(\varphi^k\Psi)^n \qquad \text{for} \quad n \ge 1. \tag{4.3}$$

Note that $\Psi = \Psi^{(1)}$. The map $\Psi^{(n)}\colon \partial\mathcal{T} \to \mathbb{R}^2$ is well defined since $\varphi^k\Psi(\partial\mathcal{T}) \subset \partial\mathcal{T}$. It is clear that $\Psi^{(n)}$ are all continuous and injective. We have for all $n \ge 1$ and $x \in \partial\mathcal{T}$:

$$\begin{aligned} |\Psi^{(n)}(x) - \Psi^{(n+1)}(x)| &= |\varphi^{-nk}(\varphi^k\Psi)^n(x) - \varphi^{-(n+1)k}(\varphi^k\Psi)^{n+1}(x)| \\ &= |\varphi^{-nk}((\varphi^k\Psi)^n(x) - \Psi(\varphi^k\Psi)^n(x))| \le \lambda^{-nk}\varepsilon, \end{aligned} \tag{4.4}$$

by Proposition 4.2(i), since φ is a similitude with expansion size λ. This implies that the sequence $\Psi^{(n)}$ converges uniformly to a continuous function $\Psi^\infty\colon \partial\mathcal{T} \to \mathbb{R}^2$, and, moreover,

$$|x - \Psi^\infty(x)| \le \frac{\varepsilon}{1 - \lambda^{-k}} < 2\varepsilon \qquad \text{for all} \quad x \in \partial\mathcal{T}. \tag{4.5}$$

By the definition of $\Psi^{(n)}$ we have $\Psi^{(n+1)} = \varphi^{-k}\Psi^{(n)}\varphi^k\Psi$, hence

$$\Psi^\infty = \varphi^{-k}\Psi^\infty\varphi^k\Psi. \tag{4.6}$$

This has several useful consequences. First, observe that

$$\varphi^k(\Psi^\infty(\partial\mathcal{T})) \subset \Psi^\infty(\partial\mathcal{T}). \tag{4.7}$$

Second, we claim that Ψ^∞ is injective, provided that $\varepsilon > 0$ is sufficiently small. Indeed, suppose that $x, y \in \partial\mathcal{T}$ with $\Psi^\infty(x) = \Psi^\infty(y)$. By (4.5) we have that $|x - y| \le 4\varepsilon$. We see from (4.6) that it is also true that $\Psi^\infty((\varphi^k\Psi)^n x) = \Psi^\infty((\varphi^k\Psi)^n y)$ for all $n \in \mathbb{N}$, and so it must be that $|(\varphi^k\Psi)^n x - (\varphi^k\Psi)^n y| \le 4\varepsilon$ as well. Assume that $\varepsilon \in (0, 1)$ is such that

$$4\varepsilon < \min\{\mathrm{dist}(e_1, e_2)\colon e_1, e_2 \in E(\partial\mathcal{T}),\ e_1 \cap e_2 = \emptyset\}.$$

Then $(\varphi^k\Psi)^n x$ and $(\varphi^k\Psi)^n y$ are always on the same edge or adjacent edges in $\partial\mathcal{T}$. Denote $\tilde\Psi := \varphi^k\Psi$. Suppose first that $\tilde\Psi^n x$ and $\tilde\Psi^n y$ are on the same edge ℓ_0. Then $\tilde\Psi^{n-1}x$ and $\tilde\Psi^{n-1}y$ are on some edge ℓ_1 of $\partial\mathcal{T}$, the points $\tilde\Psi^{n-2}x$ and $\tilde\Psi^{n-2}y$ are on the same edge ℓ_2 of $\partial\mathcal{T}$, etc., until we get that x and y are on some edge ℓ_n of $\partial\mathcal{T}$. By Proposition 4.2(ii), $\tilde{\Psi}^n|_{\ell_n}$ is linear and $|x-y| \le \rho^n|\ell_n| \le \rho^n \max\{|e|\colon e \in E(\partial\mathcal{T})\}$. If $\tilde{\Psi}^n x$ and $\tilde{\Psi}^n y$ lie on adjacent edges, we get $|x-y| \le 2\rho^n \max\{|e|\colon e \in E(\partial\mathcal{T})\}$ by a similar argument, considering the common vertex of these edges. In either case, letting $n \to \infty$, we obtain that $x = y$. This proves that Ψ^∞ is injective.

Now we are ready to define the tiling which we will prove is self-similar. Since Ψ^∞ is injective, for each tile $T = (A, l) \in \mathcal{T}$ the image $\Psi^\infty(\partial A)$ is a Jordan curve. By the Jordan curve theorem, $\Psi^\infty(\partial A)$ separates the plane into precisely two components of which $\Psi^\infty(\partial A)$ is the common boundary. Denote the closure of the bounded component by A'. It is homeomorphic to the closed disk and will be the support of a new tile. For future reference note that

$$A' \subset N_{2\varepsilon}(A) \tag{4.8}$$

by (4.5). Next we give a labeling to this tile. Recall that R is the radius of LD of Ψ from $\mathcal{T}$. Clearly, Ψ is LD from $\mathcal{T}$ with any larger radius, so we can choose R as large as we wish without loss of generality. Assume that

$$R > (r + 2\lambda^k + d_M + 2)/(\lambda^k - 1), \tag{4.9}$$

where r is the radius of LD of $\varphi^k\mathcal{T}$ onto $\mathcal{T}$. The label l' of A' is defined as the translational equivalence class of the patch $[N_R(A)]^{\mathcal{T}}$ with the marked tile T (see Definition 2.2).

