



α_i -Metric Graphs: Radius, Diameter and all Eccentricities

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Abstract

We extend known results on chordal graphs and distance-hereditary graphs to much larger graph classes by using only a common metric property of these graphs. Specifically, a graph is called α_i -metric ($i \in \mathcal{N}$) if it satisfies the following α_i -metric property for every vertices u, w, v and x : if a shortest path between u and w and a shortest path between x and v share a terminal edge vw , then $d(u, x) \geq d(u, v) + d(v, x) - i$. Roughly, gluing together any two shortest paths along a common terminal edge may not necessarily result in a shortest path but yields a “near-shortest” path with defect at most i . It is known that α_0 -metric graphs are exactly ptolemaic graphs, and that chordal graphs and distance-hereditary graphs are α_i -metric for $i = 1$ and $i = 2$, respectively. We show that an additive $O(i)$ -approximation of the radius, of the diameter, and in fact of all vertex eccentricities of an α_i -metric graph can be computed in total linear time. Our strongest results are obtained for α_1 -metric graphs, for which we prove that a central vertex can be computed in subquadratic time, and even better in linear time for so-called (α_1, Δ) -metric graphs (a superclass of chordal graphs and of plane triangulations with inner vertices of degree at least 7). The latter answers a question raised in Dragan (Inf Probl Lett 154:105873, 2020), 2020). Our algorithms follow from new results on centers and metric intervals of α_i -metric graphs. In particular, we prove that the diameter of the center is at most $3i + 2$ (at most 3, if $i = 1$). The latter partly answers a question raised in Yushmanov and Chepoi (Math Probl Cybernet 3:217–232, 1991).

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1 Introduction

Euclidean spaces have the following nice property: if the geodesic between u and w contains v , and the geodesic between v and x contains w , then their union must be the geodesic between u and x . In 1991, Chepoi and Yushmanov introduced α_i -metric properties ($i \in \mathcal{N}$), as a way to quantify by how much a graph is close to satisfy this above requirement [53] (see also [14, 15] for earlier use of α_1 -metric property). All graphs $G = (V, E)$ occurring in this paper are connected, finite, unweighted, undirected, loopless and without multiple edges. The *length* of a path between two vertices u and v is the number of edges in the path. The *distance* $d_G(u, v)$ is the length of a shortest path connecting u and v in G . The *interval* $I_G(u, v)$ between u and v consists of all vertices on shortest (u, v) -paths, that is, it consists of all vertices (metrically) between u and v : $I_G(u, v) = \{x \in V : d_G(u, x) + d_G(x, v) = d_G(u, v)\}$. Let also $I_G^o(u, v) = I_G(u, v) \setminus \{u, v\}$. If no confusion arises, we will omit subindex G .

α_i -metric property: if $v \in I(u, w)$ and $w \in I(v, x)$ are adjacent, then $d(u, x) \geq d(u, v) + d(v, x) - i = d(u, v) + 1 + d(w, x) - i$.

Roughly, gluing together any two shortest paths along a common terminal edge may not necessarily result in a shortest path (unlike in the Euclidean space) but yields a “near-shortest” path with defect at most i . A graph is called α_i -metric if it satisfies the α_i -metric property. α_i -Metric graphs were investigated in [14, 15, 53]. In particular, it is known that α_0 -metric graphs are exactly the distance-hereditary chordal graphs, also known as ptolemaic graphs [45]. Furthermore, α_1 -metric graphs contain all chordal graphs [14] and all plane triangulations with inner vertices of degree at least 7 [32]. α_2 -Metric graphs contain all distance-hereditary graphs [53] and, even more strongly, all HHD-free graphs [18]. Evidently, every graph is an α_i -metric graph for some i . Chepoi and Yushmanov in [53] also provided a characterization of all α_1 -metric graphs: They are exactly the graphs where all disks are convex and the graph W_6^{++} from Fig. 1 is forbidden as an isometric subgraph (see [53] or Theorem 5). This nice characterization was heavily used in [4] in order to characterize δ -hyperbolic graphs with $\delta \leq 1/2$.

The *eccentricity* $e_G(v)$ of a vertex v in G is defined by $\max_{u \in V} d_G(u, v)$, i.e., it is the distance to a most distant vertex. The *diameter* of a graph is the maximum over the eccentricities of all vertices: $\text{diam}(G) = \max_{u \in V} e_G(u) = \max_{u, v \in V} d_G(u, v)$. The *radius* of a graph is the minimum over the eccentricities of all vertices: $\text{rad}(G) = \min_{u \in V} e_G(u)$. The *center* $C(G)$ of a graph G is the set of its vertices with minimum eccentricity, i.e., $C(G) = \{u \in V : e_G(u) = \text{rad}(G)\}$. Each vertex from $C(G)$ is called a *central* vertex. In this paper, we investigate the radius, diameter, and all eccentricities computation problems in α_i -metric graphs. Understanding the eccentricity function of a graph and being able to efficiently compute or estimate the diameter, the radius, and all vertex eccentricities is of great importance. For example, in the analysis of social networks (e.g., citation networks or recommendation networks),

biological systems (e.g., protein interaction networks), computer networks (e.g., the Internet or peer-to-peer networks), transportation networks (e.g., public transportation or road networks), etc., the eccentricity of a vertex is used in order to measure its importance in the network: the *eccentricity centrality index* of v [46] is defined as $\frac{1}{e(v)}$. Furthermore, the problem of finding a central vertex is one of the most famous facility location problems in Operation Research and in Location Science. In [53], the following nice relation between the diameter and the radius of an α_i -metric graph G was established: $diam(G) \geq 2rad(G) - i - 1$. Recall that for every graph G , $diam(G) \leq 2rad(G)$ holds. Authors of [53] also raised a question¹ whether the diameter of the center of an α_i -metric graph can be bounded by a linear function of i . It is known that the diameters of the centers of chordal graphs or of distance-hereditary graphs are at most 3 [15, 53].

Related work on computing or estimating the radius, diameter, or all eccentricities. A naive algorithm which runs a BFS from each vertex to compute its eccentricity and then (in order to compute the radius, the diameter and a central vertex) picks the smallest and the largest eccentricities and a vertex with smallest eccentricity has running time $O(nm)$ on an n -vertex m -edge graph. Interestingly, this naive algorithm is conditionally optimal for general graphs as well as for some restricted families of graphs [1, 6, 20, 49] since, under plausible complexity assumptions, neither the diameter nor the radius can be computed in truly subquadratic time (i.e., in $O(n^a m^b)$ time, for some positive a, b such that $a + b < 2$) on those graphs. Already for split graphs (a subclass of chordal graphs), computing the diameter is roughly equivalent to DISJOINT SETS, *a.k.a.*, the monochromatic ORTHOGONAL VECTOR problem [17]. Under the Strong Exponential-Time Hypothesis (SETH), we cannot solve DISJOINT SETS in truly subquadratic time, and so neither we can compute the diameter of split graphs in truly subquadratic time [6].

In a quest to break this quadratic barrier (in the size $n+m$ of the input), there has been a long line of work presenting more efficient algorithms for computing the diameter and/or the radius, or even better all eccentricities, on some special graph classes, by exploiting their geometric and tree-like representations and/or some forbidden pattern (e.g., excluding a minor [41], or a family of induced subgraphs). For example, faster algorithms for all eccentricities computation are known for distance-hereditary graphs [12, 23, 29, 33], outerplanar graphs [42], planar graphs [9, 43], graphs with bounded tree-width [1, 8, 41] and, more generally, graphs with bounded clique-width [12, 38]. Linear-time algorithms for computing all eccentricities are also known for interval graphs [34, 47]. Some recent works have further studied which properties of interval graphs could imply on their own faster algorithms for diameter and all eccentricities computation. Efficient algorithms for these problems have been found for AT-free graphs [36], directed path graphs [10], strongly chordal graphs [21], dually chordal graphs [7, 22], Helly graphs and graphs of bounded Helly number [28, 39, 40]. See also [37]. Chordal graphs are another well-known generalization of interval graphs. Although the diameter of a split graph can unlikely be computed in subquadratic time, there is an elegant linear-time algorithm for computing the radius and a central vertex of a chordal graph [16]. However, until this work there has been little insight about

¹ It is conjectured in [53] that $diam(C(G)) \leq i + 2$ for every α_i -metric graph G .

how to extend this nice result to larger graph classes (a notable exception being the work in [18]). This intriguing question is partly addressed in our paper.

Since the exact diameter or radius computation in subquadratic time is impossible (unless the SETH is false) even for simple families of graphs (in case of the diameter, even for split graphs), a large volume of work was also devoted to approximation algorithms. It is known [13] that the diameter of any graph with n vertices and m edges can be approximated within a multiplicative factor of $3/2$ in $\tilde{O}(m^{3/2})$ time. Furthermore, unless the SETH is false, no $O(n^{2-\epsilon})$ time algorithm can achieve an approximation factor better than $3/2$ in sparse graphs [49] and no $O(m^{3/2-\epsilon})$ time algorithm can achieve an approximation factor better than $5/3$ [3]. The eccentricities of all vertices of any graph can be approximated within a factor of $5/3$ in $\tilde{O}(m^{3/2})$ time [13] and, under the SETH, no $O(n^{2-\epsilon})$ time algorithm can achieve better than $5/3$ approximation in sparse graphs [1] and no $O(m^{3/2-\epsilon})$ time algorithm can achieve an approximation factor better than $9/5$ [3]. Authors of [3] also show that no near-linear time algorithm can achieve a better than 2 approximation for the eccentricities and provide an algorithm that approximates eccentricities within a $2 + \epsilon$ factor in $\tilde{O}(m/\epsilon)$ time for any $0 < \epsilon < 1$. On planar graphs, there is an approximation scheme with near linear running time [51]. Authors of [13] additionally address a more challenging question of obtaining an additive c -approximation for the diameter $diam(G)$ of a graph G , i.e., an estimate D such that $diam(G) - c \leq D \leq diam(G)$. A simple $\tilde{O}(mn^{1-\epsilon})$ time algorithm achieves an additive n^ϵ -approximation and, for any $\epsilon > 0$, getting an additive n^ϵ -approximation algorithm for the diameter running in $O(n^{2-\epsilon'})$ time for any $\epsilon' > 2\epsilon$ would falsify the SETH.

Much better additive approximations can be achieved for graphs with bounded (metric) parameters, including chordal graphs, HHD-free graphs, k -chordal graphs, and more generally all δ -hyperbolic graphs (see [10, 11, 16, 19, 20, 24, 25, 29–31, 34]). For example, a vertex furthest from an arbitrary vertex has eccentricity at least $diam(G) - 2$ for chordal graphs [16] and at least $diam(G) - \lfloor k/2 \rfloor$ for k -chordal graphs [11]. Hence, the diameter in those graphs can be approximated within a small additive error in linear time by a BFS. In fact, the last vertex visited by a LexBFS has eccentricity at least $diam(G) - 1$ for chordal graphs [34] as well as for HHD-free graphs [24]. Thus, although the existence of a subquadratic algorithm for computing the exact diameter of a chordal graph would falsify the SETH, a vertex with eccentricity at least $diam(G) - 1$ can be found in linear time by a LexBFS. Later, those results were generalized to all δ -hyperbolic graphs [19, 20, 30, 31]. Note that chordal graphs and distance-hereditary graphs are 1-hyperbolic, while k -chordal graphs are $\lfloor k/2 \rfloor/2$ -hyperbolic [52]. Gromov [44] defines δ -hyperbolic graphs via a simple 4-point condition: for any four vertices u, v, w, x , the two largest of the three distance sums $d(u, v) + d(w, x)$, $d(u, w) + d(v, x)$, and $d(u, x) + d(v, w)$ differ by at most $2\delta \geq 0$. Such graphs have become of recent interest due to the empirically established presence of a small hyperbolicity in many real-world networks.

For every δ -hyperbolic graph, a vertex furthest from an arbitrary vertex has eccentricity at least $diam(G) - 2\delta$ [19]. Furthermore, for any m -edge δ -hyperbolic graph G , a vertex with eccentricity at most $rad(G) + 2\delta$ can be found in $O(\delta m)$ time and a vertex with eccentricity at most $rad(G) + 3\delta$ can be found in $O(m)$ time [19, 20, 30, 31]. Three approximation schemes for all eccentricities were presented in [30]:

an approximate eccentricity function \hat{e} , constructible in $O(\delta m)$ time, which satisfies $e(v) - 2\delta \leq \hat{e}(v) \leq e(v)$, for all $v \in V$, and two spanning trees T , one constructible in $O(\delta m)$ time and the other in $O(m)$ time, which satisfy $e_G(v) \leq e_T(v) \leq e_G(v) + 4\delta + 1$ and $e_G(v) \leq e_T(v) \leq e_G(v) + 6\delta$, for all $v \in V$, respectively. Our results of Sect. 2 and Sect. 3 (and some results of [53]) show that α_i -metric graphs behave like δ -hyperbolic graphs. In a forthcoming paper [27], indeed we show that the hyperbolicity δ of an α_i -metric graph depends linearly on i . However, the constants in our Theorems 1-4 are better than those that can be obtained by combining the hyperbolicity result of [27] with the algorithmic results of [19, 20, 30] on radii, diameters and all eccentricities of δ -hyperbolic graphs. In [27], we also introduce a natural generalization of an α_i -metric, which we call a (λ, μ) -bow metric: namely, if two shortest paths $P(u, w)$ and $P(v, x)$ share a common shortest subpath $P(v, w)$ of length more than λ (that is, they overlap by more than λ), then the distance between u and x is at least $d(u, v) + d(v, w) + d(w, x) - \mu$. δ -Hyperbolic graphs are $(\delta, 2\delta)$ -bow metric and α_i -metric graphs are $(0, i)$ -bow metric.

(α_1, Δ) -Metric graphs form an important subclass of both α_1 -metric graphs and weakly bridged graphs, and they contain all chordal graphs and all plane triangulations with inner vertices of degree at least 7. In [32], it was shown that every (α_1, Δ) -metric graph admits an eccentricity 2-approximating spanning tree, i.e., a spanning tree T such that $e_T(v) - e_G(v) \leq 2$ for every vertex v . As a result, for a chordal graph, an additive 2-approximation of all eccentricities can be computed in linear time [25]. Finding similar results for general α_1 -metric graphs was left as an open problem in [32].

Our Contribution. We prove several new results on metric intervals, eccentricity function, and centers in α_i -metric graphs, and their algorithmic applications, thus answering open questions in the literature [25, 32, 53]. To list our contributions, we need to introduce on our way some additional notations and terminology.

Section 2 is devoted to general α_i -metric graphs ($i \geq 0$). The set $S_k(u, v) = \{x \in I(u, v) : d(u, x) = k\}$ is called a *slice* of the interval $I(u, v)$ where $0 \leq k \leq d(u, v)$. An interval $I(u, v)$ is said to be λ -thin if $d(x, y) \leq \lambda$ for all $x, y \in S_k(u, v)$, $0 < k < d(u, v)$. The smallest integer λ for which all intervals of G are λ -thin is called the *interval thinness* of G . We show first that, in α_i -metric graphs G , the intervals are $(i + 1)$ -thin.

The disk of radius r and center v is defined as $\{u \in V : d(u, v) \leq r\}$, and denoted by $D(v, r)$. Sometimes, $D(v, r)$ is called the r -neighborhood of v . In particular, $N[v] = D(v, 1)$ and $N(v) = N[v] \setminus \{v\}$ denote the closed and open neighbourhoods of a vertex v , respectively. More generally, for any vertex-subset S and a vertex u , we define $d(u, S) = \min_{v \in S} d(u, v)$, $D(S, r) = \bigcup_{v \in S} D(v, r)$, $N[S] = D(S, 1)$ and $N(S) = N[S] \setminus S$. We say that a set of vertices $S \subseteq V$ of a graph $G = (V, E)$ is d^k -convex if for every two vertices $x, y \in S$ with $d(x, y) \geq k \geq 0$, the entire interval $I(x, y)$ is in S . For $k \leq 2$, this definition coincides with the usual definition of convex sets in graphs [5, 14, 50]: S is *convex* if for every $x, y \in S$, the interval $I(x, y)$ is also in S . Clearly, the intersection of two d^k -convex sets is also d^k -convex. We show that, in α_i -metric graphs G , the disks (and, hence, the centers $C(G)$) are d^{2i-1} -convex. The

main result of Sect. 2.1 states that the diameter of the center $C(G)$ of G is at most $3i + 2$, thus answering a question raised in [53].

