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A Helly-Type Theorem for Hyperplane Transversals to Well-Separated Convex Sets*

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Abstract. Let $\mathcal{S}$ be a finite collection of compact convex sets in $\mathbb{R}^d$. Let $D(\mathcal{S})$ be the largest diameter of any member of $\mathcal{S}$. We say that the collection $\mathcal{S}$ is *ε-separated* if, for every $0 < k < d$, any k of the sets can be separated from any other $d - k$ of the sets by a hyperplane more than $\varepsilon D(\mathcal{S})/2$ away from all d of the sets. We prove that if $\mathcal{S}$ is an ε-separated collection of at least $N(\varepsilon)$ compact convex sets in $\mathbb{R}^d$ and every $2d+2$ members of $\mathcal{S}$ are met by a hyperplane, then there is a hyperplane meeting all the members of $\mathcal{S}$. The number $N(\varepsilon)$ depends both on the dimension d and on the separation parameter ε. This is the first Helly-type theorem known for hyperplane transversals to compact convex sets of arbitrary shape in dimension greater than one.

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1. Introduction

A *k-transversal* to a collection $\mathcal{S}$ of point sets in $\mathbb{R}^d$ is a k-flat, i.e., an affine subspace of dimension k such as a point, line, or hyperplane, that intersects every member of $\mathcal{S}$. We are interested in conditions under which a collection of compact convex sets has a k-transversal.

Vincensini [13] first posed this problem in 1935, and claimed erroneously that a finite collection of compact convex sets in $\mathbb{R}^2$ has a line transversal if and only if every six members of the collection have a line transversal. In fact, Santaló [11] showed that for any m there are finite collections of compact convex sets in $\mathbb{R}^2$ such that every m members of the collection have a line transversal, but the entire collection does not. Moreover, such counterexamples exist even when the sets are restricted to be pairwise disjoint [8], or pairwise disjoint line segments [10], or unit disks (although not pairwise disjoint unit disks).

Theorems of the form "If every k members of a collection have a property P, then the entire collection has property P" are known as "Helly-type" theorems, after Helly's theorem about the intersection of convex sets. Actually, Helly's theorem itself can be restated as a Helly-type theorem about point transversals: If every $d + 1$ members of a collection of compact convex sets in $\mathbb{R}^d$ have a point transversal, then the entire collection has a point transversal. Santaló's counterexamples show that there is no such Helly-type theorem for line transversals to collections of compact convex sets in $\mathbb{R}^2$. However, he was able to give such a theorem for line transversals to collections of axis-parallel rectangles in $\mathbb{R}^2$ and, more generally, for hyperplane transversals to axis-parallel parallelepipeds in $\mathbb{R}^d$ [11]. This led to the exploration of Helly-type theorems for transversals of other, specialized collections of compact convex sets. In 1957 Danzer [4] proved a conjecture by Hadwiger that if every five members of a collection of pairwise disjoint unit disks in $\mathbb{R}^2$ have a line transversal, then the entire collection has a line transversal.

Grünbaum [7] conjectured that Danzer's theorem generalized to any collection of pairwise disjoint translates of a single compact convex set in the plane. Little progress was made on this conjecture for the next 25 years. Finally, in 1986, Katchalski [9] proved a Helly-type theorem for line transversals of pairwise disjoint translates in the plane but with a Helly number of 128. Three years later, Tverberg [12] proved Grünbaum's conjecture, showing that if every five members of a collection of pairwise disjoint translates in $\mathbb{R}^2$ have a line transversal, then the entire collection has a line transversal. (See [6] for a more detailed history.)

Katchalski has conjectured that Danzer's theorem generalizes to line transversals of unit balls in $\mathbb{R}^3$. In other words, Katchalski conjectured that there is some Helly number m such that if every m members of a collection of pairwise disjoint unit balls in $\mathbb{R}^3$ have a line transversal, then the entire collection has a line transversal. This conjecture is still open.

Instead of generalizing to *line* transversals in $\mathbb{R}^3$, Danzer's theorem can be generalized to *plane* transversals in $\mathbb{R}^3$. The condition of pairwise disjointness is now no longer sufficient. The examples of collections of unit disks in the plane, where every m have a transversal, but the entire collection does not, can be lifted to pairwise disjoint unit balls in $\mathbb{R}^3$, where every m have a plane transversal but the entire collection does not. A stronger condition is needed. We conjecture that this condition is that the collection of unit balls has no triples with line transversals. A collection of compact convex sets in $\mathbb{R}^3$, no three of which have a line transversal, is called *separated*. Equivalently, a collection $\mathcal{S}$

of compact convex sets in $\mathbb{R}^3$ is separated if each convex set in $\mathcal{S}$ can be strictly separated from any two other sets in $\mathcal{S}$ by a plane. We conjecture that there is some number m such that if every m members of a separated collection of unit balls have a plane transversal, then the entire collection has a plane transversal [2]. This conjecture is also open.