Claim. *$\mathcal{T}' := \{T' = (A', l')\colon T \in \mathcal{T}\}$ is a tiling of $\mathbb{R}^2$.*

Proof. Denote $\mathrm{supp}(\mathcal{T}) = \{\mathrm{supp}(T)\colon T \in \mathcal{T}\}$. For any $A \in \mathrm{supp}(\mathcal{T})$, by (4.5) at least a portion of $\Psi^\infty(\partial\mathcal{T}\backslash\partial A)$ lies in the unbounded component of $\mathbb{R}^2\backslash\Psi^\infty(\partial A)$. Since $\partial\mathcal{T}\backslash\partial A$ is connected, so is $\Psi^\infty(\partial\mathcal{T}\backslash\partial A)$. Thus by the injectivity of Ψ^∞ we have that $\Psi^\infty(\partial\mathcal{T}\backslash\partial A)$ lies in the unbounded component of $\mathbb{R}^2\backslash\Psi^\infty(\partial A)$. This implies that for any $B \in \mathrm{supp}(\mathcal{T}),\ B \ne A$, we have $\mathrm{int}(A') \cap \mathrm{int}(B') = \emptyset$.

It remains to verify that $\mathrm{supp}(\mathcal{T}') := \bigcup\{A'\colon A \in \mathrm{supp}(\mathcal{T})\} = \mathbb{R}^2$. Let $A \in \mathrm{supp}(\mathcal{T})$. By the Jordan–Schoenflies theorem (see Theorem 9.25 of [8]), the homeomorphism $\Psi^\infty|_{\partial A}\colon \partial A \to \partial A'$ can be extended to the homeomorphism $h_A\colon A \to A'$. Then the function $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $h(x) := h_A(x)$ for $x \in A$, is well-defined, continuous, and one-to-one (here we use that $\mathrm{int}(A') \cap \mathrm{int}(B') = \emptyset$ for $A \ne B$ in $\mathrm{supp}(\mathcal{T})$). We also have $|h(x) - x| \le d_M + 2\varepsilon$ by (4.8). Since a one-to-one continuous mapping of a compact space is a homeomorphism, h restricted to any compact subset of $\mathbb{R}^2$ is a homeomorphism on its image. Let $z \in \mathbb{R}^2$, and consider the ball $B_t(z)$ for some $t > d_M + 2\varepsilon$. Then the image $h(B_t(z))$ is simply connected and the image of the boundary $h(\partial B_t(z))$ must contain in its bounded component the point z. Thus z is in the image of h, so it is in the support of at least one $\mathcal{T}'$-tile. Thus we have shown that $\mathcal{T}'$ tiles $\mathbb{R}^2$, as desired. $\square$

Next we show that the tiling $\mathcal{T}'$ is LD from $\mathcal{T}$ using the following lemma.

Lemma 4.3. *Let $\varepsilon \in (0, 1)$, $k \in \mathbb{N}$, Ψ, R be as in Proposition* 4.2, *with R satisfying* (4.9), *and let Φ be another continuous function from $\partial\mathcal{T}$ to $\mathbb{R}^2$, which is LD from $\mathcal{T}$ with the radius R. Then $\widetilde{\Phi} := \varphi^{-k}\Phi\varphi^k\Psi$ is LD from $\mathcal{T}$ with the same radius R.*

Proof. Suppose that $x, y \in \partial\mathcal{T}$ and

$$[B_R(x)]^{\mathcal{T}} = [B_R(y)]^{\mathcal{T}} + (x - y). \tag{4.10}$$

Applying φ^k yields

$$[B_{\lambda^k R}(\varphi^k x)]^{\varphi^k \mathcal{T}} = [B_{\lambda^k R}(\varphi^k y)]^{\varphi^k \mathcal{T}} + \varphi^k(x - y).$$

Therefore, by Lemma 2.5,

$$[B_{\lambda^k R-r}(\varphi^k x)]^{\mathcal{T}} = [B_{\lambda^k R-r}(\varphi^k y)]^{\mathcal{T}} + \varphi^k(x - y).$$

In view of Proposition 4.2(i), this implies

$$[B_{\lambda^k R-r-\lambda^k\varepsilon}(\varphi^k\Psi(x))]^{\mathcal{T}} = [B_{\lambda^k R-r-\lambda^k\varepsilon}(\varphi^k\Psi(x) + \varphi^k(y - x))]^{\mathcal{T}} + \varphi^k(x - y).$$

On the other hand, since Ψ is LD from $\mathcal{T}$ with the radius R, (4.10) implies $\Psi(x) = \Psi(y) + (x - y)$ and hence

$$\varphi^k\Psi(x) = \varphi^k\Psi(y) + \varphi^k(x - y). \tag{4.11}$$

Combined with the previous equation, this yields

$$[B_{\lambda^k R-r-\lambda^k\varepsilon}(\varphi^k\Psi(x))]^{\mathcal{T}} = [B_{\lambda^k R-r-\lambda^k\varepsilon}(\varphi^k\Psi(y))]^{\mathcal{T}} + (\varphi^k\Psi(x) - \varphi^k\Psi(y)).$$

Since $\lambda^k R - r - \lambda^k\varepsilon > R$ by (4.9), we can use that Φ is LD from $\mathcal{T}$ with the radius R and (4.11) to conclude that

$$\Phi\varphi^k\Psi(x) = \Phi\varphi^k\Psi(y) + \varphi^k(x - y).$$

Applying φ^{-k} gives that $\widetilde{\Phi}$ is LD from $\mathcal{T}$ with the radius R, as desired. □