Let $F_G(v)$ be the set of all vertices of G that are most distant from v . A pair x, y is called a pair of mutually distant vertices if $e_G(x) = e_G(y) = d_G(x, y)$, i.e., $x \in F_G(y), y \in F_G(x)$. In Sect. 2.2, we show that a vertex x that is most distant from an arbitrary vertex z has eccentricity at least $\text{diam}(G) - 3i - 2$. Furthermore, a middle vertex c of any shortest path between x and $y \in F(x)$ has eccentricity at most $\text{rad}(G) + 4i + (i + 1)/2 + 2$, and a middle vertex c^* of any shortest path between any two mutually distant vertices has eccentricity at most $\text{rad}(G) + 2i + 1$. Additionally, all central vertices of G are within a small distance from c and c^* , namely, $C(G) \subseteq D(c^*, 4i + 3)$ and $C(G) \subseteq D(c, 4i + (i + 1)/2 + 2)$. Hence, an additive $O(i)$ -approximation of the radius and of the diameter of an α_i -metric graph G with m edges can be computed in $O(m)$ time. In Sect. 2.3, we present three approximation algorithms for all eccentricities: an $O(im)$ time eccentricity approximation $\hat{e}(v)$ based on the distances from any vertex to two mutually distant vertices, which satisfies $e(v) - 3i - 2 \leq \hat{e}(v) \leq e(v)$ for all $v \in V$, and two spanning trees T , one constructible in $O(im)$ time and the other in $O(m)$ time, which satisfy $e_G(v) \leq e_T(v) \leq e_G(v) + 4i + 3$ and $e_G(v) \leq e_T(v) \leq e_G(v) + 7i + 5$, respectively. Hence, an additive $O(i)$ -approximation of all vertex eccentricities of an α_i -metric graph G with m edges can be computed in $O(m)$ time.

Section 3 is devoted to α_1 -metric graphs. The eccentricity function $e(v)$ of a graph G is said to be *unimodal*, if for every non-central vertex v of G there is a neighbor $u \in N(v)$ such that $e(u) < e(v)$ (that is, every local minimum of the eccentricity function is a global minimum). We show in Sect. 3.1 that the eccentricity function on α_1 -metric graphs is almost unimodal in the sense that the only non-central vertices that violate the unimodality (that is, do not have a neighbor with smaller eccentricity) must have their eccentricity equal to $\text{rad}(G) + 1$ and their distance from $C(G)$ must be 2. In other words, every local minimum of the eccentricity function on an α_1 -metric graph G is a global minimum or is at distance 2 from a global minimum. Such behavior of the eccentricity function was observed earlier in chordal graphs [32], in distance-hereditary graphs [29] and in all (α_1, Δ) -metric graphs [32] (note also that in Helly graphs the eccentricity function is unimodal [21]). This almost unimodality of the eccentricity function turns out to be very useful in locating a vertex with eccentricity at most $\text{rad}(G) + 1$ in a gradient descent fashion. In Sect. 3.2, using the convexity of the center $C(G)$ of an α_1 -metric graph G , we show that the diameter of $C(G)$ is at most 3. This generalizes known results for chordal graphs [15] and for (α_1, Δ) -metric graphs [32]. In Sect. 3.3, we present a local-search algorithm for finding a central vertex of an arbitrary α_1 -metric graph in subquadratic time. Our algorithm even achieves linear runtime on (α_1, Δ) -metric graphs, thus answering an open question from [32]. In Sect. 3.4, we show how to approximate efficiently all vertex eccentricities in α_1 -metric graphs.

2 General Case of α_i -Metric Graphs for Arbitrary $i \geq 0$

First we present an auxiliary lemma.

Lemma 1 *Let G be an α_i -metric graph, and let u, v, x, y be vertices such that $x \in I(u, v)$, $d(u, x) = d(u, y)$, and $d(v, y) \leq d(v, x) + k$. Then, $d(x, y) \leq k + i + 2$.*

Proof Set $v = i + k + 2$, and suppose for the sake of contradiction $d(x, y) > v$. Without loss of generality, k is the minimum value for which a counter-example can be found. We may also assume, without loss of generality, that $d(u, x)$ is minimized. Let $x' \in N(x) \cap I(x, u)$ and let $y' \in N(y) \cap I(y, u)$. Observe that $d(u, x') = d(u, y')$, $x' \in I(u, v)$ and $d(v, y') \leq d(v, y) + 1 \leq d(x, v) + 1 + k = d(x', v) + k$. Therefore, by minimality of $d(u, x)$, we have $d(x', y') \leq v$ (otherwise, we could replace x, y with x', y'). Now, there are two cases to be considered:

- Case $d(y', x') < d(y', x)$. We also have $d(v, x) < d(v, x')$. Since G is an α_i -metric graph, it implies $d(v, y') \geq d(v, x) + 1 + d(x', y') - i$. Then, $d(x', y') \leq i - 1 + d(v, y') - d(v, x) = i + d(v, y') - d(v, x') \leq i + k$. However, $v \leq d(x, y) - 1 \leq d(x', y') + 1 \leq i + k + 1 = v - 1$.
- Case $d(y', x') \geq d(y', x)$. Then, $d(x, y) \leq 1 + d(x, y') \leq 1 + d(x', y') \leq v + 1$. It implies $d(x, y) = v + 1$, $d(x, y') = d(x', y') = v$, and $y' \in I(x, y) \cap N(y)$. In particular, $d(x, y') < d(x, y)$. Furthermore, we claim that we have $d(v, y) < d(v, y')$. Indeed, suppose for the sake of contradiction $d(v, y') \leq d(v, y)$. In this situation, $d(v, y') \leq d(u, y') + d(v, y') - d(u, y') \leq d(u, y) - 1 + d(v, y) - d(u, y') \leq d(u, x) - 1 + d(v, x) + k - d(u, x') = d(u, v) - d(u, x') + k - 1 = d(v, x') + k - 1$. By minimality of k , we obtain $d(x', y') \leq i + (k - 1) + 2 = i + k + 1 < v$, and a contradiction arises. Therefore, as claimed, $d(v, y) < d(v, y')$. Since G is an α_i -metric graph, it implies $d(x, v) \geq d(x, y') + 1 + d(y, v) - i = d(x, y) + d(y, v) - i$. But then, $v = d(x, y) - 1 \leq d(x, v) - d(y, v) + i - 1 \leq i - 1$.

In both cases, we derive a contradiction. □

Lemma 1 is helpful in proving that in α_i -metric graphs the intervals are rather thin.

Lemma 2 *If G is an α_i -metric graph, then its interval thinness is at most $i + 1$.*

Proof Let $u, v, x, y \in V$ be such that $x, y \in I(u, v)$, and $d(u, x) = d(u, y)$. Suppose for the sake of contradiction $d(x, y) > i + 1$. By Lemma 1 (applied for $k = 0$), we have $d(x, y) = i + 2$. We further assume, without loss of generality, that $d(u, x)$ is minimized. In particular, let $x' \in N(x) \cap I(u, x)$, $y' \in N(y) \cap I(u, y)$. By minimality of $d(u, x)$ we have $d(x', y') \leq i + 1$. We claim that $d(x', y) \geq i + 2$. Indeed, if it were not the case, then $d(x', y) < d(x, y)$ and so, since we also have $d(v, x) < d(v, x')$, we would obtain $d(y, v) \geq d(y, x) + d(x, v) - i = d(x, v) + 2 > d(x, v) = d(y, v)$, getting a contradiction. This proves as claimed that $d(x', y) \geq i + 2$. It implies $d(x', y') \geq i + 1$, and so $d(x', y') = i + 1$. However, in this situation, $d(x', y') < d(x', y)$ and $d(v, y) < d(v, y')$. As a result, $d(v, x') \geq d(v, y') + d(x', y') - i = d(v, y') + 1 > d(v, y) = d(v, x')$, and a contradiction arises. □

2.1 Centers of α_i -Metric Graphs

In this subsection we show that the diameter of the center of an α_i -metric graph is at most $3i + 2$, hereby providing an answer to a question raised in [53] whether the

diameter of the center of an α_i -metric graph can be bounded by a linear function of i . In [53], the following relation between the diameter and the radius of an α_i -metric graph G was proven: $2rad(G) \geq diam(G) \geq 2rad(G) - i - 1$.

First we show that all disks (and hence the center $C(G)$) of an α_i -metric graph G is d^{2i-1} -convex.

Lemma 3 *Every disk of an α_i -metric graph G is d^{2i-1} -convex. In particular, the center $C(G)$ of an α_i -metric graph G is d^{2i-1} -convex.*

Proof Since $C(G) = \bigcap \{D(v, rad(G)) : v \in V\}$, it suffices to prove the d^{2i-1} -convexity of an arbitrary disk. Let $v, x, y \in V$ be such that $x, y \in D(v, r)$ for some $r \geq 0$ but $I(x, y) \not\subseteq D(v, r)$. Let $a_x \in I(x, y) \setminus D(v, r)$ be such that $d(x, a_x)$ is maximized, and let $b_x \in N(a_x) \cap I(a_x, y)$ be arbitrary. Note that $b_x \in D(x, r)$ by maximality of $d(x, a_x)$. In particular, $d(v, b_x) < d(v, a_x)$. We also have $d(x, a_x) < d(x, b_x)$ because $a_x, b_x \in I(x, y)$. Since G is an α_i -metric graph, $r \geq d(v, x) \geq d(v, a_x) + d(a_x, x) - i = r + 1 + d(a_x, x) - i$. Therefore, $d(a_x, x) \leq i - 1$. Now, let $a_y \in I(x, y) \setminus D(v, r)$ be such that $d(y, a_y)$ is maximized. We prove as before $d(a_y, y) \leq i - 1$. Furthermore, $d(a_y, y) \geq d(a_x, y)$ by maximality of $d(a_y, y)$. As a result, $d(x, y) = d(x, a_x) + d(a_x, y) \leq d(x, a_x) + d(a_y, y) \leq 2i - 2$. \square

Next auxiliary lemma is crucial in obtaining many results of this section.

Lemma 4 *Let G be an α_i -metric graph. For any $x, y, v \in V$ and any integer $k \in \{0, \dots, d(x, y)\}$, there is a vertex $c \in S_k(x, y)$ such that $d(v, c) \leq \max\{d(v, x), d(v, y)\} - \min\{d(x, c), d(y, c)\} + i$ and $d(v, c) \leq \max\{d(v, x), d(v, y)\} + i/2$. For an arbitrary vertex $z \in I(x, y)$, we have $d(z, v) \leq \max\{d(x, v), d(y, v)\} - \min\{d(x, z), d(y, z)\} + 2i + 1$ and $d(z, v) \leq \max\{d(x, v), d(y, v)\} + 3i/2 + 1$. Furthermore, $e(z) \leq \max\{e(x), e(y)\} - \min\{d(x, z), d(y, z)\} + 2i + 1$ and $e(z) \leq \max\{e(x), e(y)\} + 3i/2 + 1$ when $v \in F(z)$.*

Proof Let $z \in S_k(x, y)$, for some $k \in \{0, \dots, d(x, y)\}$, and c be a vertex of $S_k(x, y)$ closest to v . By Lemma 2, $d(c, z) \leq i + 1$.

Consider a neighbor c' of c on a shortest path from c to v . We have $d(x, c) < d(x, c')$ or $d(y, c) < d(y, c')$ since otherwise, when $d(x, c) \geq d(x, c')$ and $d(y, c) \geq d(y, c')$, c' must belong to $S_k(x, y)$ and a contradiction with the choice of c arises. Without loss of generality, assume $d(x, c) < d(x, c')$. Then $c \in I(x, c')$, and we can apply α_i -metric property to x, c, c', v and get $d(x, v) \geq d(x, c) + d(c, v) - i$, i.e., $d(c, v) \leq d(x, v) - d(x, c) + i \leq \max\{d(x, v), d(y, v)\} - \min\{d(x, c), d(y, c)\} + i$. By adding also the triangle inequality $d(c, v) \leq d(v, x) + d(x, c)$ to $d(c, v) \leq d(x, v) - d(x, c) + i$, we get $d(c, v) \leq d(x, v) + i/2 \leq \max\{d(x, v), d(y, v)\} + i/2$.

For arbitrary $z \in S_k(x, y)$, as $d(z, c) \leq i + 1$, $d(x, c) = d(x, z)$, $d(y, c) = d(y, z)$, we get $d(z, v) \leq d(z, c) + d(c, v) \leq i + 1 + d(x, v) - d(x, c) + i \leq \max\{d(x, v), d(y, v)\} - \min\{d(x, z), d(y, z)\} + 2i + 1$ and $d(z, v) \leq d(z, c) + d(c, v) \leq i + 1 + d(x, v) + i/2 \leq \max\{d(x, v), d(y, v)\} + i + i/2 + 1$. Applying both inequalities to the case in which v is furthest from z , we get $e(z) = d(z, v) \leq \max\{e(x), e(y)\} - \min\{d(x, z), d(y, z)\} + 2i + 1$ and $e(z) \leq \max\{e(x), e(y)\} + i + i/2 + 1$. \square

Lemma 4 has an immediate corollary.

Corollary 1 *Let G be an α_i -metric graph. Any vertices $x, y \in V$ and $c \in I(x, y)$ satisfy $e(c) \leq \max\{e(x), e(y)\} + 3i/2 + 1$. However, if $d(x, y) \geq 4i + 2$, then any vertex $c' \in I(x, y)$ with $d(x, c') \geq 2i + 1$ and $d(y, c') \geq 2i + 1$ satisfies $e(c') \leq \max\{e(x), e(y)\}$. Furthermore, if $d(x, y) > 4i + 3$ then any vertex $c' \in I(x, y)$ with $d(x, c') > 2i + 1$ and $d(y, c') > 2i + 1$ satisfies $e(c') < \max\{e(x), e(y)\}$.*

Proof By Lemma 4, $e(c) \leq \max\{e(x), e(y)\} + 3i/2 + 1$. Suppose that $d(x, y) \geq 4i + 2$ and consider any vertex $c' \in I(x, y)$ satisfying $d(x, c') \geq 2i + 1$ and $d(y, c') \geq 2i + 1$. By Lemma 4, $e(c') \leq \max\{e(x), e(y)\} - \min\{d(x, c'), d(y, c')\} + 2i + 1$. Hence, $e(c') \leq \max\{e(x), e(y)\}$.

Suppose now that $d(x, y) > 4i + 3$, i.e., $d(x, y) \geq 4i + 4$. Consider any vertex $c' \in I(x, y)$ satisfying $d(x, c') > 2i + 1$ and $d(y, c') > 2i + 1$. By Lemma 4, $e(c') \leq \max\{e(x), e(y)\} - \min\{d(x, c'), d(y, c')\} + 2i + 1$. Hence, $e(c') < \max\{e(x), e(y)\}$. □

Using Corollary 1 one can easily prove that the diameter of the center $C(G)$ of an α_i -metric graph G is at most $4i + 3$. Below we show that the bound can be improved.

Theorem 1 *If G is an α_i -metric graph, then $\text{diam}(C(G)) \leq 3i + 2$.*

Proof Let us write $r = \text{rad}(G)$ in what follows. Suppose by contradiction $\text{diam}(C(G)) > 3i + 2$. Since $C(G)$ is d^{2i-1} -convex, every diametral path of $C(G)$ must be fully in $C(G)$. In particular, there exist $x, y \in C(G)$ such that $d(x, y) = 3i + 3$ and $I(x, y) \subseteq C(G)$. Furthermore, for every $u \in V$ such that $\max\{d(u, x), d(u, y)\} < r$, we obtain $I(x, y) \subseteq D(u, r - 1)$ because the latter disk is also d^{2i-1} -convex. Therefore, for every $z \in I(x, y)$, we must have $F(z) \subseteq F(x) \cup F(y)$.

Let ab be an edge on a shortest xy -path such that $d(a, x) < d(b, x)$. Assume $F(b) \not\subseteq F(a)$. Let $v \in F(b) \setminus F(a)$ be arbitrary. Since G is an α_i -metric graph, $d(v, y) \geq d(v, b) + d(b, y) - i = r + (d(b, y) - i)$. Therefore, $d(b, y) \leq i$. In the same way, if $F(a) \not\subseteq F(b)$, then $d(a, x) \leq i$. By induction, we get $F(z) \subseteq F(x)$ ($F(z) \subseteq F(y)$, respectively) for every $z \in I(x, y)$ such that $d(y, z) \geq i + 1$ ($d(x, z) \geq i + 1$, respectively). In particular, for every t such that $i + 1 \leq t \leq d(x, y) - i - 1$, we must have $F(z) \subseteq F(x) \cap F(y)$ for every $z \in S_t(x, y)$.

Note that the above properties are true not only for $x, y \in C(G)$ with $d(x, y) = 3i + 3$ but also for every $x', y' \in C(G)$ with $d(x, y) \geq 2i - 1$, as d^{2i-1} -convexity argument can still be used.

Let $c \in I(x, y)$ be such that $F(c) \subseteq F(x) \cap F(y)$ and $k := |F(c)|$ is minimized. We claim that $k < |F(x) \cap F(y)|$. Indeed, let $v \in F(x) \cap F(y)$ be arbitrary. By Lemma 4, some vertex $c_v \in S_{i+1}(x, y)$ satisfies $d(c_v, v) \leq r - 1$. Then, $F(c_v) \subseteq (F(x) \cap F(y)) \setminus \{v\}$, and $k \leq |F(c_v)| \leq |F(x) \cap F(y)| - 1$ by minimality of c . Then, let $y_c \in I(x, y)$ be such that $F(y_c) \cap F(x) \cap F(y) \subseteq F(c)$ and $d(x, y_c)$ is maximized (such a vertex must exist because $c \in I(x, y)$ satisfies that condition). We have $y_c \neq y$ because $F(x) \cap F(y) \not\subseteq F(c)$. Therefore, the maximality of $d(x, y_c)$ implies the existence of some $v \in (F(x) \cap F(y)) \setminus F(c)$ such that $d(v, y_c) = r - 1$. Since

G is an α_i -metric graph, $d(v, y) \geq d(v, y_c) + d(y_c, y) - i = r + (d(y_c, y) - i - 1)$. As a result, $d(y_c, y) \leq i + 1$.