In this paper we show that if we bound the separation distance from below, we can indeed get a Helly-type theorem for plane transversals. More precisely, a collection of unit balls in $\mathbb{R}^3$ is *ε-separated* if each ball in $\mathcal{S}$ can be separated from any two other balls in $\mathcal{S}$ by a plane that lies at distance more than ε away from all three balls. We prove that if $\mathcal{S}$ is a finite ε-separated collection of at least $N(\varepsilon)$ unit balls in $\mathbb{R}^3$ and every eight members of $\mathcal{S}$ have a plane transversal, then $\mathcal{S}$ has a plane transversal.

The theorem holds for finite collections of compact convex sets if we properly generalize the definition of ε-separation. A finite collection $\mathcal{S}$ of compact convex sets in $\mathbb{R}^3$ is *ε-separated* if each set in $\mathcal{S}$ can be separated from any other two sets in $\mathcal{S}$ by a plane that lies at distance more than $\varepsilon D(\mathcal{S})/2$ away from all three sets where $D(\mathcal{S})$ is the largest diameter of any set in $\mathcal{S}$. If $\mathcal{S}$ is a finite ε-separated collection of at least $N(\varepsilon)$ compact convex sets in $\mathbb{R}^3$ and every eight members of $\mathcal{S}$ have a plane transversal, then $\mathcal{S}$ has a plane transversal.

The theorem further generalizes to hyperplane transversals in any dimension. A collection of compact convex sets in $\mathbb{R}^d$, $d \geq 2$, is called *separated* if no d of the sets have a $(d-2)$-transversal. Equivalently, a collection $\mathcal{S}$ of compact convex sets in $\mathbb{R}^d$ is separated if any k sets of $\mathcal{S}$, $0 < k < d$, can be strictly separated from any other $d-k$ sets of $\mathcal{S}$ by a hyperplane. A finite collection $\mathcal{S}$ is *ε-separated* if any k of the sets of $\mathcal{S}$, $0 < k < d$, can be strictly separated from any other $d-k$ of the sets of $\mathcal{S}$ by a hyperplane that is more than $\varepsilon D(\mathcal{S})/2$ away from all d of the sets where $D(\mathcal{S})$ is the largest diameter of any set in $\mathcal{S}$. We prove that if $\mathcal{S}$ is a finite ε-separated collection of at least $N(\varepsilon)$ compact convex sets in $\mathbb{R}^d$ and every $2d+2$ of the sets of $\mathcal{S}$ have a hyperplane transversal, then $\mathcal{S}$ has a hyperplane transversal. The number $N(\varepsilon)$ depends on both the separation parameter ε and the dimension d.

For $d = 2$, our result yields a Helly-type theorem for line transversals to ε-separated collections of pairwise disjoint convex sets in $\mathbb{R}^2$. Hence it might appear that this contradicts the Hadwiger–Debrunner examples [8] of finite collections of pairwise disjoint compact convex sets in $\mathbb{R}^2$ such that every m have a line transversal but the entire collection does not. Since the sets are compact and pairwise disjoint, each such collection is, indeed, ε-separated for some value of ε. However, in each such case the collection has fewer than $N(\varepsilon)$ members and so our theorem does not apply.

The next section is devoted to a proof of our main result:

Theorem 1. *For each dimension d and each $\varepsilon > 0$, there is a number $N(\varepsilon)$ such that if every $2d+2$ members of a finite ε-separated collection $\mathcal{S}$ of at least $N(\varepsilon)$ compact convex sets in $\mathbb{R}^d$ have a hyperplane transversal, then all the members of $\mathcal{S}$ do.*

2. Proof of the Theorem

Throughout this paper a *body* is a compact convex set. Recall that a collection of at least d bodies in $\mathbb{R}^d$ is *separated* [14] if no d of the bodies have a $(d-2)$-transversal, i.e., there is no $(d-2)$-flat that meets any d of the bodies. This is equivalent to the condition that, if $0 < k < d$, then any k of the bodies can be strictly separated from any other $d-k$ of the bodies by a hyperplane. We generalize this as follows.

Definition. Given $\varepsilon \geq 0$, a finite collection $\mathcal{S}$ of at least d bodies in $\mathbb{R}^d$ is *ε-separated* if, for every $0 < k < d$, any k of the bodies can be separated from any other $d-k$ of the bodies by a hyperplane more than $\varepsilon D(\mathcal{S})/2$ away from all d of the bodies, where $D(\mathcal{S})$ is the largest diameter of any body in $\mathcal{S}$.