Now we apply Lemma 4.3 to the function $\Phi = \Psi^{(n)}$; then $\widetilde{\Phi} = \Psi^{(n+1)}$. We get by induction that $\Psi^{(n)}$ are all LD from $\mathcal{T}$ with the radius R. Passing to the limit yields that Ψ^∞ is LD from $\mathcal{T}$ with the radius R. We are going to show that $\mathcal{T}'$ is LD from $\mathcal{T}$ with the radius $R + d_M + 2\varepsilon$. Suppose that

$$[B_{R+d_M+2\varepsilon}(x)]^{\mathcal{T}} = [B_{R+d_M+2\varepsilon}(y)]^{\mathcal{T}} + (x - y). \tag{4.12}$$

Let $T_1' = (A_1', l') \in [x]^{\mathcal{T}'}$, where $\partial A_1' = \Psi^\infty(\partial A_1)$ for some $T_1 = (A_1, l) \in \mathcal{T}$. The Hausdorff distance between A_1 and A_1' is less than 2ε by (4.5). By (4.12),

$$T_2 := (A_2, l) := T_1 + (y - x) \in \mathcal{T}. \tag{4.13}$$

We have $N_R(A_1) \subset B_{R+d_M+2\varepsilon}(x)$. Thus by (4.12), $[N_R(A_1)]^{\mathcal{T}} = [N_R(A_2)]^{\mathcal{T}} + (x - y)$ and $A_1 = A_2 + (x - y)$, which implies $\Psi^\infty(\partial A_1) = \Psi^\infty(\partial A_2) + (x - y)$ since Ψ^∞ is LD from $\mathcal{T}$ with the radius R. Therefore, $A_1' = A_2' + (x - y)$ and $y \in A_2'$. By (4.12) and (4.13), the patches with marked tiles $([N_R(A_1)]^{\mathcal{T}}, T_1)$ and $([N_R(A_2)]^{\mathcal{T}}, T_2)$ are translationally equivalent, and hence A_2' has the same label as A_1'. This proves that

$T_2' = (A_2', l') \in [y]^{\mathcal{T}'}$ and concludes the proof that $\mathcal{T}'$ is LD from $\mathcal{T}$ with the radius $R + d_M + 2\varepsilon$.

Now we show that $\mathcal{T}$ is LD from $\mathcal{T}'$ with the radius 2ε. Suppose that

$$[B_{2\varepsilon}(x)]^{\mathcal{T}'} = [B_{2\varepsilon}(y)]^{\mathcal{T}'} + (x - y). \tag{4.14}$$

Let $T_1 = (A_1, l) \in [x]^{\mathcal{T}}$ and let $T_1' = (A_1', l')$ be the corresponding $\mathcal{T}'$-tile. Then $A_1' \cap B_{2\varepsilon}(x) \neq \emptyset$ by (4.5), hence $T_1' \in [B_{2\varepsilon}(x)]^{\mathcal{T}'}$ and (4.14) implies that $T_2' = (A_2', l') := T_1' + (y - x) \in \mathcal{T}'$. Let T_2 be the $\mathcal{T}$-tile corresponding to T_2'. By the definition of $\mathcal{T}'$-labels, the patches with marked tiles $([N_R(A_1)]^{\mathcal{T}}, T_1)$ and $([N_R(A_2)]^{\mathcal{T}}, T_2)$ are translationally equivalent, hence

$$A_1 = A_2 + g \quad \text{and} \quad [N_R(A_1)]^{\mathcal{T}} = [N_R(A_2)]^{\mathcal{T}} + g \tag{4.15}$$

for some $g \in \mathbb{R}^2$. Since Ψ^∞ is LD from $\mathcal{T}$ with the radius R, we conclude that $A_1' = A_2' + g$, but we already know that $A_1' = A_2' + (x - y)$, hence $g = x - y$. We have shown that $T_1 = T_2 + (x - y)$ and $T_2 \in [y]^{\mathcal{T}}$, concluding the proof that $\mathcal{T}$ is LD from $\mathcal{T}'$.

Lastly, we show that $\mathcal{T}'$ is a self-similar tiling of $\mathbb{R}^2$ with expansion φ^k. By (4.7), for any $\mathcal{T}'$-tile (A', l'), the Jordan curve $\varphi^k(\partial A')$ is a subset of $\partial \mathcal{T}'$. The interior of every $\mathcal{T}'$-tile must lie entirely in one of the two components of $\mathbb{R}^2 \backslash \varphi^k(\partial A')$, therefore $\varphi^k(A')$ is the support of a $\mathcal{T}'$-patch $[\mathrm{int}(\varphi^k(A'))]^{\mathcal{T}'}$. Let $T_1' = (A_1', l')$ and $T_2' = (A_2', l')$ be $\mathcal{T}'$-tiles with the same label. According to Definition 2.8, we must show that if $A_2' = A_1' - g'$ for some $g' \in \mathbb{R}^2$, then

$$[\mathrm{int}(\varphi^k(A_2'))]^{\mathcal{T}'} = [\mathrm{int}(\varphi^k(A_1'))]^{\mathcal{T}'} - \varphi^k g'. \tag{4.16}$$