Then, for every $z \in S_{i+1}(x, y_c)$, since we have $d(z, y_c) = d(x, y) - i - 1 - d(y_c, y) \geq d(x, y) - 2i - 2 = i + 1$, we obtain $F(z) \subseteq F(x) \cap F(y) \cap F(y_c) \subseteq F(c)$. By minimality of k , $F(z) = F(c)$. However, let $v \in F(c)$ be arbitrary. By Lemma 4, there exists some $c' \in S_{i+1}(x, y_c)$ such that $d(c', v) \leq r - 1$, thus contradicting $F(c') = F(c)$. \square

2.2 Approximating Radii and Diameters of α_i -Metric Graphs

In this subsection, we show that a vertex with eccentricity at most $rad(G) + O(i)$ and a vertex with eccentricity at least $diam(G) - O(i)$ of an α_i -metric graph G can be found in (almost) linear time.

First we show that a middle vertex of any shortest path between two mutually distant vertices has eccentricity at most $rad(G) + 2i + 1$. Furthermore, the distance between any two mutually distant vertices is at least $diam(G) - 3i - 2$.

Lemma 5 *Let G be an α_i -metric graph, x, y be a pair of mutually distant vertices of G and z be a middle vertex of an arbitrary shortest path connecting x and y . Then, $e(z) \leq rad(G) + 2i + 1$.*

Furthermore, there is a vertex c in $S_{\lfloor d(x,y)/2 \rfloor}(x, y)$ with $e(c) \leq rad(G) + i$.

Proof Let $r = rad(G)$ and $k = \lfloor d(x, y)/2 \rfloor$. We need to show that for every vertex $z \in S_k(x, y)$, $e(z) \leq r + 2i + 1$ holds. It will be sufficient to show that for a specially chosen vertex $c \in S_k(x, y)$, we have $e(c) \leq r + i$. Then, since for any two vertices $u, v \in S_k(x, y)$, $d(u, v) \leq i + 1$ (see Lemma 2), we will get $e(z) \leq d(z, c) + e(c) \leq r + 2i + 1$.

Consider in $S_k(x, y)$ vertices c with smallest eccentricity and among all those vertices pick that one whose $|F(c)|$ is as small as possible. Let $v \in F(c)$ be a most distant vertex from c and consider a neighbor t of c on a shortest path from c to v . As x, y is a pair of mutually distant vertices, we have $d(x, y) \geq d(x, v)$ and $d(x, y) \geq d(y, v)$. We know also that $d(c, x) \leq d(c, y) \leq r$ as $d(x, y) \leq 2r$.

If $d(x, c) < d(x, t)$ then $c \in I(x, t)$. By the α_i -metric property applied to x, c, t, v , we get $d(x, y) \geq d(x, v) \geq d(x, c) + d(c, v) - i$. That is, $d(c, y) \geq d(c, v) - i$ and therefore $e(c) = d(c, v) \leq d(c, y) + i \leq r + i$. Similarly, if $d(y, c) < d(y, t)$ then $e(c) = d(c, v) \leq d(c, x) + i \leq r + i$ must hold. So, in what follows, we may assume that $d(x, c) \geq d(x, t)$ and $d(y, c) \geq d(y, t)$, i.e., $t \in S_k(x, y)$ must hold.

If $e(t) = e(c) + 1$ then for every $s \in F(t)$, we have $c \in I(t, s)$ and $d(c, s) = d(c, v)$. If $e(t) = e(c)$ then, by the choice of c , there must exist a vertex $s \in F(t) \setminus F(c)$ (as $v \in F(c) \setminus F(t)$), and we have $c \in I(t, s)$ and $d(c, s) = d(c, v) - 1$. In both cases, by the α_i -metric property applied to $c \in I(t, s)$ and $t \in I(c, v)$, we get $d(s, v) \geq d(s, c) + d(v, c) - i \geq 2d(c, v) - 1 - i = 2e(c) - 1 - i$.

Now, since $d(s, v) \leq diam(G) \leq 2r$, if $e(c) \geq r + i + 1$, we get $2r \geq d(s, v) \geq 2e(c) - 1 - i \geq 2r + 2i + 2 - i - 1 = 2r + i + 1 > 2r$, which is impossible.

This proves that $e(c) \leq r + i$ and therefore $e(z) \leq r + 2i + 1$ for every $z \in S_k(x, y)$. \square

Lemma 6 *Let G be an α_i -metric graph and x, y be a pair of mutually distant vertices of G . Then, $d(x, y) \geq 2rad(G) - 4i - 3$ and $d(x, y) \geq diam(G) - 3i - 2$. Furthermore, any middle vertex z of a shortest path between x and y satisfies $e(z) \leq \lceil d(x, y)/2 \rceil + 2i + 1$.*

Proof Let z be a middle vertex of an arbitrary shortest path connecting x and y , and let v be a vertex furthest from z . By Lemma 4, $d(z, v) \leq \max\{d(x, v), d(y, v)\} - \min\{d(x, z), d(y, z)\} + 2i + 1$. Since x, y are two mutually distant vertices, $d(x, y) \geq \max\{d(x, v), d(y, v)\}$. Hence, $e(z) = d(z, v) \leq d(x, y) - \lfloor d(x, y)/2 \rfloor + 2i + 1 = \lceil d(x, y)/2 \rceil + 2i + 1$. That is, $d(x, y) \geq 2e(z) - 4i - 3 \geq 2rad(G) - 4i - 3$.

To prove $d(x, y) \geq diam(G) - 3i - 2$, consider a diametral pair of vertices u, v , i.e., with $d(u, v) = diam(G)$, and let $k = \lfloor d(x, y)/2 \rfloor$. By Lemma 4, there are vertices $v', u' \in S_k(x, y)$ such that $d(v', v) \leq \max\{d(x, v), d(y, v)\} - \min\{d(x, v'), d(y, v')\} + i$ and $d(u', u) \leq \max\{d(x, u), d(y, u)\} - \min\{d(x, u'), d(y, u')\} + i$. By the triangle inequality and Lemma 2, $diam(G) = d(u, v) \leq d(u, u') + d(u', v') + d(v', v) \leq (\max\{d(x, v), d(y, v)\} - \lfloor d(x, y)/2 \rfloor + i) + (i + 1) + (\max\{d(x, u), d(y, u)\} - \lfloor d(x, y)/2 \rfloor + i) = \max\{d(x, v), d(y, v)\} + \max\{d(x, u), d(y, u)\} - d(x, y) + 3i + 2$. Since, x, y are mutually distant vertices, $d(x, y) \geq \max\{d(x, v), d(y, v)\}$ and $d(x, y) \geq \max\{d(x, u), d(y, u)\}$. Hence, $diam(G) \leq d(x, y) + d(x, y) - d(x, y) + 3i + 2 = d(x, y) + 3i + 2$. That is, $d(x, y) \geq diam(G) - 3i - 2$. \square

For each vertex $v \in V \setminus C(G)$ of a graph G we can define a parameter

$$loc(v) = \min\{d(v, x) : x \in V, e(x) < e(v)\}$$

and call it the *locality* of v . It shows how far from v a vertex with a smaller eccentricity than that one of v exists. In α_i -metric graphs, the locality of each vertex is at most $i + 1$.

Lemma 7 *Let G be an α_i -metric graph. Then, for every vertex v in $V \setminus C(G)$, $loc(v) \leq i + 1$.*

Proof Let x be a vertex with $e(x) = e(v) - 1$ closest to v . Consider a neighbor z of x on an arbitrary shortest path from x to v . Necessarily, $e(z) = e(x) + 1 = e(v)$. Consider a vertex $u \in F(z)$. We have $u \in F(x)$ and $x \in I(z, u), z \in I(x, v)$. By the α_i -metric property, $d(v, u) \geq d(v, x) + d(x, u) - i = d(v, x) - i + e(x)$. As $e(v) \geq d(v, u)$, we get $e(v) \geq d(v, x) - i + e(x) = d(v, x) - i + e(v) - 1$, i.e., $d(v, x) \leq i + 1$. \square

In α_i -metric graphs, the difference between the eccentricity of a vertex v and the radius of G shows how far vertex v can be from the center $C(G)$ of G .

Lemma 8 *Let G be an α_i -metric graph and k be a positive integer. Then, for every vertex v of G with $e(v) \leq rad(G) + k$, $d(v, C(G)) \leq k + i$.*

Proof Let x be a vertex from $C(G)$ closest to v . Consider a neighbor z of x on an arbitrary shortest path from x to v . Necessarily, $e(z) = e(x) + 1 = rad(G) + 1$.

Consider a vertex $u \in F(z)$. We have $d(u, x) = rad(G)$ and $x \in I(z, u), z \in I(x, v)$. By the α_i -metric property, $d(v, u) \geq d(v, x) + d(x, u) - i = d(v, x) - i + rad(G)$. As $e(v) \geq d(v, u)$ and $e(v) \leq rad(G) + k$, we get $rad(G) + k \geq e(v) \geq d(v, x) - i + rad(G)$, i.e., $d(v, x) \leq i + k$. \square

As an immediate corollary of Lemma 8 we get:

Corollary 2 *Let G be an α_i -metric graph. Then, for every vertex v of G ,*

$$d(v, C(G)) + rad(G) \geq e(v) \geq d(v, C(G)) + rad(G) - i.$$

Proof The inequality $e(v) \leq d(v, C(G)) + rad(G)$ is true for any graph G and any vertex v by the triangle inequality. If G is an α_i -metric graph then, by Lemma 8, $d(v, C(G)) \leq e(v) - rad(G) + i$. \square

So, in α_i -metric graphs, to approximate the eccentricity of a vertex v up-to an additive one-sided error i , one only needs to know $rad(G)$ and the distance $d(v, C(G))$.

Now, instead of a pair of mutually distant vertices, we consider a vertex v furthest from an arbitrary vertex. It turns out that its eccentricity is also close enough to $diam(G)$. Furthermore, a middle vertex of any shortest path between that vertex v and a vertex u furthest from v has eccentricity at most $rad(G) + O(i)$.

Lemma 9 *Let G be an α_i -metric graph and v be an arbitrary vertex. Then, for every $u \in F(v)$, $e(u) \geq 2rad(G) - 2i - diam(C(G)) \geq 2rad(G) - 5i - 2 \geq diam(G) - 5i - 2$.*

Proof By Corollary 2, $e(v) \geq d(v, C(G)) + rad(G) - i$ and $e(u) \geq d(u, C(G)) + rad(G) - i$. Let $v', u' \in C(G)$ be vertices of $C(G)$ closest to v and u , respectively. By the triangle inequality, $e(v) = d(v, u) \leq d(v, v') + d(v', u') + d(u', u)$. Combining two inequalities for $e(v)$, we get $d(v, C(G)) + rad(G) - i \leq e(v) \leq d(v, C(G)) + d(v', u') + d(u, C(G))$, i.e., $d(u, C(G)) \geq rad(G) - i - d(v', u') \geq rad(G) - i - diam(C(G))$. Taking into account inequality $e(u) \geq d(u, C(G)) + rad(G) - i$ and Theorem 1, we have $e(u) \geq 2rad(G) - 2i - diam(C(G)) \geq 2rad(G) - 5i - 2 \geq diam(G) - 5i - 2$. \square

The latter inequality in Lemma 9 can be improved if we do not involve in the proof the diameter of the center $C(G)$ but rely on the interval thinness. We get the same bound as for mutually distant vertices (see Lemma 6) on the eccentricity of a vertex most distant from an arbitrary vertex.

Lemma 10 *Let G be an α_i -metric graph and u be an arbitrary vertex. Then, for every $v \in F(u)$, $e(v) \geq diam(G) - 3i - 2$.*

Proof Let x, y be a diametral pair of vertices of G , i.e., $d(x, y) = diam(G)$. We will show that $\max\{d(v, x), d(v, y)\} \geq diam(G) - 3i - 2$, hereby getting $e(v) \geq \max\{d(v, x), d(v, y)\} \geq diam(G) - 3i - 2$.

Assume, by way of contradiction, that $\max\{d(v, x), d(v, y)\} \leq diam(G) - 3(i + 1)$. Let $k = \lfloor (d(x, y) - 3(i + 1))/2 \rfloor$ and x', y' be vertices from $S_k(v, u)$ closest to x and y , respectively. Note that, since $d(u, v) = e(u) \geq rad(G)$ and $k = \lfloor (diam(G) -$

$3(i + 1))/2] \leq \lfloor (2rad(G) - 3(i + 1))/2 \rfloor < rad(G)$, k is smaller than $d(v, u)$ and hence vertices x', y' exist. By Lemma 2, $d(x', y') \leq i + 1$.

Let $d(v, y) \leq diam(G) - 3(i + 1)$ and consider a neighbor y'' of y' on a shortest path from y' to y . As $y'' \notin S_k(v, u)$, $d(v, y'') > d(v, y')$ or $d(u, y'') > d(u, y')$. In the former case, by the α_i -metric property applied to v, y', y'', y , we get $d(v, y) \geq d(v, y') + d(y', y) - i$, i.e., $d(y', y) \leq d(v, y) - d(v, y') + i \leq d(x, y) - 3(i + 1) - \lfloor (d(x, y) - 3(i + 1))/2 \rfloor + i = \lceil (d(x, y) - 3(i + 1))/2 \rceil + i$. In the latter case (i.e., when $d(u, y'') > d(u, y')$), by the α_i -metric property applied to u, y', y'', y , we get $d(u, y) \geq d(u, y') + d(y', y) - i$, i.e., $d(y', y) \leq d(u, y) - d(u, y') + i$. We know that $d(u, v) \geq d(u, y)$ (as $v \in F(u)$) and $d(u, v) - d(u, y') = d(v, y')$. Hence, $d(y', y) \leq d(v, y') + i = k + i \leq \lceil (d(x, y) - 3(i + 1))/2 \rceil + i$. Thus, in either case, $d(y', y) \leq \lceil (d(x, y) - 3(i + 1))/2 \rceil + i$.

By symmetry, also $d(v, x) \leq diam(G) - 3(i + 1)$ implies $d(x', x) \leq \lceil (d(x, y) - 3(i + 1))/2 \rceil + i$. But then, by the triangle inequality, $d(x, y) \leq d(x, x') + d(x', y') + d(y', y) \leq \lceil (d(x, y) - 3(i + 1))/2 \rceil + i + i + 1 + \lceil (d(x, y) - 3(i + 1))/2 \rceil + i \leq d(x, y) - 1$. The contradiction obtained shows that $\max\{d(v, x), d(v, y)\} \geq diam(G) - 3i - 2$ must hold. □

Lemma 11 *Let G be an α_i -metric graph, $z \in V$, $x \in F(z)$, and $y \in F(x)$. Any vertex $c \in S_{\lfloor d(x, y)/2 \rfloor}(x, y)$ satisfies $e(c) \leq rad(G) + 4i + (i + 1)/2 + 2$ and $e(c) \leq \lceil d(x, y)/2 \rceil + 5i + 3$.*

Proof Let $v \in F(c)$ be a furthest vertex from c . Since $y \in F(x)$, $d(x, v) \leq d(x, y)$. If also $d(y, v) \leq d(x, y)$ then, by Lemma 4, $e(c) = d(c, v) \leq \max\{d(v, x), d(v, y)\} - \min\{d(x, c), d(y, c)\} + 2i + 1 \leq d(x, y) - \lfloor d(x, y)/2 \rfloor + 2i + 1 = \lceil d(x, y)/2 \rceil + 2i + 1 \leq rad(G) + 2i + 1$.

Let now $d(y, v) > d(x, y) \geq d(x, v)$. Again, by Lemma 4, $e(c) = d(c, v) \leq d(y, v) - \lfloor d(x, y)/2 \rfloor + 2i + 1$. By Lemma 9, $d(x, y) = e(x) \geq 2rad(G) - 5i - 2 \geq d(y, v) - 5i - 2$. Therefore, $e(c) \leq d(y, v) - \lfloor (2rad(G) - 5i - 2)/2 \rfloor + 2i + 1 \leq 2rad(G) - rad(G) + 4i + (i + 1)/2 + 2 = rad(G) + 4i + (i + 1)/2 + 2$. Also, by Lemma 10, $d(x, y) = e(x) \geq diam(G) - 3i - 2 \geq d(y, v) - 3i - 2$. Hence, $e(c) \leq d(y, v) - \lfloor d(x, y)/2 \rfloor + 2i + 1 \leq d(x, y) + 3i + 2 - \lfloor d(x, y)/2 \rfloor + 2i + 1 = \lceil d(x, y)/2 \rceil + 5i + 3$. □

Next we show that all central vertices are close to a middle vertex c of a shortest path between vertices x and y , provided that x is furthest from some vertex and that y is furthest from x . Namely, $D(c, 4i + (i + 1)/2 + 2) \supseteq C(G)$ holds.