Notice that for $\varepsilon = 0$ this specializes to the condition that the bodies are separated.

For any body S, let $S(\alpha)$ be the Minkowski sum of S with the closed ball of radius α centered at the origin and let $\mathcal{S}(\alpha) = \{S(\alpha) \mid S \in \mathcal{S}\}$. Then clearly $\mathcal{S}$ is ε-separated if and only if $\mathcal{S}(\varepsilon D(\mathcal{S})/2)$ is separated, so that in particular it follows that $\mathcal{S}$ is ε-separated if and only if given any $(d-2)$-flat F and any d bodies $S_1, \ldots, S_d \in \mathcal{S}$, F is more than $\varepsilon D(\mathcal{S})/2$ away from at least one of the bodies S_i, i.e., F avoids at least one of the bodies $S_i(\varepsilon D(\mathcal{S})/2)$.

Notice in particular that the definition is invariant under scaling and under rigid motions. To simplify our presentation, we assume hereafter, without loss of generality, that $D(\mathcal{S}) = 2$. In addition, notice that if $\mathcal{S}$ is a collection of ε-separated bodies, $\mathcal{S}' \subseteq \mathcal{S}$, and $0 \leq \delta \leq \varepsilon$, then $\mathcal{S}'$ is also δ-separated.

In what follows we work in $\mathbb{R}^d$, with $d \geq 2$ fixed. Thus in our notation we suppress the dependence of the various "constants" on d.

Definition. The *orientation* of the $(d+1)$-tuple $(a_1, \ldots, a_{d+1})$ of points in $\mathbb{R}^d$ is $\operatorname{sgn}\langle a_1, \ldots, a_{d+1}\rangle$, the sign of the determinant of the $(d+1) \times (d+1)$ matrix whose ith row consists of the d coordinates of a_i followed by 1.

Definition. The *orientation* of the d-tuple $(a_1, \ldots, a_d)$ of points lying in an oriented hyperplane H with normal $\mathbf{n}$ is $\operatorname{sgn}\langle 0, a_2 - a_1, \ldots, a_d - a_1, \mathbf{n}\rangle$.

We begin with a lemma which, strictly speaking, is not needed, but whose proof will make that of the lemma that follows more transparent.

Lemma 1. *Given points $a_1, \ldots, a_d, a_1', \ldots, a_d' \in \mathbb{R}^{d-1}$. If the d-tuples $(a_1, \ldots, a_d)$ and $(a_1', \ldots, a_d')$ have opposite orientation, there is a $(d-2)$-flat cutting all of the segments $a_i a_i'$.*

Proof. For $t \in [0, 1]$ let $a_i(t) = (1-t)a_i + t a_i'$. Since $\langle a_1(t), \ldots, a_d(t)\rangle$ changes sign in $[0, 1]$ it must vanish at some value of t. □

Remark. If $\mathcal{S}$ is a separated collection of bodies in $\mathbb{R}^d$, then it makes sense to speak of the orientation of the intersections of a d-tuple of bodies of $\mathcal{S}$ with an oriented hyperplane $(H, \mathbf{n})$. Indeed, no $(d-2)$-flat in H can meet all d bodies, so that (by Lemma 1) the orientation of any d-tuple of points, one from each of the d bodies, is independent of the choice of the points.

Lemma 2. *Suppose $a_1, \dots, a_d, a'_1, \dots, a'_d \in \mathbb{R}^d$ with $a_1, \dots, a_d$ (resp. $a'_1, \dots, a'_d$) in general position, and with $|a_i a'_i| \le 1$ for every i. Let H (resp. H') be the hyperplane spanned by the points a_i (resp. a'_i) for $i = 1, \dots, d$, with unit normal $\mathbf{n}$ (resp. $\mathbf{n}'$) chosen so that $(a_1, \dots, a_d)$ in $(H, \mathbf{n})$ and $(a'_1, \dots, a'_d)$ in $(H', \mathbf{n}')$ have* opposite *orientation, and let ε be the angle between $\mathbf{n}$ and $\mathbf{n}'$. Suppose $0 \le \varepsilon \le \pi/2$. Then there exists a $(d-2)$-flat F within ε of all the segments $a_i a'_i$.*