Since T_1' and T_2' have the same label, we know that (4.15) holds for some $g \in \mathbb{R}^2$, where A_i is such that $\partial A_i' = \Psi^\infty(\partial A_i)$ for $i = 1, 2$. From the fact that Ψ^∞ is LD from $\mathcal{T}$ with the radius R we conclude that $A_2' = A_1' - g$ hence $g' = g$. Now (4.15) implies

$$[N_{\lambda^k R}(\varphi^k A_2)]^{\varphi^k \mathcal{T}} = [N_{\lambda^k R}(\varphi^k A_1)]^{\varphi^k \mathcal{T}} - \varphi^k g.$$

Since $\mathcal{T}$ is LD from $\varphi^k \mathcal{T}$ with the radius r, we have from Lemma 2.5 that

$$[N_{\lambda^k R - r}(\varphi^k A_2)]^{\mathcal{T}} = [N_{\lambda^k R - r}(\varphi^k A_1)]^{\mathcal{T}} - \varphi^k g.$$

Using that $\mathcal{T}'$ is LD from $\mathcal{T}$ with the radius $R + d_M + 2\varepsilon$ and $\lambda^k R - r - R - d_M - 2\varepsilon \geq 2\lambda^k \varepsilon$ by the choice of R (see (4.9)), we conclude that

$$[N_{2\lambda^k \varepsilon}(\varphi^k A_2)]^{\mathcal{T}'} = [N_{2\lambda^k \varepsilon}(\varphi^k A_1)]^{\mathcal{T}'} - \varphi^k g'.$$

Since $\varphi^k A_i' \subset N_{2\lambda^k \varepsilon}(\varphi^k A_i),\ i = 1, 2$, by (4.8) and $\varphi^k A_2' = \varphi^k A_1' - \varphi^k g'$, this implies (4.16) as desired.

It remains to note that $\mathcal{T}'$ has FLC and is repetitive since these properties are preserved by MLD. Thus we have constructed the tiling $\mathcal{T}'$ which is MLD from $\mathcal{T}$ and which is a self-similar tiling of $\mathbb{R}^2$, finishing the proof of Theorem 3.1. $\square$

5. Proof of Proposition 4.2.

For any $a, b \in \mathbb{R}^2$, denote by $[a, b]$ the line segment from a to b. Recall that d_M is the largest diameter of a support of a $\mathcal{T}$-tile and $N_r(A)$ denotes the closed neighborhood of A of radius r.

Lemma 5.1. *Let $\mathcal{T}$ be an FLC tiling satisfying* (4.1) *and $a, b \in V(\partial\mathcal{T})$. Then there exists a continuous, one-to-one mapping $\zeta\colon [0, 1] \to \partial\mathcal{T}$ such that $\zeta(0) = a$, $\zeta(1) = b$, $\zeta([0, 1]) \subset N_{d_M}([a, b])$,*

$$|\zeta(t) - a - t(b - a)| \le 5d_M \qquad \textit{for all} \quad t \in [0, 1], \tag{5.1}$$

and for any $\ell \in \zeta([0, 1]) \cap E(\partial\mathcal{T})$ the restriction $\zeta|_{\zeta^{-1}(\ell)}$ is linear.

Note. There are several ways to prove this rather straightforward statement. We do not claim that the constant 5 is optimal.

Proof. For $x, y \in \mathbb{R}^2$ let $G[x, y]$ denote the union of boundaries of those $\mathcal{T}$-tiles whose supports intersect $[x, y]$. Clearly, $G[x, y]$ is connected; we can consider this set as a connected subgraph of $\partial\mathcal{T}$. Suppose first that $|a - b| \le 4d_M$. Then we consider the shortest path connecting a and b in $G[a, b]$. It is simple, and any piecewise linear parameterization ζ of this path will have the desired properties.

Next suppose that $|a - b| > 4d_M$. We can choose points $a_0 = a,\ a_1, \dots, a_n = b$, so that $a_i < a_{i+1}$ in the natural order on $[a, b]$ and

$$2d_M < |a_i - a_{i+1}| \le 3d_M \qquad \text{for} \quad i = 0, \dots, n - 1.$$

Fix $a'_i \in V(\partial\mathcal{T}) \cap [a_i]^{\mathcal{T}}$ for $i = 1, \dots, n - 1$ and let $a'_0 = a,\ a'_n = b$. Then a'_i and a'_{i+1} are vertices of $G[a_i, a_{i+1}]$. Let Γ_i be the shortest path connecting a'_i and a'_{i+1} in $G[a_i, a_{i+1}]$. We have that Γ_i is simple and $\Gamma_i \subset N_{d_M}([a_i, a_{i+1}])$. The union $\bigcup_{i=0}^{n-1} \Gamma_i$ connects a to b but it may fail to be a simple path, so we have to do some "pruning." Observe that

$$\Gamma_{i-1} \cap \Gamma_i \subset N_{d_M}([a_{i-1}, a_i]) \cap N_{d_M}([a_i, a_{i+1}]) = B_{d_M}(a_i),$$

and the neighborhoods $B_{d_M}(a_i)$ are disjoint for different i's by construction.

Since Γ_i is simple, it has a linear ordering with a'_i being the smallest point and a'_{i+1} being the largest point. For $i = 1, \dots, n - 1$ let a''_i be the smallest point in Γ_{i-1} such that $a''_i \in \Gamma_i$. Then a''_i is a vertex in $B_{d_M}(a_i) \cap V(\partial\mathcal{T})$ and taking the union of edges less than a''_i in Γ_{i-1} and edges greater than a''_i in Γ_i yields a simple path. Doing this for all i yields a simple path Γ connecting a and b in $N_{d_M}([a, b])$.