Lemma 12 *Let G be an α_i -metric graph, and let $z \in V$, $x \in F(z)$ and $y \in F(x)$. Let also c be a middle vertex of an arbitrary shortest path connecting x and y . Then, $C(G) \subseteq D(c, 4i + (i + 1)/2 + 2)$.*

Proof Consider an arbitrary vertex $u \in C(G)$. By Lemma 4, $d(c, u) \leq \max\{d(x, u), d(y, u)\} - \min\{d(x, c), d(y, c)\} + 2i + 1$. As $e(u) = rad(G)$, $\max\{d(x, u), d(y, u)\} \leq rad(G)$ holds. Since c is a middle vertex of a shortest path between x and y , $\min\{d(x, c), d(y, c)\} = \lfloor d(x, y)/2 \rfloor$. As $x \in F(z)$, by Lemma 9, $d(x, y) = e(x) \geq 2rad(G) - 5i - 2$. Hence, $d(c, u) \leq rad(G) - \lfloor (2rad(G) - 5i - 2)/2 \rfloor + 2i + 1 \leq 4i + (i + 1)/2 + 2$. □

A stronger result can be obtained for a middle vertex of a shortest path between two mutually distant vertices.

Lemma 13 *Let G be an α_i -metric graph and x, y be a pair of mutually distant vertices of G . Let also c be a middle vertex of an arbitrary shortest path connecting x and y . Then, $C(G) \subseteq D(c, 4i + 3)$.*

Proof The proof is analogous to that of Lemma 12. However, since x, y are mutually distant, by Lemma 6, $d(x, y) \geq 2rad(G) - 4i - 3$. Hence, for any $u \in C(G)$, $d(c, u) \leq rad(G) - \lfloor (2rad(G) - 4i - 3)/2 \rfloor + 2i + 1 \leq 4i + 3$. \square

There are several algorithmic implications of the results of this subsection. For an arbitrary connected graph G with m edges and a given vertex $z \in V$, a vertex $x \in F(z)$ most distant from z can be found in linear ($O(m)$) time by a *breadth-first-search BFS*(z) started at z . A pair of mutually distant vertices of an α_i -metric graph can be computed in $O(im)$ total time as follows. By Lemma 10, if x is a most distant vertex from an arbitrary vertex z and y is a most distant vertex from x , then $d(x, y) \geq diam(G) - 3i - 2$. Hence, using at most $O(i)$ *breadth-first-searches*, one can generate a sequence of vertices $x := v_1, y := v_2, v_3, \dots, v_k$ with $k \leq 3i + 4$ such that each v_i is most distant from v_{i-1} (with $v_0 = z$) and v_k, v_{k-1} are mutually distant vertices (the initial value $d(x, y) \geq diam(G) - 3i - 2$ can be improved at most $3i + 2$ times).

We summarize algorithmic results of this section in the following theorem.

Theorem 2 *There is a linear ($O(m)$) time algorithm which finds vertices v and c of an m -edge α_i -metric graph G such that $e(v) \geq diam(G) - 3i - 2$, $e(c) \leq rad(G) + 4i + (i + 1)/2 + 2$ and $C(G) \subseteq D(c, 4i + (i + 1)/2 + 2)$. Furthermore, there is an almost linear ($O(im)$) time algorithm which finds a vertex c of G such that $e(c) \leq rad(G) + 2i + 1$ and $C(G) \subseteq D(c, 4i + 3)$.*

Corollary 3 *An additive $O(i)$ -approximation of the radius and of the diameter of an α_i -metric graph G with m edges can be computed in $O(m)$ time.*

2.3 Approximating all Eccentricities in α_i -Metric Graphs

In this subsection, we show that the eccentricities of all vertices of an α_i -metric graph G can be approximated with an additive one-sided error at most $O(i)$ in (almost) linear total time.

Interestingly, the distances from any vertex v to two mutually distant vertices give a very good estimation on the eccentricity of v .

Lemma 14 *Let G be an α_i -metric graph and x, y be a pair of mutually distant vertices of G . Any vertex $v \in V$ satisfies $\max\{d(x, v), d(y, v)\} \leq e(v) \leq \max\{d(x, v), d(y, v)\} + 3i + 2$.*

Proof The inequality $e(v) \geq \max\{d(x, v), d(y, v)\}$ holds for any three vertices by definition of eccentricity. To prove the upper bound on $e(v)$ for any $v \in V$, consider a furthest vertex $u \in F(v)$ and let $k = \lfloor d(x, y)/2 \rfloor$. Note that, as x and y are mutually

distant, $d(x, y) \geq \max\{d(x, u), d(y, u)\}$. By Lemma 4, there are vertices $v', u' \in S_k(x, y)$ such that $d(v', v) \leq \max\{d(x, v), d(y, v)\} - \min\{d(x, v'), d(y, v')\} + i$ and $d(u', u) \leq \max\{d(x, u), d(y, u)\} - \min\{d(x, u'), d(y, u')\} + i$. By the triangle inequality and Lemma 2, $e(v) = d(u, v) \leq d(u, u') + d(u', v') + d(v', v) \leq (\max\{d(x, v), d(y, v)\} - \lfloor d(x, y)/2 \rfloor + i) + (i + 1) + (\max\{d(x, u), d(y, u)\} - \lfloor d(x, y)/2 \rfloor + i) = \max\{d(x, v), d(y, v)\} + \max\{d(x, u), d(y, u)\} - d(x, y) + 3i + 2 \leq \max\{d(x, v), d(y, v)\} + d(x, y) - d(x, y) + 3i + 2 = \max\{d(x, v), d(y, v)\} + 3i + 2$. \square

By Lemma 14, we get the following left-sided additive approximations of all vertex eccentricities. Let x, y be a pair of mutually distant vertices of G . For every vertex $v \in V$, set $\hat{e}(v) := \max\{d(x, v), d(y, v)\}$.

Theorem 3 *Let G be an α_i -metric graph with m -edges. There is an algorithm which in total almost linear ($O(im)$) time outputs for every vertex $v \in V$ an estimate $\hat{e}(v)$ of its eccentricity $e(v)$ such that $e(v) - 3i - 2 \leq \hat{e}(v) \leq e(v)$.*

If the minimum integer i for a graph G so that G is an α_i -metric graph is known in advance, then we can transform \hat{e} into a right-sided additive $(3i + 2)$ -approximation by setting $\hat{e}(v) := \max\{d(x, v), d(y, v)\} + 3i + 2$. Unfortunately, for a given graph to find the minimum i such that G is an α_i -metric graph is not an easy problem. We observe that even checking whether a graph is α_1 -metric is at least as hard as checking if a graph has an induced subgraph isomorphic to C_4 . Indeed, take an arbitrary graph G and add a universal vertex to it. Let the resulting graph be G' . Then, G' is an α_1 -metric graph if and only if G is C_4 -free.

In what follows, we present two right-sided additive eccentricity approximation schemes for all vertices, using a notion of *eccentricity approximating spanning tree* introduced in [48] and investigated in [20, 25, 30, 32, 35]. We get for m -edge α_i -metric graphs a $O(m)$ time right-sided additive $(9i + 5)$ -approximation and a $O(im)$ time right-sided additive $(4i + 2)$ -approximation.

A spanning tree T of a graph G is called an *eccentricity k -approximating spanning tree* if for every vertex v of G $e_T(v) \leq e_G(v) + k$ holds [48]. All (α_1, Δ) -metric graphs (including chordal graphs and the underlying graphs of 7-systolic complexes) admit eccentricity 2-approximating spanning trees [32]. An eccentricity 2-approximating spanning tree of a chordal graph can be computed in linear time [25]. An eccentricity k -approximating spanning tree with minimum k can be found in $O(nm)$ time for any n -vertex, m -edge graph G [35]. It is also known [20, 30] that if G is a δ -hyperbolic graph, then G admits an eccentricity $(4\delta + 1)$ -approximating spanning tree constructible in $O(\delta m)$ time and an eccentricity (6δ) -approximating spanning tree constructible in $O(m)$ time.

Lemma 15 *Let G be an α_i -metric graph with m edges. If c is a middle vertex of any shortest path between a pair x, y of mutually distant vertices of G and T is a $BFS(c)$ -tree of G , then, for every vertex v of G , $e_G(v) \leq e_T(v) \leq e_G(v) + 4i + 3$. That is, G admits an eccentricity $(4i + 3)$ -approximating spanning tree constructible in $O(im)$ time.*

Proof Let $e_G(v)$ ($e_T(v)$) be the eccentricity of v in G (in T , respectively). The eccentricity in T of any vertex v can only increase compared to its eccentricity in G . Hence, $e_G(v) \leq e_T(v)$. By the triangle inequality and the fact that all graph distances from vertex c are preserved in T , $e_T(v) \leq d_T(v, c) + e_T(c) = d_G(v, c) + e_G(c)$. We know that $e_G(v) \geq \max\{d_G(y, v), d_G(x, v)\}$. By Lemma 4, also $d_G(v, c) - \max\{d_G(y, v), d_G(x, v)\} \leq 2i + 1 - \min\{d_G(y, c), d_G(x, c)\}$ holds. Since c is a middle vertex of a shortest path between x and y , necessarily $\min\{d_G(y, c), d_G(x, c)\} = \lfloor d_G(x, y)/2 \rfloor$ and, by Lemma 6, $e_G(c) \leq \lceil d_G(x, y)/2 \rceil + 2i + 1$. Combining all these, we get $e_T(v) - e_G(v) \leq d_G(v, c) + e_G(c) - e_G(v) \leq d_G(v, c) - \max\{d_G(y, v), d_G(x, v)\} + e_G(c) \leq 2i + 1 - \min\{d_G(y, c), d_G(x, c)\} + e_G(c) \leq 2i + 1 - \lfloor d_G(x, y)/2 \rfloor + e_G(c) \leq 2i + 1 - \lfloor d_G(x, y)/2 \rfloor + \lceil d_G(x, y)/2 \rceil + 2i + 1 \leq 4i + 3$. \square

Lemma 16 *Let G be an α_i -metric graph with m edges, and let $z \in V$, $x \in F(z)$ and $y \in F(x)$. If c is a middle vertex of any shortest path between x and y and T is a $BFS(c)$ -tree of G , then, for every vertex v of G , $e_G(v) \leq e_T(v) \leq e_G(v) + 7i + 5$. That is, G admits an eccentricity $(7i + 5)$ -approximating spanning tree constructible in $O(m)$ time.*

Proof The proof follows the proof of Lemma 15 with one adjustment: replace the application of Lemma 6 which yields $e_G(c) \leq \lceil d_G(x, y)/2 \rceil + 2i + 1$ with Lemma 11 which yields $e_G(c) \leq \lceil d_G(x, y)/2 \rceil + 5i + 3$. Hence, $e_T(v) - e_G(v) \leq 2i + 1 - \lfloor d_G(x, y)/2 \rfloor + \lceil d_G(x, y)/2 \rceil + 5i + 3 \leq 7i + 5$. \square

Note that the eccentricities of all vertices in any tree $T = (V, U)$ can be computed in $O(|V|)$ total time. It is a folklore by now that for trees the following facts are true: (1) The center $C(T)$ of any tree T consists of one vertex or two adjacent vertices; (2) The center $C(T)$ and the radius $rad(T)$ of any tree T can be found in linear time; (3) For every vertex $v \in V$, $e_T(v) = d_T(v, C(T)) + rad(T)$. Hence, using $BFS(C(T))$ on T one can compute $d_T(v, C(T))$ for all $v \in V$ in total $O(|V|)$ time. Adding now $rad(T)$ to $d_T(v, C(T))$, one gets $e_T(v)$ for all $v \in V$. Consequently, by Lemma 15 and Lemma 16, we get the following additive approximations for the vertex eccentricities in α_i -metric graphs.

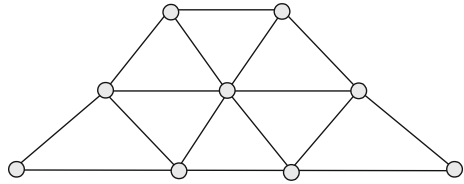
Theorem 4 *Let G be an α_i -metric graph with m edges. There is an algorithm which in total linear ($O(m)$) time outputs for every vertex $v \in V$ an estimate $\hat{e}(v)$ of its eccentricity $e(v)$ such that $e(v) \leq \hat{e}(v) \leq e(v) + 7i + 5$. Furthermore, there is an algorithm which in total almost linear ($O(im)$) time outputs for every vertex $v \in V$ an estimate $\hat{e}(v)$ of its eccentricity $e(v)$ such that $e(v) \leq \hat{e}(v) \leq e(v) + 4i + 3$.*

Corollary 4 *An additive $O(i)$ -approximation of all vertex eccentricities of an α_i -metric graph G with m edges can be computed in $O(m)$ time.*

3 Graphs with α_1 -Metric

Now we concentrate on α_1 -metric graphs, which contain all chordal graphs and all plane triangulations with inner vertices of degree at least 7 (see, e.g., [14, 15, 32,

Fig. 1 Forbidden isometric subgraph W_6^{++}



[53]). For them we get much sharper bounds. It is known that for α_1 -metric graphs the following relation between the diameter and the radius holds: $2rad(G) \geq diam(G) \geq 2rad(G) - 2$ [53].

First we recall some known results and give an auxiliary lemma.

Lemma 17 ([4]) *Let G be an α_1 -metric graph. Let x, y, v, u be vertices of G such that $v \in I(x, y), x \in I(v, u)$, and x and v are adjacent. Then $d(u, y) = d(u, x) + d(v, y)$ holds if and only if there exist a neighbor x' of x in $I(x, u)$ and a neighbor v' of v in $I(v, y)$ with $d(x', v') = 2$; in particular, x' and v' lie on a common shortest path of G between u and y .*

Theorem 5 ([53]) *G is an α_1 -metric graph if and only if all disks $D(v, k)$ ($v \in V, k \geq 1$) of G are convex and G does not contain the graph W_6^{++} from Fig. 1 as an isometric subgraph.*

Lemma 18 ([50]) *All disks $D(v, k)$ ($v \in V, k \geq 1$) of a graph G are convex if and only if for every vertices $x, y, z \in V$ and $v \in I(x, y), d(v, z) \leq \max\{d(x, z), d(y, z)\}$.*

Letting z to be from $F(v)$, we get:

Corollary 5 *If all disks $D(v, k)$ ($v \in V, k \geq 1$) of a graph G are convex then for every vertices $x, y \in V$ and $v \in I(x, y), e(v) \leq \max\{e(x), e(y)\}$.*

Lemma 19 ([32]) *Let G be an α_1 -metric graph and x be an arbitrary vertex with $e(x) \geq rad(G) + 1$. Then, for every vertex $z \in F(x)$ and every neighbor v of x in $I(x, z), e(v) \leq e(x)$ holds.*

We will need also the following auxiliary lemma.

Lemma 20 *Let G be an α_1 -metric graph. Then, for every shortest path $P = (x_1, \dots, x_l)$ with $l \leq 4$ and a vertex u of G with $d(u, x_i) = k \geq 2$ for all $i \in \{1, \dots, l\}$, there exists a vertex u' at distance 2 from each x_i ($i \in \{1, \dots, l\}$) and at distance $k - 2$ from u .*

Proof We prove by induction on l . If $l = 1$, the statement is clearly correct. Assume now that there is a vertex u' that is at distance 2 from each x_i ($i \in \{1, \dots, l - 1\}$) and at distance $k - 2$ from u . Assume that $d(u', x_l)$ is greater than 2, i.e., $d(u', x_l) = 3$. Consider a common neighbor a of u' and x_{l-1} . We have $x_{l-1} \in I(x_l, a)$ and $a \in I(u, x_{l-1})$. Then, by the α_1 -metric property, $k = d(x_l, u) \geq 1 + k - 1 = k$, and therefore, by Lemma 17, there must exist vertices b and c such that $bx_l, cb, ca \in E$

and $d(c, u) = k - 2$. As $d(c, u) = d(u', u) = k - 2$ and $d(u, a) = k - 1$, by convexity of disk $D(u, k - 2)$, vertices u' and c must be adjacent. If c is at distance 2 from x_1 then, by convexity of disk $D(c, 2)$, each vertex x_i ($1 \leq i \leq l$) is at distance 2 from c , and we are done. If c is at distance 3 from x_1 , then from $u' \in I(c, x_1)$ and $c \in I(u', x_l)$, by the α_1 -metric property, we get $d(x_1, x_l) \geq 2 + 2 = 4$, which is impossible since $d(x_1, x_l) \leq 3$. \square

Corollary 6 *Let G be an α_1 -metric graph. Then, for every edge $xy \in E$ and a vertex $u \in V$ with $d(u, x) = d(u, y) = k$, either there is a common neighbor u' of x and y at distance $k - 1$ from u or there exists a vertex u' at distance 2 from x and y and at distance $k - 2$ from u such that, for every $z \in N(x) \cap N(u')$ and $w \in N(y) \cap N(u')$, the sequence (x, z, u', w, y) forms an induced C_5 in G .*

Proof We may assume that $k \geq 2$. By Lemma 20, there exists a vertex u' at distance 2 from x and y and at distance $k - 2$ from u . Consider a common neighbor z of x and u' and a common neighbor w of y and u' . If $zy, wx \notin E$ then, by distance requirements, either (x, z, w, y) induces a C_4 (which is impossible) or (x, z, u', w, y) induces a C_5 . \square

3.1 The Eccentricity Function on α_1 -Metric Graphs is Almost Unimodal

The goal of this section is to prove the following theorem.