Proof. If $\varepsilon > 0$, the two hyperplanes H and H' partition $\mathbb{R}^d$ into four quadrants, one of which, Q_{+-}, lies on the positive side of H and on the negative side of H'. Let $H(t)$, $0 \le t \le 1$, be a hyperplane that rotates about the $(d-2)$-flat $H \cap H'$ from H to H' through Q_{+-} (and its opposite quadrant Q_{-+}), with $H(0) = H$ and $H(1) = H'$. Let $\mathbf{n}(t)$ be the normal to $H(t)$, chosen so that it varies continuously with t, with $\mathbf{n}(0) = \mathbf{n}$ and $\mathbf{n}(1) = \mathbf{n}'$. Suppose the segments $a_1 a'_1, \dots, a_k a'_k$ lie in Q_{+-} or Q_{-+}, while $a_{k+1} a'_{k+1}, \dots, a_d a'_d$ lie in the remaining two quadrants, Q_{++} and Q_{--} (see Fig. 1). For each $t \in [0, 1]$, we choose points $b_i(t) \in H(t)$, $i = 1, \dots, d$, as follows. For $i = 1, \dots, k$, let $b_i(t)$ be the *intersection* of the segment $a_i a'_i$ with $H(t)$; for $i = k+1, \dots, d$, let $b_i(t)$ be the *projection* of $a_i(t)$ on $H(t)$, where (for $i = k+1, \dots, d$) $a_i(t)$ is any continuous parametrization of the segment $a_i a'_i$ with $a_i(0) = a_i$ and $a_i(1) = a'_i$.

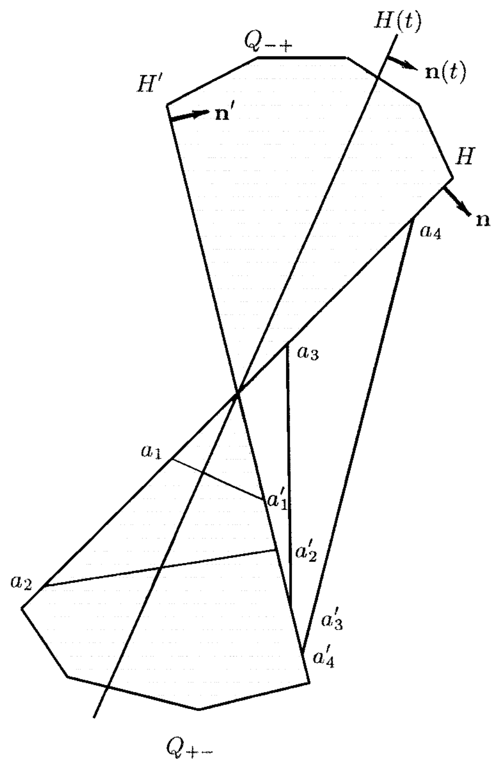

Fig. 1. The four quadrants of Lemma 2.

As t moves from 0 to 1, the d-tuple $(b_1(t), \ldots, b_d(t))$ in $(H(t), \mathbf{n}(t))$ goes from one orientation to the other. Hence for some $t^* \in (0, 1)$, as in Lemma 1, we must have a $(d-2)$-flat F passing through all the points $b_1(t^*), \ldots, b_d(t^*)$. We claim that F lies within ε of all the segments $a_1a'_1, \ldots, a_da'_d$.

For $i = 1, \ldots, k$, this is clear, since F actually *meets* these segments. Suppose $i > k$. Since the segment $a_ia'_i$ lies either in Q_{++} or in Q_{--}, and its length is bounded by 1, its extreme points and all the points between them lie no more than one unit from $H \cap H'$ (this is where we are using the assumption that $\varepsilon \le \pi/2$), hence within a distance of $\sin\varepsilon < \varepsilon$ from their projections into $H(t)$ for each $t \in [0, 1]$. However, for $t = t^*$, these projections lie in F. Hence the distance from F to each segment $a_ia'_i$ is no more than ε.

Finally, if $\varepsilon = 0$, the two hyperplanes H and H' are parallel, and all the segments are of the first type. Let $H(t)$ move monotonically from H to H', remaining parallel to H, and apply the argument of Lemma 1; the result is a flat F cutting *all* of the segments $a_ia'_i$ $(i = 1, \ldots, d)$. □

Recall that the *width*, width X, of a compact set X is the least $w \ge 0$ for which there is a hyperplane H with unit normal $\mathbf{n}$ such that X lies between H and $H + w\mathbf{n}$. Notice that, by this definition, if the affine dimension of X is smaller than the dimension of the ambient space, its width is automatically 0. Sometimes, however, we need to measure the *width* $\text{width}_F\, X$ *of* X *relative to a* k*-flat* $F \supset X$—we define it to be the minimum distance between two parallel $(k-1)$-flats in F containing X between them.