Let $a''_0 = a, a''_n = b$, and define $\varphi\colon [a, b] \to \Gamma$ to be a continuous one-to-one mapping such that $\varphi([a_i, a_{i+1}])$ is mapped to the part of Γ from a''_i to a''_{i+1}, for $i = 0, \dots, n - 1$, and φ is linear on the preimage of each edge. Then

$$|\varphi(x) - x| \le d_M + |a_i - a_{i+1}| \le 4d_M \qquad \text{for all} \quad x \in [a_i, a_{i+1}].$$

The function $\zeta(t) = \varphi(a + t(b - a))$ has all the desired properties and the proof is complete. $\square$

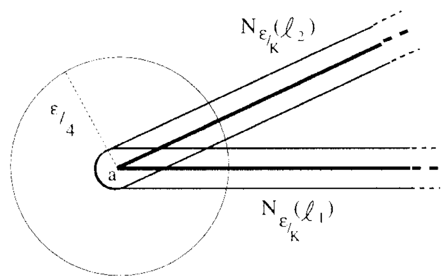

Fig. 2. The choice of K.

Proof of Proposition 4.2. We can take ε as small as we wish, so we assume that ε is less than the minimal distance between the vertices of $\partial\mathcal{T}$. By FLC, we can choose $K \geq 10$ so that for any two edges ℓ_1, ℓ_2 with a common vertex a,

$$N_{\varepsilon/K}(\ell_1) \cap N_{\varepsilon/K}(\ell_2) \subset B_{\varepsilon/4}(a), \tag{5.2}$$

as seen in Fig. 2. Choose $k \in \mathbb{N}$ so that

$$\lambda^{-k} d_M < \varepsilon/(2K) \leq \varepsilon/20. \tag{5.3}$$

For each vertex $a \in V(\partial\mathcal{T})$ choose a vertex $a' \in V(\varphi^{-k}\partial\mathcal{T})$ near a, so that $|a' - a| \leq \lambda^{-k} d_M$. Using that $\varphi^{-k}\mathcal{T}$ is LD from $\mathcal{T}$ we make sure that the choice of a' depends only on a $\mathcal{T}$-patch of some fixed radius around a (more precisely, there exists an $R > 0$ such that for any $a, b \in V(\partial\mathcal{T})$, if $[B_R(a)]^{\mathcal{T}} = [B_R(b)]^{\mathcal{T}} + (a - b)$ then $a' = b' + (a - b)$. We can take R as the radius of LD of $\mathcal{T}$ onto $\varphi^{-k}\mathcal{T}$).

Next consider an edge $\ell = [a, b] \in E(\partial\mathcal{T})$. Apply Lemma 5.1 to the tiling $\varphi^{-k}\mathcal{T}$ to find a simple path $\Gamma(\ell) \subset N_{\lambda^{-k} d_M}([a', b'])$, with a parameterization $\zeta_\ell\colon [0, 1] \to \Gamma(\ell)$ satisfying

$$|\zeta_\ell(t) - a' - t(b' - a')| \leq 5\lambda^{-k} d_M \qquad \text{for} \quad t \in [0, 1],$$

which implies

$$|\zeta_\ell(t) - a - t(b - a)| \leq 6\lambda^{-k} d_M \qquad \text{for} \quad t \in [0, 1] \tag{5.4}$$

by the choice of a', b'. The same thing is done for all edges, again using LD to make sure that the choice of $\Gamma(\ell)$ and ζ_ℓ depends only on a $\mathcal{T}$-patch of some fixed radius around the edge. Since we are considering *directed edges* here, we also use LD to ensure that

$$\text{if} \quad \ell = [a, b] \quad \text{and} \quad \bar{\ell} := [b, a], \quad \text{then} \quad \zeta_\ell(t) = \zeta_{\bar{\ell}}(1 - t) \quad \text{and} \quad \Gamma(\bar{\ell}) = \Gamma(\ell). \tag{5.5}$$

Observe that

$$\Gamma(\ell) \subset N_{\lambda^{-k} d_M}([a', b']) \subset N_{2\lambda^{-k} d_M}([a, b]) \subset N_{\varepsilon/K}([a, b])$$

by (5.3). If ℓ_1 and ℓ_2 are two edges with a common vertex a, then $\Gamma(\ell_1) \cap \Gamma(\ell_2) \subset B_{\varepsilon/4}(a)$ by (5.2).

Recall that each vertex of $\partial\mathcal{T}$ has degree 3 by (4.2). Fix a vertex a and let ℓ_1, ℓ_2, ℓ_3 be the edges coming out of it. One can "prune" some edges from $\Gamma(\ell_1) \cup \Gamma(\ell_2) \cup \Gamma(\ell_3)$ in order to make this union into a tree with three branches stemming from a vertex in $\varphi^{-k}\,\partial\mathcal{T}$ near a'. We show how to do it below (but note that there are many alternatives).