Theorem 6 *Let G be an α_1 -metric graph and v be an arbitrary vertex of G . If*

- (i) $e(v) > rad(G) + 1$ or
- (ii) $e(v) = rad(G) + 1$ and $diam(G) < 2rad(G) - 1$,

then there must exist a neighbor w of v with $e(w) < e(v)$.

If $e(v) = rad(G) + k$ for some integer $k > 0$, then $d(v, C(G)) \leq k + 1$.

Theorem 6 says that if a vertex v with $loc(v) > 1$ exists in an α_1 -metric graph G then $diam(G) \geq 2rad(G) - 1$, $e(v) = rad(G) + 1$ and $d(v, C(G)) = 2$. That is, only in the case when $diam(G) \in \{2rad(G) - 1, 2rad(G)\}$, the eccentricity function may fail to be unimodal and yet all local minima of the eccentricity function are concentrated around the center $C(G)$ of G (they are at distance 2 from $C(G)$). Two α_1 -metric graphs depicted in Fig. 2 show that this result is sharp.

We will split the proof of Theorem 6 into a series of lemmas of independent interest. By Lemma 7, Lemma 8 and Corollary 2, we already know that every vertex v of an α_1 -metric graph has locality at most 2, is at distance at most $k + 1$ from $C(G)$, provided that its eccentricity $e(v)$ is at most $rad(G) + k$, and satisfies $d(v, C(G)) + rad(G) \geq e(v) \geq d(v, C(G)) + rad(G) - 1$. In the following lemmas, two specific properties of α_1 -metric graphs stated in Lemma 17 and Theorem 5 are heavily used.

Lemma 21 *Let G be an α_1 -metric graph and v be a vertex of G with $loc(v) = 2$. Then, $e(v) \leq rad(G) + 2$. Furthermore, if $e(v) = rad(G) + 2$, then $diam(G) = 2rad(G)$.*

Proof Let $k := e(v)$ and x be a vertex with $d(x, v) = 2$ and $e(x) = k - 1$ such that $|F(x)|$ is as small as possible. Consider a common neighbor z of x and v and a vertex

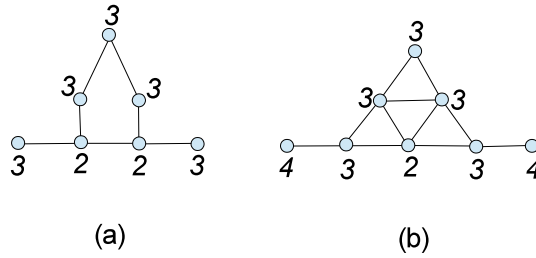


Fig. 2 Sharpness of the result of Theorem 6. **a** An α_1 -metric graph G with $diam(G) = 2rad(G) - 1$ and a vertex (topmost) with locality 2. **b** A chordal graph (and hence an α_1 -metric graph) G with $diam(G) = 2rad(G)$ and a vertex (topmost) with locality 2. The number next to each vertex indicates its eccentricity

$u \in F(z)$. Necessarily, $e(z) = e(v) = e(x) + 1$ and $u \in F(x)$. We have $x \in I(z, u)$ and $z \in I(x, v)$. By the α_1 -metric property, $d(v, u) \geq d(v, z) + d(x, u) = d(u, x) + 1 = e(z) = k$. As $e(v) \geq d(v, u) \geq e(z) = e(v)$, i.e., $k = d(v, u)$, by Lemma 17, there must exist vertices w and t such that $wv, wt, tx \in E$ and $d(t, u) = d(u, x) - 1 = k - 2$.

If $e(t) = e(x) + 1$ then for every $s \in F(t)$, we have $x \in I(t, s)$ and $d(x, s) = k - 1$. If $e(t) = e(x)$ then, by the choice of x , there must exist a vertex $s \in F(t) \setminus F(x)$ (as $u \in F(x) \setminus F(t)$), and we have $x \in I(t, s)$ and $d(x, s) = k - 2$. In both cases, by the α_1 -metric property applied to $x \in I(t, s)$ and $t \in I(x, u)$, we get $d(s, u) \geq d(s, x) + d(u, t) \geq k - 2 + k - 2 = 2k - 4$.

Since $d(s, u) \leq diam(G) \leq 2rad(G)$, we have $k \leq rad(G) + 2$ and if $k = rad(G) + 2$ then $d(s, u) = diam(G) = 2rad(G)$. □

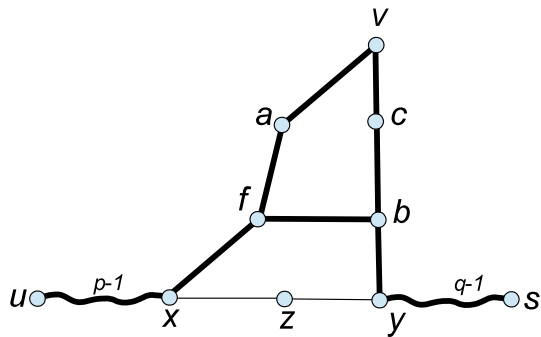
Lemma 22 *Let G be an α_1 -metric graph and v be a vertex of G with $e(v) = rad(G) + 2$. Then, $loc(v) = 1$.*

Proof Assume, by way of contradiction, that $loc(v) = 2$. Then, by Lemma 8, $d(v, C(G)) \leq 3$. However, since $e(v) = rad(G) + 2$, v cannot have a vertex from $C(G)$ at distance 2 or less as that would imply that a neighbor of v on a shortest path to $C(G)$ has eccentricity at most $rad(G) + 1$, contradicting with $loc(v) = 2$. Thus, $d(v, C(G)) = 3$.

Let x be a vertex from $C(G)$ closest to v and $P = (x, z, y, v)$ be a shortest path between x and v chosen in such a way that the neighbor y of v in P has $|F(y)|$ as small as possible. Necessarily, $e(x) = rad(G) = e(z) - 1$ and $e(z) + 1 = e(y) = e(v) = rad(G) + 2$. Consider a vertex $u \in F(y)$. Since $d(u, y) = rad(G) + 2$, $d(u, x) \leq rad(G)$ and $d(u, z) \leq rad(G) + 1$, we have $d(u, x) = rad(G) = d(u, z) - 1 = d(u, y) - 2$, i.e., $x \in I(u, z)$. Applying the α_1 -metric property to $x \in I(u, z)$ and $z \in I(x, v)$, we get $d(v, u) \geq d(v, z) + d(x, u) = rad(G) + 2$. By Lemma 17, there exist vertices f, w, t such that $fv, fz, fw, wt, tx \in E$ and $d(t, u) = rad(G) - 1$. Notice that $f \neq y$ since $d(u, y) = rad(G) + 2$ and $d(u, f) = rad(G) + 1$. To avoid an induced C_4 , vertices f and y must be adjacent. As $loc(v) = 2$, we have also $e(f) \geq e(v)$.

Now, we have $f \in S_2(x, v)$ and $u \in F(y) \setminus F(f)$. By the choice of y , there must exist a vertex s which is in $F(f)$ but not in $F(y)$. Hence, $y \in I(f, s)$. Since also $f \in I(y, u)$, by the α_1 -metric property, $d(u, s) \geq d(f, u) + d(s, y) \geq rad(G) +$

Fig. 3 Illustration to the proof of Lemma 23



$1 + rad(G) + 1 = 2rad(G) + 2 > diam(G)$, which is impossible. Thus, $loc(v) = 1$ must hold. □

Lemma 23 *Let G be an α_1 -metric graph. Let v, c, b, f, a, u, s be vertices of G such that v, c, b, f, a form an induced C_5 , $p = d(u, f) = d(v, u) - 2 = d(c, u) - 2$ and $q = d(s, b) = d(v, s) - 2 = d(a, s) - 2$ (See Fig. 3). Then, either $d(u, s) = p + q + 1$ or there is a vertex h which is adjacent to all vertices of $C_5 = (v, c, b, f, a)$.*

Proof By distance requirements, $f \in I(b, u)$ and $b \in I(f, s)$ hold. Hence, by α_1 -metric property, we have $d(u, s) \geq p + q$. Assume, in what follows, that $d(u, s) = p + q$. Then, by Lemma 17, there must exist vertices x, y, z such that $xf, xz, yb, yz \in E$ and $d(x, u) = p - 1$ and $d(y, s) = q - 1$. See Fig. 3 for an illustration.

As $d(v, x) = d(v, y) = 3$ and $z \in I(x, y)$, by convexity of disk $D(v, 3)$, we get $d(v, z) \leq 3$. Vertex z cannot be adjacent to v, c, a as $d(v, y) = d(a, y) = 3 = d(v, x) = d(c, x)$. So, $2 \leq d(v, z) \leq 3$.

First consider the case when $d(v, z) = 2$. Consider a common neighbor h of v and z . As $d(v, x) = d(v, y) = 3$ and $d(v, f) = d(v, b) = d(v, z) = 2$, by convexity of disk $D(v, 2)$, we get $zf, zb \in E$. Convexities of disks $D(y, 2)$ and $D(x, 2)$ imply $hc, ha \in E$ (notice that v is at distance 3 from both y and x , vertices a, h are at distance 2 from x , vertices h, c are at distance 2 from y). To avoid a forbidden induced C_4 , h must be adjacent to f and b as well. So, h is adjacent to all vertices of $C_5 = (v, c, b, f, a)$.

Now consider the case when $d(v, z) = 3$. Consider a path (v, h, g, z) between v and z . If z is adjacent to f then it is adjacent to b (to avoid an induced C_4), and vice versa. But if $fz, zb \in E$, by convexity of $D(z, 2)$, a and c must be adjacent, contradicting with $C_5 = (v, c, b, f, a)$ being an induced cycle. So, $zf, zb \notin E$, and hence (x, z, y, b, f) forms an induced C_5 and $g \neq f, b$.

Vertices c and z cannot be at distance 2 from each other since then convexity of disk $D(c, 2)$ and $d(c, x) = 3$ will imply $zf \in E$, which is impossible. Hence, $d(z, c) = 3$ and, in particular, $gc \notin E$ and $h \neq c$. Similarly, $d(a, z) = 3$ holds and, in particular, $ga \notin E$ and $h \neq a$.

We claim that $d(h, x) = d(h, y) = 2$. If $d(h, y) = 3$ then using also $d(c, z) = 3$ we get $z \in I(h, y)$, $y \in I(z, c)$. By the α_1 -metric property, we obtain $d(h, c) \geq d(h, z) + d(c, y) = 4$, contradicting with $d(h, c) \leq 2$. Similarly, $d(h, x) = 2$ must hold.

Now, convexity of disks $D(y, 2)$ and $D(x, 2)$ gives first $hc, ha \in E$ (as $d(y, h) = d(y, c) = 2 = d(y, v) - 1$ and $d(x, h) = d(x, a) = 2 = d(x, v) - 1$) and then $hf, hb \in E$ (as $d(y, h) = d(y, f) = 2 = d(y, a) - 1$ and $d(x, h) = d(x, b) = 2 = d(x, c) - 1$). So, h is adjacent to all vertices of $C_5 = (v, c, b, f, a)$. \square

Lemma 24 *If an α_1 -metric graph G has a vertex v with $loc(v) > 1$ and $e(v) = rad(G) + 1$, then $diam(G) \geq 2rad(G) - 1$.*

Proof Set $r := rad(G)$. Assume, by way of contradiction, that $diam(G) \leq 2r - 2$. Let v be an arbitrary vertex with $loc(v) > 1$ and $e(v) = r + 1$. Consider a vertex $s \in F(v)$ and a vertex $c \in S_1(v, s)$ with $|F(c)|$ as small as possible. Since $loc(v) > 1$, $e(c) \geq e(v)$. Let also u be an arbitrary vertex from $F(c)$.

We claim that $e(c) = d(u, c) = d(u, v) = e(v)$. If $d(u, c) > d(u, v)$, then α_1 -metric property applied to $v \in I(c, u)$ and $c \in I(s, v)$ gives $d(s, u) \geq d(s, c) + d(v, u) \geq e(v) - 1 + e(v) - 1 = 2r$, which is impossible. Hence, $e(c) = d(u, c) = d(u, v) = e(v) = r + 1$ must hold.

Assume that a vertex g exists in G such that $gc, gv \in E$ and $d(g, u) = r = d(u, c) - 1$. If $d(g, s) > d(c, s)$ then, by α_1 -metric property applied to $c \in I(g, s)$ and $g \in I(u, c)$, we get $d(s, u) \geq d(s, c) + d(g, u) = e(v) - 1 + r = 2r$, which is impossible. If $d(g, s) \leq d(c, s)$, then $g \in S_1(v, s)$ and, by the choice of c , there must exist a vertex $t \in F(g) \setminus F(c)$ (recall that $u \in F(c) \setminus F(g)$ as $e(g) \geq r + 1$). So, α_1 -metric property applied to $c \in I(g, t)$ and $g \in I(u, c)$, gives $d(t, u) \geq d(t, c) + d(g, u) \geq 2r$, which is impossible.

So, in what follows, we can assume that no common neighbor g of c and v with $d(g, u) = r = d(u, c) - 1$ can exist in G . Since, $d(u, c) = d(u, v)$, by Corollary 6, there is a vertex f which is at distance 2 from c and v , at distance $d(u, c) - 2 = r - 1$ from u and forms with any $b \in N(c) \cap N(f)$ and any $a \in N(v) \cap N(f)$ an induced $C_5 = (c, b, f, a, v)$.

We claim that $d(s, a) = d(s, v)$ for every $a \in N(v) \cap N(f)$. Indeed, if $d(s, a) < d(s, v)$ then, by convexity of disk $D(s, r)$, vertices a and c need to be adjacent (as both are adjacent to v with $d(v, s) = r + 1$), contradicting with $ac \notin E$. If $d(s, a) > d(s, v)$ then, α_1 -metric property applied to $v \in I(a, s)$ and $a \in I(u, v)$, gives $d(s, u) \geq d(s, v) + d(a, u) \geq 2r + 1$, which is impossible.

From $d(s, a) = d(s, v)$, we have also $d(s, f) \geq d(s, c)$. Assume $d(s, f) = d(s, c) + 1$. If $d(s, b) = d(s, c)$ then $b \in I(s, f)$. Since also $f \in I(b, u)$, α_1 -metric property implies $d(s, u) \geq d(s, b) + d(f, u) = d(s, c) + r - 1 = 2r - 1$, which is impossible. If now $d(s, b) = d(s, c) + 1$, then $c \in I(s, b)$. Since also $b \in I(c, u)$, α_1 -metric property implies $d(s, u) \geq d(s, c) + d(b, u) = 2r$, which is impossible. The last two contradictions show that $d(s, f) = d(s, c) + 1$ is impossible, i.e., $d(s, f) = d(s, c)$ must hold. By convexity of disk $D(s, r)$, $d(s, b) \leq r$ holds for every $b \in I(f, c)$. We distinguish between two cases. In both cases we get contradictions.

Case 1: There is a common neighbor $bof f$ and c which is at distance $r - 1 = d(s, c) - 1$ from s .

We have $d(v, u) = r + 1 = d(c, u) = d(u, f) + 2$ and $d(v, s) = r + 1 = d(a, s) = d(s, b) + 2$. By Lemma 23, either $d(u, s) = 2r - 1$ (which is impossible) or there is a vertex h which is adjacent to all vertices of $C_5 = (b, c, v, a, f)$. The latter contradicts

with our earlier claim that $d(s, a) = d(s, v)$ for every $a \in N(v) \cap N(f)$ (observe that $h \in N(v) \cap N(f)$ and $d(s, h) = d(s, b) + 1 = d(s, v) - 1$).

Case 2: There is no common neighbor of f and c which is at distance $r - 1 = d(s, c) - 1$ from s .

So, for every $b \in I(f, c)$, $d(b, s) = r$. By Lemma 20, there exists a vertex s' at distance 2 from c and f and at distance $r - 2$ from s . Consider a common neighbor x of c and s' and a common neighbor y of f and s' . We may assume that $yc, xf \notin E$ (otherwise, we are in Case 1). We claim that $d(y, v) = 2$. Assume, $d(y, v) = 3$. If also $d(a, x) = 3$, then $a \in I(y, v)$ and $v \in I(a, x)$ and α_1 -metric property implies $d(y, x) \geq d(y, a) + d(v, x) = 4$, which is impossible. So, $d(a, x) = 2$ must hold and therefore, by convexity of disk $D(a, 2)$, vertices x and y are adjacent (observe that $x, y \in D(a, 2)$ and $s' \notin D(a, 2)$). As $y \in I(f, x)$, $y \notin D(v, 2)$ and $f, x \in D(v, 2)$, we get a contradiction with convexity of disk $D(v, 2)$. Thus, $d(y, v) = 2$ must hold. By convexity of disk $D(v, 2)$, vertices x and y must be adjacent.

Consider a common neighbor w of y and v . By convexity of disk $D(s', 2)$, vertices w and c are adjacent. To avoid an induced cycle C_4 , w and x are also adjacent. If $d(w, u) = d(u, c) - 1$ then, by the choice of c , there exists a vertex $t \in F(w) \setminus F(c)$ (recall that $u \in F(c) \setminus F(w)$ as $e(w) \geq r + 1$). So, α_1 -metric property applied to $c \in I(w, t)$ and $w \in I(u, c)$, gives $d(t, u) \geq d(t, c) + d(w, u) \geq 2r$, which is impossible. So, $d(w, u) = d(u, c) = r + 1$ must hold. In particular, vertices w and f cannot be adjacent. Note also that $d(a, y) = d(v, y) = 2$ (as $d(a, s) = d(v, s)$) and $wa \notin E$ (to avoid an induced C_4). Hence, (v, w, y, f, a) induce a C_5 in G .