Throughout the remainder of this paper, by the width of a collection of sets we mean the width of their union.

Lemma 3. *Any compact set* X *of affine dimension* d *in* $\mathbb{R}^d$ *contains a subset of size* $d + 1$ *and width at least* $(\text{width}\, X)/c$, *where* $c = c(d) > 0$ *is a dimension-dependent constant.*

Proof. We explicitly construct the $(d+1)$-tuple $P = \{p_1, \ldots, p_{d+1}\}$ of points incrementally. Start with an arbitrary point $p_1 \in X$. Suppose we have already constructed $p_1, \ldots, p_i$, for some i, $1 \le i \le d$. Then let p_{i+1} be a point of X farthest from the $(i-1)$-flat $F_i = \text{aff}\{p_1, \ldots, p_i\}$. Let $h_i = d(p_{i+1}, F_i)$, where $d(\cdot,\cdot)$ denotes the Euclidean distance.

We proceed to prove that P has the desired property. For convenience, let $\Delta = \text{conv}\, P$. First, observe that $h_i \ge h_{i+1}$, for $1 \le i \le d-1$. Secondly, $\text{vol}\, \Delta = (1/d!) \prod_{i=1}^{d} h_i$.

We now construct a hyperrectangle R containing X, as follows: We start with a sequence of cylindrical objects C_i, $i = 1, \ldots, d$, each containing X. Specifically, let

$$C_i = \{x \in \mathbb{R}^d \mid d(x, F_i) \le h_i\}.$$

We now approximate C_i by a parallel slab $S_i \supseteq C_i$ whose medial hyperplane H_i passes through F_i and whose width is $2h_i$. This does not fully specify S_i: there is still some freedom in choosing its orientation. We arrange it so that the slabs S_i are mutually orthogonal. Indeed, S_d is fully specified, as its medial hyperplane $H_d = F_d$ is fixed. H_{d-1} and H_d meet in F_{d-1}, and H_{d-1} can be rotated around this $(d-2)$-flat. We rotate it so that it is orthogonal to H_d. The process is then repeated: Having fixed the position for all but the first i slabs, we fix the position of S_i by noting that H_i, as all

H_j for $j > i$, passes through the $(i-1)$-flat F_i and $H_{i+1}, \ldots, H_d$ have been chosen to be mutually orthogonal. Hence there exists one more hyperplane orthogonal to all of them and containing F_i—that is how H_i is chosen. The intersection of the d slabs is a hyperrectangle R with $\operatorname{vol} R = \prod_{i=1}^{d} 2h_i = 2^d d! \operatorname{vol} \Delta$.

As the smallest dimension of R is $2h_d$, $\operatorname{width} X \leq \operatorname{width} R = 2h_d$. On the other hand, $\operatorname{width} P = \operatorname{width} \Delta \geq 2r$, where r is the radius of the largest sphere inscribed in Δ. In fact, $\operatorname{vol} \Delta = r \cdot \operatorname{area} \Delta / d$, where $\operatorname{area} \Delta$ is the surface area of Δ, i.e., the $(d-1)$-dimensional volume of the boundary of Δ. As $\Delta \subset R$ and both are convex, $\operatorname{area} \Delta \leq \operatorname{area} R \leq 2d \prod_{i=1}^{d-1} 2h_i = d(\operatorname{vol} R)/h_d$. Hence

$$\operatorname{width} P \geq 2r = \frac{2d \operatorname{vol} \Delta}{\operatorname{area} \Delta} \geq \frac{2d(\operatorname{vol} R/2^d d!)}{d(\operatorname{vol} R/h_d)} = \frac{2h_d}{2^d d!} \geq \frac{\operatorname{width} X}{2^d d!},$$

as claimed. □

For a hyperplane H, let π_H denote orthogonal projection to H. We then have:

Lemma 4. *There exists a sufficiently large $c = c(d) \geq 2\pi$, so that, for any $\varepsilon < \pi/2$, the following holds*: *If $\mathcal{P}$ is a set of d bodies in $\mathbb{R}^d$ each of diameter at most* 2 *and such that* $\operatorname{width}_H(H \cap \mathcal{P}) > c/\varepsilon$ *for every hyperplane transversal H of $\mathcal{P}$, then any two hyperplane transversals meeting the bodies in the same orientation are within an angle ε of each other.*

Proof. At first glance, the lemma appears ill-stated, since the definition of orientation for a collection of *d bodies* in an oriented hyperplane requires separation. Hence we begin by observing that $\mathcal{P}$ is indeed separated. If that were not the case, $\mathcal{P}$ would have a $(d-2)$-transversal, and since the diameter of any set in $\mathcal{P}$ is at most 2, $\mathcal{P}$ would be contained in the cylinder C of radius 2 around this $(d-2)$-transversal, so that for *any* hyperplane H, we would have $\operatorname{width}_H(H \cap \mathcal{P}) \leq \operatorname{width}_H \pi_H(\mathcal{P}) \leq \operatorname{width}_H \pi_H(C) = \operatorname{width} C = 4 < c/\varepsilon$, contradicting our width assumption.