Let $\zeta_i = \zeta_{\ell_i}$ for $i \leq 3$. Recall that $\zeta_i(0) = a'$ for $i \leq 3$. Let

$$t_2 = \max\{t \in [0, 1]\colon\ \zeta_2(t) \in \Gamma(\ell_1)\}$$

and consider $\Gamma'(\ell_2) = \zeta_2([t_2, 1])$, so that $\Gamma(\ell_1)$ and $\Gamma'(\ell_2)$ intersect in a single point. Next, let

$$t_3 = \max\{t \in [0, 1]\colon\ \zeta_3(t) \in \Gamma(\ell_1) \cup \Gamma'(\ell_2)\}$$

and consider $\Gamma'(\ell_3) = \zeta_3([t_3, 1])$, so that $\Gamma'(\ell_3)$ intersects in a single point with $\Gamma'(\ell_2) \cup \Gamma(\ell_1)$. Finally, let

$$t_1 = \min\{t \in [0, 1]\colon\ \zeta_1(t) \in \Gamma'(\ell_2) \cup \Gamma'(\ell_3)\}$$

and consider $\Gamma'(\ell_1) = \zeta_1([t_1, 1])$, eliminating a possible "dead branch" (see Fig. 3). It is easy to see that $\Gamma'_a := \Gamma'(\ell_1) \cup \Gamma'(\ell_2) \cup \Gamma'(\ell_3)$ is a subgraph of $\varphi^{-k}\,\partial\mathcal{T}$ homeomorphic to $\ell_1 \cup \ell_2 \cup \ell_3$. It has a unique vertex of degree 3 which we denote $\tilde{a}$; a typical scenario is depicted in Fig. 3. It is clear that $\tilde{a} \in V(\varphi^{-k}\partial\mathcal{T}) \cap B_{\varepsilon/4}(a)$ and all the modifications occurred in $B_{\varepsilon/4}(a)$. These neighborhoods for different vertices a are disjoint by the choice of ε. We make sure that this procedure is performed the same way near every vertex depending only on a $\mathcal{T}$-patch of some fixed radius around the vertex. Let $\Gamma' = \bigcup_{a \in V(\partial\mathcal{T})} \Gamma'_a$. This is a subset of $\varphi^{-k}\,\partial\mathcal{T}$ homeomorphic to $\partial\mathcal{T}$, and we define the map Ψ as a specific homeomorphism from $\partial\mathcal{T}$ to Γ'.

Let $\ell = [a, b] \in E(\partial\mathcal{T})$ and let $\gamma(\ell)$ be the simple path in Γ' connecting $\tilde{a}$ with $\tilde{b}$ obtained from $\Gamma(\ell)$ by the above described modification near a and b. Note that $\Gamma(\ell) \cap \gamma(\ell)$ forms the bulk of $\gamma(\ell)$ by construction. Equip $\gamma(\ell)$ with the linear ordering making $\tilde{a}$ the smallest element (thus, we consider ℓ as a directed edge). Let

$$a'' = \min[\gamma(\ell) \cap V(\varphi^{-k}\partial\mathcal{T}) \backslash B_{\varepsilon/4}(a)],$$

that is, a'' is the first vertex on $\gamma(\ell)$ outside $B_{\varepsilon/4}(a)$. Consider $\gamma(\ell, a'') = \{x \in \gamma(\ell)\colon\ x \leq a''\}$, the initial part of $\gamma(\ell)$ from $\tilde{a}$ to a''. Then $\gamma(\ell, a'')$ is nonempty and $\gamma(\ell, a'') \subset$

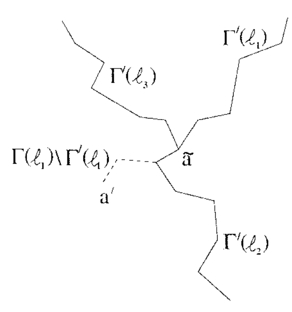

Fig. 3. Defining $\Gamma'(\ell_i)$ and $\tilde{a}$.

$B_{\varepsilon/3}(a)$ since $\lambda^{-k}d_M < \varepsilon/(2K) < \varepsilon/3 - \varepsilon/4$. Since $a'' \notin B_{\varepsilon/4}(a)$, we have $a'' \in \Gamma(\ell)$ and there is $t_{\ell,a} \in (0,1)$ such that $\zeta_\ell(t_{\ell,a}) = a''$. When we do this procedure for $\bar{\ell} = [b,a]$, we similarly get $b'' \in \gamma(\bar{\ell}) \cap \Gamma(\bar{\ell}) = \gamma(\ell) \cap \Gamma(\ell)$ near b and $t_{\bar{\ell},b} \in (0,1)$. Now let

$$\Psi(a + t(b-a)) = \zeta_\ell(t) \qquad \text{for} \quad t_{\ell,a} \le t \le t_{\bar{\ell},b}. \tag{5.6}$$

This is unambiguous due to (5.5). Next we define Ψ on $[a, a + t_{\ell,a}(b-a)]$ to be a continuous one-to-one map onto $\gamma(\ell, a'')$ such that $\Psi(a) = \tilde{a}$, $\Psi(a + t_{\ell,a}(b-a)) = a''$, and Ψ is linear on the preimage of every edge in $E(\partial\varphi^{-k}\mathcal{T})$. Again we make sure that everything depends only on a $\mathcal{T}$-patch of some fixed radius around the edge, which is possible since $\varphi^{-k}\mathcal{T}$ is LD from $\mathcal{T}$. Notice that Ψ will be defined on the whole of ℓ, since the part of the edge near b is taken care of when we consider $\bar{\ell}$. Since $\partial\mathcal{T}$ is a union of its edges, $\Psi\colon\ \partial\mathcal{T} \to\ \Gamma'$ is now completely defined.