Now, we have $d(v, u) = r + 1 = d(w, u) = d(f, u) + 2$ and $d(v, s) = r + 1 = d(a, s) = d(s, y) + 2$. By Lemma 23, either $d(u, s) = 2r - 1$ (which is impossible) or there is a vertex h which is adjacent to all vertices of $C_5 = (y, w, v, a, f)$. The latter contradicts with our earlier claim that $d(s, a) = d(s, v)$ for every $a \in N(v) \cap N(f)$ (observe that $h \in N(v) \cap N(f)$ and $d(s, h) = d(s, y) + 1 = d(s, v) - 1$).

The contradictions obtained prove the lemma. □

Now Theorem 6 follows from Lemma 8, Lemma 21, Lemma 22 and Lemma 24. Here we formulate three interesting corollaries of Theorem 6.

Corollary 7 *Let G be an α_1 -metric graph. Then,*

- (i) *if $\text{diam}(G) < 2\text{rad}(G) - 1$ (i.e., $\text{diam}(G) = 2\text{rad}(G) - 2$) then every local minimum of the eccentricity function on G is a global minimum.*
- (ii) *if $\text{diam}(G) \geq 2\text{rad}(G) - 1$ then every local minimum of the eccentricity function on G is a global minimum or is at distance 2 from a global minimum.*

Corollary 8 *For every α_1 -metric graph G and any vertex v , the following formula is true:*

$$d(v, C(G)) + \text{rad}(G) \geq e(v) \geq d(v, C(G)) + \text{rad}(G) - \epsilon,$$

where $\epsilon \leq 1$, if $\text{diam}(G) \geq 2\text{rad}(G) - 1$, and $\epsilon = 0$, otherwise.

A path $(v = v_0, \dots, v_k = x)$ of a graph G from a vertex v to a vertex x is called *strictly decreasing* (with respect to the eccentricity function) if for every i

$(0 \leq i \leq k - 1), e(v_i) > e(v_{i+1})$. It is called *decreasing* if for every i $(0 \leq i \leq k - 1), e(v_i) \geq e(v_{i+1})$. An edge $ab \in E$ of a graph G is called *horizontal* (with respect to the eccentricity function) if $e(a) = e(b)$.

Corollary 9 *Let G be an α_1 -metric graph and v be its arbitrary vertex. Then, there is a shortest path $P(v, x)$ from v to a closest vertex x in $C(G)$ such that:*

- (i) *if $diam(G) < 2rad(G) - 1$ (i.e., $diam(G) = 2rad(G) - 2$) then $P(v, x)$ is strictly decreasing;*
- (ii) *if $diam(G) \geq 2rad(G) - 1$ then $P(v, x)$ is decreasing and can have only one horizontal edge, with an end-vertex adjacent to x .*

3.2 Diameters of Centers of α_1 -Metric Graphs

Observe that the center $C(G)$ of a graph $G = (V, E)$ can be represented as the intersection of all the disks of G of radius $rad(G)$, i.e., $C(G) = \bigcap \{D(v, rad(G)) : v \in V\}$. Consequently, the center $C(G)$ of an α_1 -metric graph G is convex (in particular, it is connected), as the intersection of convex sets is always a convex set. In general, any set $C_{\leq i}(G) := \{z \in V : ecc(z) \leq rad(G) + i\}$ is a convex set of G as $C_{\leq i}(G) = \bigcap \{D(v, rad(G) + i) : v \in V\}$.

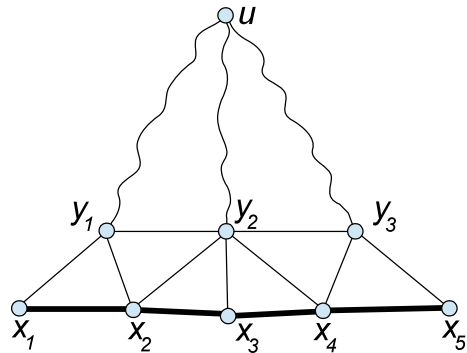
Corollary 10 *In an α_1 -metric graph G , all sets $C_{\leq i}(G), i \in \{0, \dots, diam(G) - rad(G)\}$, are convex. In particular, the center $C(G)$ of an α_1 -metric graph G is convex.*

In this section, we provide sharp bounds on the diameter and the radius of the center of an α_1 -metric graph. Previously, it was known that the diameter (the radius) of the center of a chordal graph is at most 3 (at most 2, respectively) [15]. To prove our result, we will need a few technical lemmas.

Lemma 25 *Let G be an α_1 -metric graph. Then, for every shortest path $P = (x_1, x_2, x_3, x_4, x_5)$ and a vertex u of G with $d(u, x_i) = k$ for all $i \in \{1, \dots, 5\}$, there exist vertices t, w, s such that $d(t, u) = d(s, u) = k - 1, k - 2 \leq d(w, u) \leq k - 1$, and t is adjacent to x_1, x_2, w and s is adjacent to x_4, x_5, w .*

Proof By Lemma 20, there is a vertex u' that is at distance 2 from each x_i $(1 \leq i \leq 4)$ and at distance $k - 2$ from u . Assume that $d(u', x_5)$ is greater than 2, i.e., $d(u', x_5) = 3$. Consider a common neighbor a of u' and x_4 . We have $x_4 \in I(x_5, a)$ and $a \in I(u, x_4)$. Then, by the α_1 -metric property, $k = d(x_5, u) \geq 1 + k - 1 = k$, and therefore, by Lemma 17, there must exist vertices b and c such that $bx_5, cb, ca \in E$ and $d(c, u) = k - 2$. As $d(c, u) = d(u', u) = k - 2$ and $d(u, a) = k - 1$, by convexity of disk $D(u, k - 2)$, vertices u' and c must be adjacent. If c is at distance 2 from x_1 then, by convexity of disk $D(c, 2)$, each vertex x_i $(1 \leq i \leq 5)$ is at distance 2 from c , and we can replace u' with c (see the case when $d(u', x_5) = 2$ below). If c is at distance 3 from x_1 , then from $u' \in I(c, x_1)$ and $c \in I(u', x_5)$, by the α_1 -metric property, we get $4 = d(x_1, x_5) \geq 2 + 2 = 4$. Now, by Lemma 17, there must exist vertices t, w and s such that $sx_5, sc, sw, tu', tx_1, tw \in E$. Furthermore, since $d(x_5, t) = d(x_5, x_2) = 3$ and $d(x_5, x_1) = 4$, by convexity of disk $D(x_5, 3)$, vertices t and x_2 must also be

Fig. 4 Illustration to Lemma 26



adjacent. Similarly, $sx_4 \in E$. Necessarily, $d(u, t) = d(u, s) = k - 1$. By convexity of disk $D(u, k - 1)$, either $d(u, w) = k - 1$ or $d(u, w) = k - 2$. In the latter case, w has all vertices $x_i, 1 \leq i \leq 5$, at distance 2, by convexity of disk $D(w, 2)$.

The remaining case, when $d(u', x_5) = 2$, can be handled in a similar (even simpler) manner. Set $w := u', t$ is any common neighbor of u' and x_1 and s is any common neighbor of u' and x_5 . By convexity of disks $D(x_1, 3)$ and $D(x_5, 3)$, the existence of edges tx_2 and sx_4 follows. □

Lemma 26 *Let G be an α_1 -metric graph. Then, for every shortest path $P = (x_1, x_2, x_3, x_4, x_5)$ and a vertex u of G with $d(u, x_i) = k$ for all $i \in \{1, \dots, 5\}$, there exists a shortest path $Q = (y_1, y_2, y_3)$ such that $d(u, y_i) = k - 1$, for each $i \in \{1, \dots, 3\}$, and $N(y_1) \cap P = \{x_1, x_2\}$, $N(y_2) \cap P = \{x_2, x_3, x_4\}$ and $N(y_3) \cap P = \{x_4, x_5\}$ (see Fig. 4).*

Proof By Lemma 25, there exist vertices t, s such that $d(t, u) = d(s, u) = k - 1$, t is adjacent to x_1, x_2 and s is adjacent to x_4, x_5 . If $d(x_4, t) = 3$, then $x_2 \in I(t, x_4)$ and $t \in I(u, x_2)$ and, by the α_1 -metric property, $d(x_4, u) \geq 2 + k - 1 = k + 1$, which is impossible. Hence, $d(x_4, t) = 2$ must hold. Let v be a common neighbor of t and x_4 . If $d(v, u) = k$, then again the α_1 -metric property applied to $v \in I(t, x_5)$ and $t \in I(u, v)$ gives $d(x_5, u) \geq 2 + k - 1 = k + 1$, which is impossible. Thus, $d(v, u) = k - 1$ must hold. As $d(v, u) = k - 1 = d(s, u)$ and $d(u, x_4) = k$, by convexity of $D(u, k - 1)$, vertices v and s must be adjacent. Notice also that x_3 cannot be adjacent to t or to s . If, for example, $x_3s \in E$ (the case when $tx_3 \in E$ can be handled in a similar way) then the α_1 -metric property applied to $s \in I(u, x_3)$ and $x_3 \in I(s, x_1)$ will give $d(x_1, u) \geq 2 + k - 1 = k + 1$, which is impossible. Now, convexity of disk $D(x_1, 2)$ implies edge vx_3 and convexity of disk $D(x_3, 1)$ implies edge vx_2 . Letting $y_1 := t, y_2 := v$ and $y_3 := s$, we get the required path Q . □

Theorem 7 *Let G be an α_1 -metric graph. For every pair of vertices s, t of G with $d(s, t) \geq 4$ there exists a vertex $c \in I^o(s, t)$ such that $e(c) < \max\{e(s), e(t)\}$.*

Proof It is sufficient to prove the statement for vertices s, t with $d(s, t) = 4$.

We know, by Corollary 5, that $e(c) \leq \max\{e(s), e(t)\}$ for every $c \in I(s, t)$. Assume, by way of contradiction, that there is no vertex $c \in I^o(s, t)$ such that

$e(c) < \max\{e(s), e(t)\}$. Let, without loss of generality, $e(s) \leq e(t)$. Then, for every $c \in I^o(s, t)$, $e(c) = e(t)$. Consider a vertex $c \in S_1(s, t)$. If $e(c) > e(s)$, then $e(c) = e(s) + 1$. Consider a vertex z from $F(c)$. Necessarily, $z \in F(s)$. Applying the α_1 -metric property to $c \in I(s, t)$, $s \in I(c, z)$, we get $e(c) = e(t) \geq d(t, z) \geq d(c, t) + d(s, z) = 3 + e(s) = 2 + e(c)$, which is impossible. So, $e(s) = e(c) = e(t)$ for every $c \in I^o(s, t)$.

Consider an arbitrary shortest path $P = (s = x_1, x_2, x_3, x_4, x_5 = t)$ connecting vertices s and t .

We claim that for any vertex $u \in F(x_3)$ all vertices of P are at distance $k := d(u, x_3) = e(x_3)$ from u . As $e(x_i) = e(x_3)$, we know that $d(u, x_i) \leq k$ ($1 \leq i \leq 5$). Assume $d(u, x_i) = k - 1$, $d(u, x_{i+1}) = k$, and $i \leq 2$. Then, the α_1 -metric property applied to $x_i \in I(u, x_{i+1})$ and $x_{i+1} \in I(x_i, x_{i+3})$ gives $d(x_{i+3}, u) \geq k - 1 + 2 = k + 1$, which is a contradiction with $d(u, x_{i+3}) \leq k$. So, $d(u, x_1) = d(u, x_2) = k$. By symmetry, also $d(u, x_4) = d(u, x_5) = k$.

Hence, by Lemma 26, for the path $P = (x_1, x_2, x_3, x_4, x_5)$, there exists a shortest path $Q = (y_1, y_2, y_3)$ such that $d(u, y_i) = k - 1$, for each $i \in \{1, \dots, 3\}$, and $N(y_1) \cap P = \{x_1, x_2\}$, $N(y_2) \cap P = \{x_2, x_3, x_4\}$ and $N(y_3) \cap P = \{x_4, x_5\}$ (see Fig. 4). As $y_i \in I^o(x_1, x_5) = I^o(s, t)$ for each $i \in \{1, \dots, 3\}$, we have $e(y_i) = e(x_3) = k$.

All the above holds for every shortest path $P = (s = x_1, x_2, x_3, x_4, x_5 = t)$ connecting vertices s and t . Now, assume that P is chosen in such a way that, among all vertices in $S_2(s, t)$, the vertex x_3 has the minimum number of furthest vertices, i.e., $|F(x_3)|$ is as small as possible. As y_2 also belongs to $S_2(s, t)$ and has u at distance $k - 1$, by the choice of x_3 , there must exist a vertex $u' \in F(y_2)$ which is at distance $k - 1$ from x_3 . Applying the previous arguments to the path $P' := (s = x_1, x_2, y_2, x_4, x_5 = t)$, we will have $d(x_i, u') = d(y_2, u') = k$ for $i = 1, 2, 4, 5$ and, by Lemma 26, get two more vertices v and w at distance $k - 1$ from u' such that $vx_1, vx_2, wx_4, wx_5 \in E$ and $vy_2, wy_2 \notin E$ (see Fig. 5). By convexity of disk $D(u', k - 1)$, also $vx_3, wx_3 \in E$. Now consider the disk $D(x_2, 2)$. Since y_3, w are in the disk and x_5 is not, vertices w and y_3 must be adjacent. But then vertices y_2, x_3, w, y_3 form a forbidden induced cycle C_4 .

Thus, a vertex $c \in I^o(s, t)$ with $e(c) < \max\{e(s), e(t)\}$ must exist. □

Corollary 11 *Let G be an α_1 -metric graph. Then, $\text{diam}(C(G)) \leq 3$ and $\text{rad}(C(G)) \leq 2$.*

Proof Assume, by way of contradiction, that there are vertices $s, t \in C(G)$ such that $d(s, t) \geq 4$. Then, by Theorem 7, there must exist a vertex $c \in I^o(s, t)$ such that $e(c) < \max\{e(s), e(t)\}$. The latter means that $e(c) < \text{rad}(G)$, which is impossible. So, $\text{diam}(C(G)) \leq 3$ must hold. As $C(G)$ is a convex set of G , the subgraph of G induced by $C(G)$ is also an α_1 -metric graph. According to [53], $\text{diam}(G) \geq 2\text{rad}(G) - 2$ holds for every α_1 -metric graph G . Hence, for a graph induced by $C(G)$, we have $3 \geq \text{diam}(C(G)) \geq 2\text{rad}(C(G)) - 2$, i.e., $\text{rad}(C(G)) \leq 2$. □

See Fig. 6 for the sharpness of these bounds. As chordal graphs are α_1 -metric graphs, we get also the following old result.

Corollary 12 ([15]) *Let G be a chordal graph. Then, $\text{diam}(C(G)) \leq 3$ and $\text{rad}(C(G)) \leq 2$.*

Fig. 5 Illustration to the proof of Theorem 7

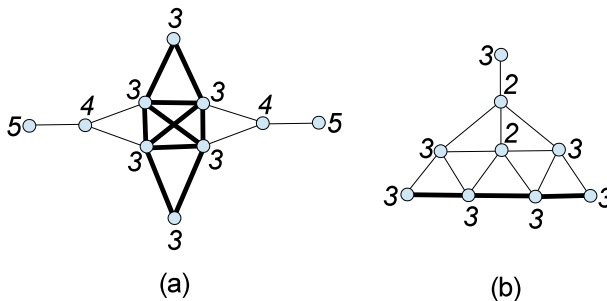
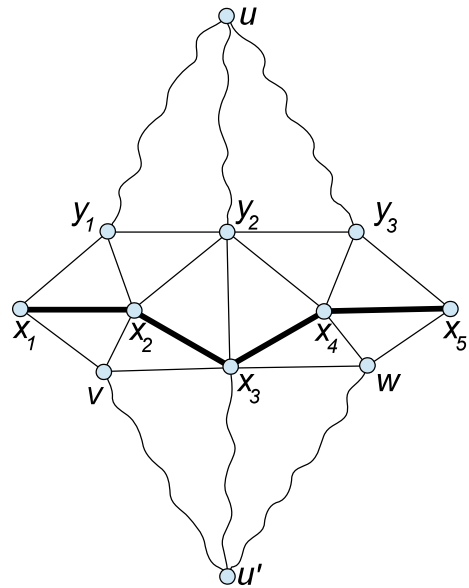


Fig. 6 Sharpness of results of Theorem 7 and Corollary 11. **a** A chordal graph with $diam(C(G)) = 3$ and $rad(C(G)) = 2$. **b** An α_1 -metric graph with a pair of vertices at distance 3 for which no vertex with smaller eccentricity exists in a shortest path between them. The number next to each vertex indicates its eccentricity

3.3 Finding a Central Vertex of an α_1 -Metric Graph

We present a local-search algorithm in order to compute a central vertex of an arbitrary α_1 -metric graph in subquadratic time (Theorem 8). Our algorithm even achieves linear runtime on an important subclass of α_1 -metric graphs, namely, (α_1, Δ) -metric graphs (Theorem 9), thus answering an open question from [32] where this subclass was introduced. The (α_1, Δ) -metric graphs are exactly the α_1 -metric graphs that further satisfy the so-called triangle condition: for every vertices u, v, w such that u and v are adjacent, and $d(u, w) = d(v, w) = k$, there must exist some common neighbour $x \in N(u) \cap N(v)$ such that $d(x, w) = k - 1$. Chordal graphs, and plane triangulations with inner vertices of degree at least 7, are (α_1, Δ) -metric graphs (see [14, 15, 32, 53]).