Fix an arbitrary *reference point* in each set of $\mathcal{P}$. Let $(H, \mathbf{n})$ be the oriented hyperplane spanned by the d reference points, and let $(H', \mathbf{n}')$ be any oriented hyperplane transversal to $\mathcal{P}$ such that the orientation in $(H', \mathbf{n}')$ of the (ordered) collection $H' \cap \mathcal{P}$ agrees with the orientation of (the ordered set of) the reference points in $(H, \mathbf{n})$. We argue that the angle between $\mathbf{n}$ and $\mathbf{n}'$ does not exceed $\varepsilon/2$, if c is large enough. This will finish the proof.

Let F be the $(d-2)$-flat $H \cap H'$. (If $H' \parallel H$, $\mathbf{n}' = \pm\mathbf{n}$. By a continuity argument similar to the one given below $\mathbf{n}' = -\mathbf{n}$ is impossible.) Project H, H', $\mathcal{P}$, and F orthogonally to the 2-flat (i.e., plane) $F^{\perp}$. Refer to Fig. 2. Since $\operatorname{width}_H H \cap \mathcal{P}$ is at least c/ε, the projections of the reference points span a segment of length at least $c/\varepsilon - 4$ on the line $\ell = H \cap F^{\perp}$. Thus ℓ and $\ell' = H' \cap F^{\perp}$ are both line transversals of the projection of $\mathcal{P}$, which is a collection $\mathcal{P}' = \pi_{F^{\perp}}(\mathcal{P})$ of convex sets each of which fits in a disk of radius 2 centered at a point of ℓ; disk centers are spread out for a distance at least $c/\varepsilon - 4$ along ℓ. Hence the angle θ between the two lines (and thus the two hyperplanes) is such that $\sin\theta \leq 4/(c/\varepsilon - 4)$. Since $\theta \leq \pi/2$, we conclude that $\theta \leq (\pi/2)\sin\theta \leq 2\pi/(c/\varepsilon - 4) < \varepsilon/2$ for an appropriate choice of c.

Without loss of generality, suppose that ℓ is horizontal and (the projection of) $\mathbf{n}$ points vertically upward. To finish the argument, we must show that (the projection of) $\mathbf{n}'$ also

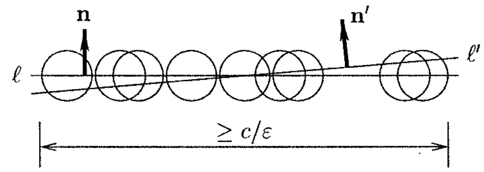

Fig. 2. Orthogonal projection to $F^{\perp}$; the figure illustrates the case where each set is a unit ball and the reference points coincide with ball centers, for ease of visualization.

points into the upper halfplane, which, together with the fact that the directions of ℓ and ℓ' are within $\varepsilon/2$ of each other, implies that the same holds for $\mathbf{n}$ and $\mathbf{n}'$.

To argue this, we replace each body of $\mathcal{P}$ with a ball of radius 2 centered at the corresponding reference point, obtaining a collection $\mathcal{Q}$ of d sets in $\mathbb{R}^d$. The argument that $\mathcal{P}$ is separated given in the beginning of this proof is easily seen to apply to $\mathcal{Q}$ as well, with a slightly larger value of c (the only difference being that it takes a cylinder of radius 4, and not 2, to enclose $\mathcal{Q}$ if it is not separated). Let $\mathcal{Q}' = \pi_{F^{\perp}}(\mathcal{Q})$. It is a collection of disks of radius 2 centered at points on the line ℓ. Line ℓ' also meets all disks. It is easy to verify that under these assumptions ℓ' can be rotated into ℓ around the point of their intersection while remaining a transversal of $\mathcal{Q}'$—the rotation is through the smaller angle between the lines, i.e., at most $\varepsilon/2$. This means that H' can be rotated into H while remaining a transversal of $\mathcal{Q}$. However, by continuity, such a rotation cannot change the orientation of the intersection of $\mathcal{Q}$ with the hyperplane, for otherwise $\mathcal{Q}$ would have a $(d-2)$-transversal. Hence $(H, \mathbf{n})$ can be obtained from $(H', \mathbf{n}')$ by a rotation through an angle of at most $\varepsilon/2$, as claimed. □

Corollary 1. *The conclusion of Lemma* 4 *also holds for collections* $\mathcal{P}$ *of* d *bodies* (*each of diameter at most* 2) *with the property that there is* some *hyperplane* G (*not necessarily a transversal of* $\mathcal{P}$) *with* $\text{width}_G\, \pi_G(\mathcal{P}) \ge c'/\varepsilon$, *for some constant* $c' > c$, *where* c *is the constant of Lemma* 4.