We claim that Ψ has all the desired properties. We check condition (i) of Proposition 4.2. By (5.4) and (5.3) we have that $|\Psi(x) - x| < \varepsilon$ on the part of the edge where (5.6) applies. Let $x = a + t(b-a) \in \ell$, with $0 \le t \le t_{\ell,a}$. Since $a'' = \zeta_\ell(t_{\ell,a})$, we have by (5.4) that $|a'' - a - t_{\ell,a}(b-a)| \le 6\lambda^{-k}d_M$, and so the triangle inequality yields $|t_{\ell,a}(b-a)| - |a'' - a| \le 6\lambda^{-k}d_M$. Since $a'' \in B_{\varepsilon/3}(a)$, by (5.3),

$$|x - a| = |t(b-a)| \le |t_{\ell,a}(b-a)| \le \varepsilon/3 + 6\lambda^{-k}d_M \le \varepsilon/3 + 6\varepsilon/20 < 2\varepsilon/3.$$

On the other hand, $\Psi([a, a + t_{\ell,a}(b-a)]) = \gamma(\ell, a'') \subset B_{\varepsilon/3}(a)$, so $|\Psi(x) - a| \le \varepsilon/3$ and hence $|\Psi(x) - x| < \varepsilon$. This concludes the verification of Proposition 4.2(i). It is clear that the preimage of any $\ell \in \Psi(\partial\mathcal{T})$ is a proper subset of the edge $e \in \partial\mathcal{T}$ it is inside; by the finiteness property of our construction we can choose ρ as the largest proportion of $|\Psi^{-1}(\ell)|$ to $|e|$; clearly, $\rho < 1$. By construction, Ψ is linear on the preimage of every edge in $E(\varphi^{-k}\partial\mathcal{T})$. It remains to note that Ψ is LD from $\mathcal{T}$ since our construction depended only on a $\mathcal{T}$-patch of fixed radius in a translation-invariant way. □

6. Proof of Theorem 3.2

Recall the construction of the derived Voronoï family $\mathcal{F}(\mathcal{T})$ from Definition 2.10. We restate Theorem 3.2:

A nonperiodic, repetitive tiling of $\mathbb{R}^2$ is pseudo-self-similar if and only if its derived Voronoï family is ψ-finite for an expanding, orientation-preserving similitude ψ.

Proof. The sufficient condition is proved in Theorem 5.2 of [14]. We show that if $\mathcal{T}$ is a nonperiodic pseudo-self-similar tiling with expansion map φ, then the family $\mathcal{F}(\mathcal{T})$ is φ^m-finite for some $m \in \mathbb{N}$.

We use Theorem 3.1 to obtain an integer $k \ge 1$ and a self-similar tiling $\mathcal{T}'$ with expansion map φ^k that is MLD with $\mathcal{T}$. Suppose that $R > 0$ is chosen to be the radius of MLD between the two tilings, and let ρ be the radius of MLD of $\varphi^k\mathcal{T}'$ with $\mathcal{T}'$ (the latter are MLD by [19] since $\mathcal{T}'$ is nonperiodic).

We use a "core argument" like that used in [14] to show that every tiling in $\mathcal{F}(\mathcal{T})$ is the same, up to similarity, as one given by a finite list of "extensions" that come from $\mathcal{T}'$. The core is given by $E = [0]^{\mathcal{T}'}$, which will be present inside any central patch $P'_r = [B_r(0)]^{\mathcal{T}'}$. Write the locator set for a patch $P \subset \mathcal{T}'$ as $\mathcal{L}'(P) = \{q \in \mathbb{R}^2$ such that there exists $P' \subset \mathcal{T}'$ with $P = P' - q\}$. We establish that there is a list of $\mathcal{T}'$-patches $F_1, F_2, \ldots, F_m$ and vectors $q_1, q_2, \ldots, q_m$ so that for all r sufficiently large there exist $i_1, \ldots, i_n, l \in \mathbb{Z}$ with

$$\mathcal{L}_r = \varphi^{lk}\left(\bigcup_{j=1}^{n} \mathcal{L}'(F_{i_j}) + q_{i_j}\right) \tag{6.1}$$

and that the tiles of $\mathcal{T}_r$ can be relabeled (in a one-to-one fashion) by the elements of $2^{\{i_1,\ldots,i_n\}}$.

Fix $r \geq R + \rho$, and find the $l \in \mathbb{N}$ such that

$$R + \sum_{i=0}^{l} \lambda^{ik}\rho \leq r < R + \sum_{i=0}^{l+1} \lambda^{ik}\rho, \tag{6.2}$$

where $\lambda = \|\varphi\|$. (We do not need to consider $r < R + \rho$, as the number of derived Voronoï tilings with such r is finite by FLC.)

Now $P_r = [B_r(0)]^{\mathcal{T}}$ forces $P'_{r-R} = [B_{r-R}(0)]^{\mathcal{T}'}$ in the sense that for any $x \in \mathbb{R}^2$ such that $P_r + x$ appears in $\mathcal{T}$, we have that $P'_{r-R} + x$ appears in $\mathcal{T}'$. Note that this implies $\mathcal{L}_r \subset \mathcal{L}'(P'_{r-R})$. The $\mathcal{T}'$-patch P'_{r-R} in turn forces the central $\varphi^k\mathcal{T}'$-patch of radius $r - R - \rho$, which forces the central $\varphi^{2k}\mathcal{T}'$-patch of radius $r - R - \rho - \lambda^k\rho$, and so on until finally the $\varphi^{lk}\mathcal{T}'$-patch of $B_\eta(0)$ for some $\eta \geq 0$ is determined. In particular, the $\mathcal{T}$-patch P_r forces the core patch $\varphi^{lk}(E)$ in $\varphi^{lk}\mathcal{T}'$, and so we have that $\mathcal{L}_r \subset \varphi^{lk}(\mathcal{L}'(E))$.