3.3.1 Distance- k Gates

We first introduce the required new notations and terminology for this part. In what follows, let $\text{proj}(v, A) = \{a \in A : d(v, a) = d(v, A)\}$ denote the metric projection of a vertex v to a vertex subset A . For every k such that $0 \leq k \leq d(v, A)$, we define $S_k(A, v) = \bigcup\{S_k(a, v) : a \in \text{proj}(v, A)\}$. A *distance- k gate* of v with respect to A is a vertex v^* such that $v^* \in \bigcap\{I(a, v) : a \in \text{proj}(v, A)\}$ and $d(v^*, A) \leq k$. If $k = 1$, then following [16] we simply call it a gate. Note that every vertex v such that $d(v, A) \leq k$ is its own distance- k gate. We study the existence of distance- k gates, for some $k \leq 2$, with respect to neighbour-sets and cliques. The latter are a cornerstone of our main algorithms. They are also, we think, of independent interest.

Lemma 27 *Let x and v be vertices in an α_1 -metric graph G such that $d(x, v) \geq 3$. For every vertices $u, u' \in S_3(x, v)$, the metric projections $\text{proj}(u, D(x, 1))$ and $\text{proj}(u', D(x, 1))$ are comparable by inclusion.*

Proof Suppose by contradiction that $U = \text{proj}(u, D(x, 1))$ and $U' = \text{proj}(u', D(x, 1))$ are incomparable by inclusion. Let $a \in U \setminus U'$ and $a' \in U' \setminus U$. Observe that $d(u, a) < d(u', a)$ and similarly $d(u', a') < d(u, a')$. Furthermore, $a, a' \in N(x) \cap I(x, v) (= D(x, 1) \cap D(v, d(v, x) - 1))$ must be adjacent because, according to Theorem 5, every disk is convex, and the intersection of two convex sets is also convex. As a result, applying α_1 -metric property to u, a, a', u' , we get $d(u, u') \geq d(u, a) + d(a', u') = 4$. A contradiction arises because $u, u' \in S_3(x, v)$ and, by Lemma 2, the interval thinness of G is at most 2. \square

Lemma 28 *Let x and u be vertices in an α_1 -metric graph G such that $d(x, u) = 3$. For every vertices $w, w' \in N(u) \cap I(x, u)$, the metric projections $\text{proj}(w, D(x, 1))$, $\text{proj}(w', D(x, 1))$ are comparable by inclusion.*

Proof Since $w, w' \in N(u) \cap I(x, u)$, which is the intersection of two disks, and by Theorem 5 every disk must be convex, the vertices w and w' must be adjacent. Suppose by contradiction that $\text{proj}(w, D(x, 1)), \text{proj}(w', D(x, 1))$ are not comparable by inclusion. Let $a, a' \in N(x)$ be such that $a \in N(w) \setminus N(w')$ and $a' \in N(w') \setminus N(w)$. We prove similarly as above that $a, a' \in N(x) \cap I(x, u)$ must be adjacent. But then, vertices a, a', w', w induce a C_4 , which is impossible. \square

Corollary 13 *Let x be an arbitrary vertex of an α_1 -metric graph G . Every vertex v of G has a gate v^* with respect to $D(x, 1)$.*

Proof If $d(v, x) \leq 2$, then we can choose $v^* = v$. From now on, $d(x, v) \geq 3$. Let $u \in S_3(x, v)$ be such that $|\text{proj}(u, D(x, 1))|$ is maximized. By Lemma 27, $\text{proj}(v, D(x, 1)) = \text{proj}(u, D(x, 1))$. Then, let $v^* \in N(u) \cap I(x, u)$ be such that $|\text{proj}(v^*, D(x, 1))|$ is maximized. By Lemma 28, $\text{proj}(v^*, D(x, 1)) = \text{proj}(u, D(x, 1)) = \text{proj}(v, D(x, 1))$. \square

We now turn our attention to cliques. A difference appears between general α_1 -metric graphs and (α_1, Δ) -metric graphs, which partly justifies the better runtime achieved for computing a central vertex in the latter subclass.

Lemma 29 ([32]) *Let K be a clique in an (α_1, Δ) -metric graph G . Every vertex v has a gate v^* with respect to K .*

Lemma 29 does not hold for general α_1 -metric graphs. For example, if one takes an edge in C_5 , then the vertex at distance two to both end-vertices has no gate with respect to this edge. Nevertheless, we prove next the existence of *distance-two gates*.

Lemma 30 *Let K be a clique in an α_1 -metric graph G . Every vertex v has a distance-two gate v^* with respect to K .*

Proof We may assume, without loss of generality, $d(v, K) \geq 3$. Let $v^* \in S_2(K, v)$ be maximizing $|\text{proj}(v^*, K)|$. Suppose by contradiction that $\text{proj}(v^*, K) \neq \text{proj}(v, K)$. Let $x \in \text{proj}(v, K) \setminus \text{proj}(v^*, K)$ be arbitrary, and let $w \in S_2(x, v)$. By maximality of $|\text{proj}(v^*, K)|$, there exists a $y \in \text{proj}(v^*, K) \setminus \text{proj}(w, K)$. Since G is an α_1 -metric graph, $d(v^*, w) \geq d(v^*, y) + d(x, w) = 4$. We also have $d(v^*, w) < d(v^*, y) + 1 + d(x, w) = 5$ because otherwise, the disk $D(v, d(v, K) - 2)$ could not be convex, thus contradicting Theorem 5. Therefore, $d(v^*, w) = 4$. By Lemma 17, there exist $y' \in N(y) \cap N(v^*)$ and $x' \in N(x) \cap N(w)$ such that $d(x', y') = 2$. However, since we have $x', y' \in I(v^*, w) \setminus D(v, d(v, K) - 2)$, the latter still contradicts the convexity of disk $D(v, d(v, K) - 2)$. \square

3.3.2 Computation of Gates and Distance-two Gates

The problem of computing gates has already attracted some attention, e.g., see [16]. We use this routine in the design of our main algorithms.

Lemma 31 ([39]) *Let A be an arbitrary subset of vertices in some graph G with m edges. In total $O(m)$ time, we can map every vertex $v \notin A$ to some vertex $v^* \in D(v, d(v, A) - 1) \cap N(A)$ such that $|N(v^*) \cap A|$ is maximized. Furthermore, if v has a gate with respect to A , then v^* is a gate of v .*

The efficient computation of distance-two gates is more challenging. We present a subquadratic-time procedure that only works in our special setting.

Lemma 32 *Let K be a clique in some α_1 -metric graph G with m edges. In total $O(m^{1.41})$ time, we can map every vertex $v \notin K$ to some distance-two gate v^* with respect to K . Furthermore, in doing so we can also map v^* to some independent set $J_K(v^*) \subseteq D(v^*, 1)$ such that $\text{proj}(v^*, K)$ is the disjoint union of neighbour-sets $N(w) \cap K$, for every $w \in J_K(v^*)$.*

Proof For short, let us write $p_K(v) = |\text{proj}(v, K)|$ for every vertex v . If $v \in N(K)$, then we can set $v^* = v$, $J_K(v) = \{v\}$ and $p_K(v) = |N(v) \cap K|$. This can be done in total $O(m)$ time for every vertex of $N(K)$. Thus from now on we only consider vertices v such that $d(v, K) \geq 2$.

We compute $J_K(u)$ for every vertex u such that $d(u, K) = 2$. For that, we order the vertices of $N(u) \cap N(K)$ by nonincreasing p_K -value. For every $x \in N(u) \cap N(K)$, if there is a triangle xyu for some $y \in N(K)$ ordered before x , then we claim that $N(x) \cap K \subseteq N(y)$, and so we can ignore x . Indeed, since x, y are adjacent, $N(x) \cap K$

and $N(y) \cap K$ must be comparable by inclusion (otherwise, since K is a clique, there would exist an induced C_4 with x, y and arbitrary vertices of $(N(x) \setminus N(y)) \cap K, (N(y) \setminus N(x)) \cap K$). Since y was ordered before $x, p_K(y) \geq p_K(x)$, which implies as claimed $N(x) \cap K \subseteq N(y)$. Else, for any $y \in N(K) \cap N(u)$ ordered before x , we claim that $N(x) \cap K$ and $N(y) \cap K$ must be disjoint. Indeed, otherwise, there would be an induced C_4 with u, x, y and an arbitrary vertex of $N(x) \cap N(y) \cap K$. Then, we put vertex x in $J_K(u)$. Overall, we are left solving a variation of the well-known problem of deciding, for every edge in a graph, whether it is part of a triangle. This problem can be solved in $O(m^{1.41})$ time [2].

Finally, we consider all vertices v such that $d(v, K) \geq 2$ by nondecreasing distance to K . If $d(v, K) = 2$, then we set $v^* = v$ and $p_K(v) = \sum\{p_K(x) : x \in J_K(v)\}$. Otherwise, we pick some neighbour $u \in N(v)$ such that $d(u, K) = d(v, K) - 1$ and $p_K(u)$ is maximized. Then, we set $v^* = u^*$ and $p_K(v) = p_K(u)$. This procedure is correct assuming that a distance-two gate always exists, which follows from Lemma 30. □

3.3.3 Local-search Algorithms

Now that we proved the existence of gates and distance-two gates, and of efficient algorithms in order to compute them, we turn our attention to the following subproblem: being given a vertex x in an α_1 -metric graph G , either compute a neighbour y such that $e(y) < e(x)$, or assert that x is a local minimum for the eccentricity function (but not necessarily a central vertex). Our analysis of the next algorithms essentially follows from the results of Sect. 3.1. We first present the following special case, of independent interest, and for which we obtain a better runtime than for the more general Lemma 34.

Lemma 33 *Let x be an arbitrary vertex in an α_1 -metric graph G with m edges. If $e(x) \geq rad(G) + 2$, then $\bigcap\{N(x) \cap I(x, z) : z \in F(x)\} \neq \emptyset$, and every neighbour y in this subset satisfies $e(y) < e(x)$. In particular, there is an $O(m)$ -time algorithm that either outputs a $y \in N(x)$ such that $e(y) < e(x)$, or asserts that $e(x) \leq rad(G) + 1$.*

Proof If $e(x) \geq rad(G) + 2$, then by Theorem 6, there exists a $y \in N(x)$ such that $e(y) < e(x)$. In particular, $\bigcap\{N(x) \cap I(x, z) : z \in F(x)\} \neq \emptyset$. Let y' be an arbitrary neighbour of x in this subset. Suppose by contradiction $e(y') \geq e(x)$. By Lemma 19, $e(y') \leq e(x)$. Hence, $e(x) = e(y') \geq rad(G) + 2$. Furthermore, $F(x) \cap F(y') = \emptyset$ because we assumed that $y' \in N(x) \cap I(x, z)$ for every $z \in F(x)$. But then, let $z \in F(x), z' \in F(y')$ be arbitrary. Since G is an α_1 -metric graph, $d(z, z') \geq d(z, y') + d(x, z') = e(y') - 1 + e(x) - 1 \geq 2(rad(G) + 1) > diam(G)$. The latter is impossible.

Finally, we describe our $O(m)$ -time algorithm for an arbitrary vertex x (of unknown eccentricity). We assume without loss of generality $e(x) \geq 2$. We compute gates z^* with respect to $D(x, 1)$ for every $z \in F(x)$, whose existence follows from Corollary 13. By Lemma 31, this can be done in $O(m)$ time. Then, we compute $K = \bigcap\{N(x) \cap I(x, z) : z \in F(x)\}$. Note that $K = \bigcap\{N(x) \cap N(z^*) : z \in F(x)\}$, and therefore, we can also compute K in $O(m)$ time. If $K = \emptyset$, then we can assert $e(x) \leq rad(G) + 1$.

Else, let $y \in K$ be arbitrary. If $e(x) \leq e(y)$, then again we can assert that $e(x) \leq \text{rad}(G) + 1$. Otherwise, we are done outputting y . \square

We can strengthen Lemma 33 as follows, at the expenses of a higher runtime.

Lemma 34 *Let x be an arbitrary vertex in an α_1 -metric graph G with m edges. There is an $O(m^{1.41})$ -time algorithm that either outputs a $y \in N(x)$ such that $e(y) < e(x)$, or asserts that x is a local minimum for the eccentricity function. If G is (α_1, Δ) -metric, then its runtime can be lowered down to $O(m)$.*

Proof If $e(x) \leq 2$ then x is always a local minimum for the eccentricity function, unless $e(x) = 2$ and x is adjacent to some universal vertex y . Therefore, we assume for the remainder of the proof $e(x) \geq 3$. First we compute $K = \bigcap \{N(x) \cap I(x, z) : z \in F(x)\}$. This can be done in $O(m)$ time by using the exact same approach as for Lemma 33. If $K = \emptyset$, then clearly x is a local minimum for the eccentricity function. Otherwise, we are left deciding whether there exists a vertex of K with eccentricity smaller than $e(x)$. For that, we claim that we only need to consider the vertices v such that $d(v, K) \geq e(x) - 1 \geq 2$. Indeed, by Theorem 5, the disks of G must be convex, which implies that K is a clique. In particular, every vertex v such that $d(v, K) \leq e(x) - 2$ is at a distance at most $e(x) - 1$ to every vertex of K . Therefore, the claim is proved. Now, if $d(v, K) = e(x)$ for at least one vertex v , then every vertex of K must have eccentricity at least $e(x)$, hence we can assert that x is a local minimum for the eccentricity function. Otherwise, let $F(K) = \{v \in V : d(v, K) = e(x) - 1\}$. For every $y \in K$, in order to decide whether $e(y) < e(x)$, it suffices to decide whether $y \in \bigcap \{\text{proj}(v, K) : v \in F(K)\}$, or even more strongly to compute the number of vertices $v \in F(K)$ such that $y \in \text{proj}(v, K)$.

For general α_1 -metric graphs, we compute distance-two gates v^* for every $v \in F(K)$, whose existence follows from Lemma 30. By Lemma 32, this can be done in $O(m^{1.41})$ time. Being also given the independent sets $J_K(v^*) \subseteq N(v^*)$, for every $v \in F(K)$, as in Lemma 32, we compute the following weight function α on $N(K)$: $\alpha(w) = |\{v \in F(K) : w \in J_K(v^*)\}|$. This can be done in $O(m)$ time. Recall that for every $v \in F(K)$, $\text{proj}(v, K)$ is the disjoint union of $N(w) \cap K$ for every $w \in J_K(v^*)$. As a result, for every $y \in K$, the number of vertices $v \in F(K)$ such that $y \in \text{proj}(v, K)$ is exactly $\sum \{\alpha(w) : w \in N(y) \setminus K\}$, which can be calculated in total $O(m)$ time for every vertex of K .

Finally, assume for the remainder of the proof that G is (α_1, Δ) -metric, and let us modify this last part of the procedure as follows. We compute gates v^* for every $v \in F(K)$, whose existence follows from Lemma 29. By Lemma 31, this can be done in $O(m)$ time. For every $w \in N(K)$, let $\alpha(w) = |\{v \in F(K) : v^* = w\}|$. As before, for every $y \in K$, the number of vertices $v \in F(K)$ such that $y \in \text{proj}(v, K)$ is exactly $\sum \{\alpha(w) : w \in N(y) \setminus K\}$, which can still be calculated in total $O(m)$ time for every vertex of K . \square

3.3.4 The Main Procedures

We start presenting our algorithm for the general α_1 -metric graphs.

Theorem 8 *If G is an α_1 -metric graph with m edges, then a vertex x_0 such that $e(x_0) \leq \text{rad}(G) + 1$ can be computed in $O(m)$ time. Furthermore, a central vertex can be computed in $O(m^{1.71})$ time.*

Proof By Theorem 2, we can compute in $O(m)$ time a vertex x_0 such that $e(x_0) \leq \text{rad}(G) + 3$. We repeatedly apply Lemma 33 until we can further assert that $e(x_0) \leq \text{rad}(G) + 1$ (and, hence, by Theorem 6, $d(x_0, C(G)) \leq 2$). Since there are at most two calls to this local-search procedure, the runtime is in $O(m)$. Then, we apply the following procedure, starting from x_0 and $X_0 := V$, until we can assert that the current vertex x_i (considered at the i^{th} iteration) is central.