Proof. Let P be a set of d points, one from each body of $\mathcal{P}$. Let $H = \text{aff}\, P$. Note that the affine dimension of P cannot be smaller than $d-1$, for otherwise $\mathcal{P}$ would fit into a cylinder of radius 2 around H and no projection of $\mathcal{P}$ to a hyperplane would have width larger than 4—this would contradict our width assumption, since $c' > c \ge 2\pi$. Hence H is a hyperplane.

It is now sufficient to prove that $\text{width}_H\, P \ge c/\varepsilon$. Indeed, it is easy to see that $\text{width}_G\, \pi_G(\mathcal{P}) \le \text{width}_G\, \pi_G(P) + 4$ and that $\text{width}_G\, \pi_G(P) \le \text{width}_H\, P$. Hence $\text{width}_H\, H \cap \mathcal{P} \ge \text{width}_H\, P \ge c'/\varepsilon - 4 \ge c/\varepsilon$, for an appropriate choice of $c' > c$ (namely $c' = c + 2\pi$). □

Proposition 1. *For each* $\varepsilon \ge 0$, *there is a number* $N(\varepsilon)$ *such that every* ε*-separated collection of at least* $N(\varepsilon)$ *bodies in* $\mathbb{R}^d$ *contains* d *bodies such that any two oriented hyperplanes each meeting these* d *bodies in positively oriented sets make an angle of less than* ε.

Proof. In view of the fact that ε-separation implies δ-separation for $0 \le \delta < \varepsilon$, we may assume without loss of generality that $0 \le \varepsilon \le \pi/2$. Let $w = c/\varepsilon$, for an appropriate choice of $c = c(d)$, to be specified below.

Let $\mathcal{S}$ be an ε-separated collection, as above, of size at least $N(\varepsilon) = (d-1)(c + 2\varepsilon^2)^2/(\pi\varepsilon^4)$. We first argue that its orthogonal projection $\pi_H(\mathcal{S})$ to some hyperplane H has width, relative to H, at least w.

Indeed, suppose there is no such hyperplane. Then $\mathcal{S}$ lies in some parallel slab of width less than w. Let H be one of the bounding hyperplanes of the slab. Since $\text{width}_H\, \pi_H(\mathcal{S}) < w$, it follows that $\mathcal{S}$ is contained in an open region which is the Cartesian product of a $w \times w$ square with some $(d-2)$-flat F. By the definition of ε-separation, no $(d-2)$-flat can meet more than $d-1$ sets of $\mathcal{S}(\varepsilon)$. Hence, if we project $\mathcal{S}(\varepsilon)$ to the orthogonal complement of F, the projected collection is confined to a region of area less than $(w + 2\varepsilon)^2$, covers it no more than $(d-1)$-fold, and consists of sets of area at least $\pi\varepsilon^2$ each. So $|\mathcal{S}| < (d-1)(w+2\varepsilon)^2/(\pi\varepsilon^2) = (d-1)(c+2\varepsilon^2)^2/(\pi\varepsilon^4) = N(\varepsilon)$, a contradiction.

Thus, for a large enough ε-separated collection $\mathcal{S}$, there always exists a hyperplane H with $\text{width}_H\, \pi_H(\mathcal{S}) \ge w$. By Lemma 3, there is a subset P of d points in $\pi_H(\mathcal{S})$ with $\text{width}_H\, P \ge w/(2^d d!)$. Pick d distinct sets of $\mathcal{S}$, each containing a point whose projection belongs to P; note that since P has large width in H, the projection of no single body of $\mathcal{S}$ could contain more than one point of P. Let $\mathcal{P}$ be the resulting set of d bodies. Hence $\text{width}_H\, \pi_H(\mathcal{P}) \ge \text{width}_H\, P > c/(2^d d!\, \varepsilon)$, so that the corollary applies, provided c is large enough. This is the desired set of d bodies. □

The following definition is not the standard one, but the "Local Realizability Criterion" on p. 140 of [3] shows that they are equivalent, at least in the case we are interested in, where the sets are in general position.