We can put an upper bound on the repetitivity radius $R(r)$ of the $\mathcal{T}$-patch P_r as a constant multiple of r since $\mathcal{T}$ is linearly repetitive by Corollary 3.3. So we choose a number $A > 0$ so that $R(r) \leq \lambda^{lk}A$ for all r. We use this number to define a radius M for the extensions as follows:

$$\lambda^{lk}M - R \geq 2\lambda^{lk}A \geq 2R(r) \qquad \text{for all} \quad l \in \mathbb{N}. \tag{6.3}$$

The extensions $F_1, F_2, \ldots, F_n$ are defined to be representatives of the translation equivalence classes of $[B_M(q)]^{\mathcal{T}'}$ for all $q \in \mathcal{L}'(E)$. Let $q_1, q_2, \ldots, q_m$ be the elements of $\mathcal{L}'(E)$ with $F_i - q_i \cap E = E$. Note that

$$\mathcal{L}'(E) = (\mathcal{L}'(F_1) + q_1) \cup (\mathcal{L}'(F_2) + q_2) \cup \cdots \cup (\mathcal{L}'(F_m) + q_m). \tag{6.4}$$

Since we know that $\mathcal{L}_r \subset \varphi^{lk}(\mathcal{L}'(E))$, we can examine each $\varphi^{lk}(F_i)$ to determine whether or not it forces a copy of P_r in $\mathcal{T}$. Since $F_i = [B_M(q_i)]^{\mathcal{T}'}$, the $\varphi^{lk}\mathcal{T}'$-patch $\varphi^{lk}F_i$ forces the patch $[B_{\lambda^{lk}M}(\varphi^{lk}q_i)]^{\mathcal{T}'}$ (here we use that $\mathcal{T}'$ is self-similar, so any $\varphi^k\mathcal{T}'$-patch subdivides in a prescribed way into a $\mathcal{T}'$-patch), which in turn forces the patch $[B_{\lambda^{lk}M-R}(\varphi^{lk}q_i)]^{\mathcal{T}}$. By (6.3), $\lambda^{lk}M - R \geq 2R(r)$, and hence $\varphi^{lk}F_i$ not only determines whether there is a copy of P_r at $\varphi^{lk}q_i$, but if so, it also determines the label of the tile $t_{\varphi^{lk}q_i}$ in $\mathcal{T}_r$. Let $i_1, \ldots, i_n$ be the indices which correspond to extensions that force P_r. We see now that (6.1) holds for this choice of indices, and the labeling of the $\mathcal{T}_r$-tiles is also determined. (Note that several distinct patches $[B_{\lambda^{lk}M}(\varphi^{lk}q_{i_j})]^{\mathcal{T}'}$ may force $[B_{2R(r)}(\varphi^{lk}q_i)]^{\mathcal{T}}$, while the translational equivalence class of the latter is the label of $t_{\varphi^{lk}q_i} \in \mathcal{T}_r$. We can relabel by the subset of $\{i_1, \ldots, i_n\}$ consisting of those i_j which force $[B_{2R(r)}(\varphi^{lk}q_i)]^{\mathcal{T}}$; clearly this is a one-to-one relabeling.) Since φ^{lk} is a similitude, the Voronoï tilings corresponding to any locator set $\mathcal{L}$ and its expansion $\varphi^{lk}\mathcal{L}$ are φ^{lk}-similar, and there are only a finite number of ways the relabelings could have been chosen for tiles in $\mathcal{T}_r$. Since the choice of extensions was independent of r, we see that the family $\mathcal{F}(\mathcal{T})$ is φ^{lk}-finite. □

7. Concluding Remarks

1. We believe that our results generalize to tilings of $\mathbb{R}^d$ when $d > 2$. Derived Voronoï tilings exist in higher dimensions, and they keep their convenient geometric and combinatorial properties, but the addition of more dimensions brings more elements to the combinatorial structure than the usual vertices, edges, and facets. It is probably possible to extend the algorithm which redraws the edges of tiles to a redrawing of higher-dimensional tile boundaries, but the details can get quite involved.

2. It would be interesting to extend our results to the case of tilings in which rotations of prototiles are allowed along with translations. Of course, if only rational (modulo π) rotations occur, we can increase the number of prototiles and still deal with translational equivalence classes. Radin and others have contributed to the theory of substitution tilings in which prototiles occur in infinitely many orientations; see [15], [16], and references therein.

Note that FLC would now have to be defined up to translation and rotation, and the definition of LD would have to be modified. Let G be the group of Euclidean motions of the plane preserving orientation; G is generated by translations and rotations. We would say that $\mathcal{T}_2$ is LD from $\mathcal{T}_1$ if there exists $R > 0$ such that for every $x, y \in \mathbb{R}^2$ and $g \in G$ with $g(y) = x$,

$$[B_R(x)]^{\mathcal{T}_1} = g\,[B_R(y)]^{\mathcal{T}_1} \quad \Rightarrow \quad [x]^{\mathcal{T}_2} = g\,[y]^{\mathcal{T}_2}.$$

In the definition of a self-similar tiling (Definition 2.8) replace the last condition by

(ii′) if $T' = g(T)$ for some $g \in G$, then $P(\varphi T') = (\varphi g \varphi^{-1})P(\varphi T)$.

Much of the proof of Theorem 3.1 extends to this setting, but there are some difficulties, so the question remains open. Note that throughout the paper we referred to a number of results on tilings in translational-FLC setting, among them the characterization of expansion constants for self-similar tilings, the (absence of) mixing results for associated dynamical systems, the unique composition property, and the implication *finite Voronoï family* ⇒ *pseudo-self-similar*. None of these is known in the more general setting discussed here.

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