Definition. A *rank-r oriented matroid* on a finite set M consists of a positive or negative orientation assigned to each r-tuple of distinct elements of M so that r-tuples that differ by an even (resp. odd) permutation have the same (resp. opposite) orientation and so that each subset of size $r+2$ is *realizable* in $\mathbb{R}^{r-1}$. This means that every $(r+2)$-subset M' of M is in 1–1 correspondence with an $(r+2)$-subset P' of points in $\mathbb{R}^{r-1}$ so that corresponding r-tuples have the same orientation.

The oriented matroid structure derived from a finite set of points in $\mathbb{R}^d$ is called the *order type* of the set.

We recall the following "generalized Hadwiger theorem" of Goodman and Pollack from [5].

Generalized Hadwiger Theorem. *A finite separated collection $\mathcal{S}$ of bodies in $\mathbb{R}^d$ has a hyperplane transversal if and only if there is an oriented matroid of rank d on $\mathcal{S}$ such that every $d+1$ members of $\mathcal{S}$ are met by an oriented hyperplane consistently with that oriented matroid*.

We are now able to present our proof of Theorem 1.

Proof of Theorem 1. We may assume without loss of generality that $\varepsilon \leq \pi/2$, since ε-separated bodies are also δ-separated for every $\delta < \varepsilon$. This being the case, it follows in particular that $\varepsilon \leq \pi - \varepsilon$.

By Proposition 1, we may choose d bodies $S_1^*, \ldots, S_d^*$ from $\mathcal{S}$ such that any two oriented hyperplanes each meeting them in positively oriented sets make an angle smaller than ε. For every $d+2$ bodies $S_{i_1}, \ldots, S_{i_{d+2}}$ of $\mathcal{S}$, there is a transversal $T = T(i_1, \ldots, i_{d+2})$ to $S_1^*, \ldots, S_d^*, S_{i_1}, \ldots, S_{i_{d+2}}$ (if some indices are repeated in this list, enlarge this collection to contain $2d+2$ distinct sets of $\mathcal{S}$ in an *arbitrary* way and then pick a transversal); fix it, choosing a unit normal vector $\mathbf{n}(i_1, \ldots, i_{d+2})$ so that $S_1^* \cap T, \ldots, S_d^* \cap T$ have positive orientation. Since any two transversals, $T(i_1, \ldots, i_d, i_{d+1}, i_{d+2})$ and $T(i_1, \ldots, i_d, i'_{d+1}, i'_{d+2})$, make an angle smaller than ε, it follows (by Lemma 2) that they meet $S_{i_1}, \ldots, S_{i_d}$ with the same orientation—since the presence of two transversals (meeting them in opposite orientations) with "nearby" normal vectors would imply the existence of a $(d-2)$-flat that lies within distance ε of each of $S_{i_1}, \ldots S_{i_d}$. (Note that here we use the fact that $\varepsilon \leq \pi - \varepsilon$.) Thus, for each d-tuple $i_1, \ldots, i_d$, we have a distinguished orientation, and the collection of these orientations determines an oriented matroid M, since restricted to any $d+2$ they agree with the order type in which the corresponding bodies are met by the transversal in our collection for those $d+2$ bodies.

Now by the Generalized Hadwiger Theorem, since every $d+1$ of our bodies, $S_{i_1}, \ldots, S_{i_{d+1}}$, have a transversal (just take $T(i_1, \ldots, i_{d+2})$ for *any* choice of i_{d+2}) such that all the order types are consistent with those of M, it follows that *all* the bodies have a common transversal. □

3. Remarks

In Theorem 1 we gave a Helly-type theorem with a fixed Helly number (namely, $2d+2$). Note that the conclusion holds only for ε-separated collections of cardinality that grows rapidly with decreasing ε. Using similar methods, it is possible to give a different Helly-type theorem, which applies to collections of much smaller cardinality, but at the cost of having the Helly number depend on ε.

In [1] Amenta showed a connection between Helly-type theorems and linear-time algorithms. Perhaps our Helly-type theorem suggests a linear-time algorithm for finding hyperplane transversals to ε-separated convex sets under a suitable model of computation.

Katchalski's conjecture that there is a Helly-type theorem for line transversals to collections of pairwise disjoint unit balls in $\mathbb{R}^3$ remains open. Similarly, the conjecture that there is a Helly-type theorem for plane transversals to separated collections of unit balls in $\mathbb{R}^3$ is also open. More generally, are there such Helly-type theorems for line transversals to collections of pairwise disjoint translates or plane transversals to separated collections of translates in $\mathbb{R}^3$?

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