1. If $\text{deg}(x_i) \leq m^{29}$, then we output a vertex of minimum eccentricity within $D(x_i, 2)$, and then we stop. This is done by applying Lemma 34 to every vertex of $D(x_i, 1)$. Otherwise ($\text{deg}(x_i) > m^{29}$), we go to Step 2.
2. Let $z_i \in F(x_i)$ be arbitrary. We set $X_{i+1} := X_i \cap D(x_i, 5) \cap D(z_i, e(x_i) - 1)$. If $X_{i+1} = \emptyset$, then we output x_i . Otherwise, we pick an arbitrary vertex $y \in X_{i+1}$, and then we go to Step 3.
3. We consider several cases in what follows.
 - (a) If $e(y) < e(x_i)$, then we output y ;
 - (b) Else, if $e(y) = e(x_i)$, then we set $x_{i+1} := y$ and we go back to Step 1;
 - (c) Else, there are three subcases. If $\bigcap \{X_{i+1} \cap N(y) \cap I(y, w) : w \in F(y)\} = \emptyset$, then we output x_i . Otherwise, we pick an arbitrary neighbour y' in this subset. If $e(y') \geq e(y)$, then we also output x_i . Else, we set $y := y'$ and we repeat Step 3.

We stress that at the i^{th} iteration we ensure at Step 2 that $x_i \in X_i \setminus X_{i+1}$. In particular, $X_0 \supset X_1 \supset \dots \supset X_i$. It implies that our procedure eventually halts at some iteration T because we always stop at Step 2 if $X_{i+1} = \emptyset$. Suppose by contradiction that we do not output a central vertex. For every i such that $0 \leq i < T$, we ensure at Step 3b that $e(x_{i+1}) = e(x_i)$. Therefore, $e(x_T) = e(x_{T-1}) = \dots = e(x_0) \leq \text{rad}(G) + 1$. Since at the T^{th} iteration, we output a vertex of eccentricity at most $e(x_T)$, we must have $e(x_T) \geq \text{rad}(G) + 1$. It implies $e(x_i) = \text{rad}(G) + 1$ for every i such that $0 \leq i \leq T$. Then, by Theorem 6, $d(x_i, C(G)) \leq 2$ for every i such that $0 \leq i \leq T$. Let us now consider the last step executed during iteration T . It cannot be Step 1 because a minimum eccentricity vertex within $D(x_T, 2)$ must be central. Suppose by contradiction that we halt at Step 2. Then, $X_{T+1} = \emptyset$. We prove by induction that $C(G) \subseteq X_i$ for every i with $0 \leq i \leq T + 1$, obtaining a contradiction. This is true for $i = 0$ because $X_0 = V$. Assume now this is true for some i with $0 \leq i \leq T$. Recall that $d(x_i, C(G)) \leq 2$. By Corollary 11, $\text{diam}(C(G)) \leq 3$. As a result, $C(G) \subseteq D(x_i, 5)$. Furthermore, $C(G) \subseteq D(z_i, e(x_i) - 1)$ because we have $\text{rad}(G) = e(x_i) - 1$. Hence, $C(G) \subseteq X_i \cap D(x_i, 5) \cap D(z_i, e(x_i) - 1) = X_{i+1}$. We deduce from the above that we must halt at Step 3. Since we suppose that we output a vertex of eccentricity $e(x_T) = \text{rad}(G) + 1$, it implies that we halt at Step 3c. In particular, we found a vertex $y \in X_{T+1}$ such that $e(y) \geq e(x_T) + 1$, and either $\bigcap \{X_{T+1} \cap N(y) \cap I(y, w) : w \in F(y)\} = \emptyset$, or $e(y') \geq e(y)$ for some arbitrary neighbour of y in this subset. However, by Corollary 9, there exists a shortest $yC(G)$ -path that is decreasing and such that the only horizontal edge, if any, must have one end

that is adjacent to the end-vertex in $\text{proj}(y, C(G))$ (i.e., it must be the penultimate edge of the path, starting from y). In particular, the neighbour y'' of y on this shortest path must satisfy $e(y'') < e(y)$. Since $y \in X_{T+1}$, $C(G) \subseteq X_{T+1}$ and according to Theorem 5 the subset X_{T+1} must be convex (as intersection of convex disks), we get that $y'' \in X_{T+1}$. Then, $\bigcap\{X_{T+1} \cap N(y) \cap I(y, w) : w \in F(y)\} \neq \emptyset$. Furthermore, since we assume $e(y) \geq \text{rad}(G) + 2$, by Lemma 33, every neighbour y' in this subset must satisfy $e(y') < e(y)$. A contradiction occurs.

We end up analysing the runtime of the procedure. Recall that all vertices x_0, x_1, \dots, x_T are pairwise different. Since all such vertices, except maybe the last one, have degree more than $m^{.29}$, the number of iterations is in $O(m/m^{.29}) = O(m^{.71})$. Let us consider an arbitrary iteration i , for some i such that $0 \leq i \leq T$. During Step 1, either we do nothing or (only if $i = T$) we call Lemma 34 at most $O(m^{.29})$ times and then we stop. In the latter case, the runtime is in $O(m^{1.7})$. Step 2 takes $O(m)$ time. Then, during Step 3, we only consider vertices $y \in D(x_i, 5)$. Since every such vertex y satisfies $e(y) \leq e(x_i) + 5$, we can execute Step 3c at most five times. For every execution of Step 3c, we can slightly modify the algorithm of Lemma 33 in order to perform all computations in $O(m)$ time. Therefore, Step 3 also takes $O(m)$ time. Overall, the total runtime of any iteration is in $O(m)$, except maybe for the last iteration whose runtime can be $O(m^{1.7})$. Since there are only $O(m^{.71})$ iterations, the final runtime is in $O(m^{1.71})$. \square

To lower the runtime to $O(m)$ for the (α_1, Δ) -metric graphs, we use a different approach that is based on the following additional properties of these graphs. Unfortunately, these properties crucially depend on the triangle condition.

Lemma 35 ([32]) *Let G be an (α_1, Δ) -metric graph. Then, in every slice $S_k(y, z)$, there is a vertex w that is universal to that slice, i.e., $S_k(y, z) \subseteq D(w, 1)$.*

Lemma 36 *Let x, y be adjacent vertices in an (α_1, Δ) -metric graph G such that both x, y are local minima for the eccentricity function and $e(x) = e(y) = \text{rad}(G) + 1$. For every $z \in \text{proj}(x, C(G))$, there exists a $u \in N(x) \cap N(z)$ such that $F(u) \subseteq F(x) \cap F(y)$.*

Proof First we prove that $z \in \text{proj}(y, C(G))$. Suppose by contradiction it is not the case. By Theorem 6, $d(x, C(G)) = d(y, C(G)) = 2$. Therefore, $d(y, z) = 3$ and $x \in N(y) \cap I(y, z)$. Let $w \in N(x) \cap N(z)$ be arbitrary. Note that $e(w) \leq e(z) + 1 = e(x)$. It implies $e(w) = e(x) = \text{rad}(G) + 1$ because x is assumed to be a local minimum for the eccentricity function. Then, let $v \in F(w)$ be arbitrary. Since G is α_1 -metric, $d(v, y) \geq d(v, z) + d(w, y) = \text{rad}(G) + 2$. The latter is in contradiction with $e(y) = \text{rad}(G) + 1$.

Since x, y are adjacent and $d(x, z) = d(y, z) = 2$, the triangle condition implies the existence of some common neighbour $u \in N(x) \cap N(y) \cap N(z)$. Recall that $e(u) = e(x) = e(y) = \text{rad}(G) + 1$. Suppose by contradiction that there exists a $v \in F(u) \setminus F(x)$. Since G is an α_1 -metric graph, $d(z, v) \geq d(z, u) + d(x, v) = 1 + \text{rad}(G)$, which is impossible for $z \in C(G)$. As a result, $F(u) \subseteq F(x)$. We prove similarly that $F(u) \subseteq F(y)$, and so $F(u) \subseteq F(x) \cap F(y)$. \square

Theorem 9 *If G is an (α_1, Δ) -metric graph with m edges, then a central vertex can be computed in $O(m)$ time.*

Proof The algorithm starts from a vertex x which is a local minimum for the eccentricity function of G . To compute such a vertex in $O(m)$ time, we first apply Theorem 2 in order to compute a vertex of eccentricity at most $rad(G) + 3$. Then, we apply Lemma 34 at most three times.

We run a core procedure which either outputs two adjacent vertices $u, v \in D(x, 1)$ such that $e(u) = e(v) = e(x)$ and $F(u), F(v)$ are not comparable by inclusion, or outputs a central vertex. In the former case, let $y \in F(u) \setminus F(v)$ and $z \in F(v) \setminus F(u)$ be arbitrary. Since G is an α_1 -metric graph, $d(y, z) \geq d(y, v) + d(u, z) = 2e(x) - 2$. If $d(y, z) \geq 2e(x) - 1$, then $rad(G) > e(x) - 1$, and therefore x is a central vertex. From now on, we assume that $d(y, z) = 2e(x) - 2$. Then, any vertex of eccentricity $e(x) - 1$ must be contained in $S_{e(x)-1}(y, z)$. Let w be such that $S_{e(x)-1}(y, z) \subseteq D(w, 1)$, whose existence follows from Lemma 35. We apply Lemma 34 in order to compute a minimum eccentricity vertex x' within $D(w, 1)$. Finally, we output any of x, x' that has minimum eccentricity. To complete the description of our algorithm, we now present the core procedure.

1. We compute gates z^* with respect to $D(x, 1)$ for every $z \in F(x)$, whose existence follows from Corollary 13. By Lemma 31, this can be done in $O(m)$ time. Then, for every $y \in N(x)$, we compute $f_x(y) = |\{z \in F(x) : y \in N(x) \cap I(x, z)\}|$. This can also be done in $O(m)$ time by enumerating the gates z^* in $N(y)$ for every $y \in N(x)$.
2. We choose a neighbour y_1 such that $f_x(y_1)$ is maximized. There are three cases.
 - (a) If $e(y_1) > e(x)$ then, by Lemma 19, we can assert that x is a central vertex. From now on, $e(y_1) \leq e(x)$. Since x is a local minimum for the eccentricity function, $e(x) = e(y_1)$.
 - (b) If y_1 is not a local minimum for the eccentricity function of G , then by Lemma 34 we can compute a central vertex within $N(y_1)$ in $O(m)$ time. From now on both x, y_1 are local minima for the eccentricity function.
 - (c) If $F(y_1) \setminus F(x) \neq \emptyset$, then we can output $u = x, v = y_1$. Indeed, we also have $F(x) \setminus F(y_1) \neq \emptyset$ because $y_1 \in N(x) \cap I(x, z)$ for some $z \in F(x)$. In what follows, we assume that $F(x), F(y_1)$ are comparable by inclusion, and so, $F(y_1) \subset F(x)$.
3. Let $z_1 \in F(y_1)$ be arbitrary. We choose some neighbour y_2 within $N(x) \cap I(x, z_1)$ which maximizes $f_x(y_2)$. Note that in order to compute $N(x) \cap I(x, z_1)$, and so y_2 , in $O(m)$ time, it suffices to only consider the vertices of $N(x) \cap N(z_1^*)$. There are two cases.
 - (a) If $e(y_2) > e(x)$ then, by Lemma 19, we can assert that x is a central vertex. From now on we assume $e(y_2) = e(x)$.
 - (b) If y_2 is not a local minimum for the eccentricity function of G , then by Lemma 34 we can compute a central vertex within $N(y_2)$ in $O(m)$ time. From now on vertices x, y_1, y_2 are local minima for the eccentricity function.

4. Due to the maximality of $f_x(y_1)$ and the existence of a $z_1 \in F(x) \cap F(y_1)$, we have $f_x(y_2) \leq f_x(y_1) < |F(x)|$. Furthermore, both $F(x) \setminus F(y_1), F(x) \setminus F(y_2)$ are nonempty. For each $i \in \{1, 2\}$, we compute $B_i = \bigcap \{N(x) \cap I(x, z) : z \in F(x) \setminus F(y_i)\}$. Since $B_i = \bigcap \{N(x) \cap N(z^*) : z \in F(x) \setminus F(y_i)\}$, it can be done in $O(m)$ time. There are now two cases.

- (a) Assume the existence of an edge uv where $u \in B_1, v \in B_2$. If $\max\{e(u), e(v)\} > e(x)$ then, by Lemma 19, we can assert that x is a central vertex. Otherwise, we prove next that $F(u), F(v)$ are not comparable by inclusion, and therefore we can output u, v . Indeed, $F(y_1), F(x) \cap F(y_2)$ are not comparable by inclusion because

$$|F(x) \cap F(y_2)| = |F(x)| - f_x(y_2) \geq |F(x)| - f_x(y_1) = |F(y_1)|$$

and $z_1 \in F(y_1) \setminus F(y_2)$. Furthermore, $F(x) \cap F(u) \subseteq F(y_1) (F(x) \cap F(v) \subseteq F(y_2)$, respectively) because $u \in B_1 (v \in B_2)$, respectively). Then, $F(u) \cap F(x) = F(y_1)$ and $F(v) \cap F(x) = F(y_2) \cap F(x)$, because otherwise this would contradict the maximality of either $f_x(y_1)$ or $f_x(y_2)$.

- (b) Finally, assume that every vertex of B_1 is nonadjacent to every vertex of B_2 . We claim that x is a central vertex. Suppose by contradiction $e(x) = rad(G) + 1$. Let $z \in \text{proj}(x, C(G))$ be arbitrary. By Lemma 36, there exist vertices $u, v \in N(x) \cap N(z)$ such that $F(u) \subseteq F(x) \cap F(y_1) = F(y_1), F(v) \subseteq F(x) \cap F(y_2)$. Note that $\max\{e(u), e(v)\} \leq e(z) + 1 = rad(G) + 1$. Since x is a local minimum for the eccentricity function, we obtain $e(u) = e(v) = rad(G) + 1$. Then, $u \in B_1, v \in B_2$. In particular, u, v must be nonadjacent, thus contradicting the convexity of $D(x, 1) \cap D(z, 1)$, and so, Theorem 5.

□

We leave as an open question whether there exists a linear-time algorithm for computing a central vertex in an α_1 -metric graph.

3.4 Approximating all Eccentricities in α_1 -Metric Graphs

It follows from the results of Sects. 3.3 and 2.2 that a vertex with eccentricity at most $rad(G) + 1$ and a vertex with eccentricity at least $diam(G) - 5$ can be found in linear time for every α_1 -metric graph G . Furthermore, all vertex eccentricities with an additive one-sided error at most 5 in an α_1 -metric graph can be computed in total linear time (see Sect. 2.3). Here, we present two immediate consequences of the results of Sect. 3, hereby answering some open questions from [25, 32].

Theorem 10 *Every α_1 -metric graph G admits an eccentricity 3-approximating spanning tree. Furthermore, an additive 4-approximation of all vertex eccentricities in G can be computed in subquadratic total time.*

Proof It is sufficient to show that any breadth-first-search tree T of G rooted at a vertex $c \in C(G)$ is an eccentricity 4-approximating spanning tree of G and any breadth-first-search tree T of G rooted at a vertex $c \in C(C(G))$ (i.e., a central vertex of the subgraph

of G that is induced by $C(G)$) is an eccentricity 3-approximating spanning tree of G . We do not know how to find efficiently a vertex $c \in C(C(G))$ but, by Theorem 8, a central vertex c of G with m edges can be computed in $O(m^{1.71})$ time.

Consider an arbitrary vertex v in G and let v' be a vertex of $C(G)$ closest to v . By Corollary 8, $e(v) \geq d(v, v') + rad(G) - \epsilon$, where $\epsilon \leq 1$. Since T is a shortest path tree and c is a central vertex of G , $e_T(v) \leq d_T(v, c) + e_T(c) = d_G(v, c) + e_G(c) = d_G(v, c) + rad(G)$. Hence, by the triangle inequality, $e_T(v) - e_G(v) \leq d_G(v, c) + rad(G) - d_G(v, v') - rad(G) + \epsilon \leq d_G(c, v') + \epsilon \leq d_G(c, v') + 1$.

By Corollary 11, we know $diam(C(G)) \leq 3$ and $rad(C(G)) \leq 2$. Hence, if $c \in C(C(G))$ then $d_G(c, v') \leq 3$, and if $c \in C(C(G))$ then $d_G(c, v') \leq 2$. \square

For (α_1, Δ) -metric graphs this result can be strengthened further.

Theorem 11 *Every (α_1, Δ) -metric graph G admits an eccentricity 2-approximating spanning tree. Furthermore, an additive 3-approximation of all vertex eccentricities in G with m edges can be computed in total $O(m)$ time.*

Proof Let T be a $BFS(c)$ -tree, where c is a central vertex of G . We can follow the proof of Theorem 10 and get $e_T(v) - e_G(v) \leq d_G(c, v') + \epsilon$, where v' is a vertex of $C(G)$ closest to v . In an (α_1, Δ) -metric graph G , we have $\epsilon \leq 1$, if $diam(G) = 2rad(G)$, and $\epsilon = 0$, otherwise [32]. Furthermore, when $diam(G) = 2rad(G)$, $diam(C(G)) \leq 2$ and $rad(C(G)) \leq 1$ must hold [32]. Thus, if $c \in C(G)$ then $e_T(v) - e_G(v) \leq d_G(c, v') + \epsilon \leq diam(C(G)) + \epsilon \leq 3$ (i.e., $\leq 3 + 0$ or $\leq 2 + 1$), and if $c \in C(C(G))$ then $e_T(v) - e_G(v) \leq d_G(c, v') + \epsilon \leq rad(C(G)) + \epsilon \leq 2$ (i.e., $\leq 2 + 0$ or $\leq 1 + 1$). Note also that a central vertex of an (α_1, Δ) -metric graph can be computed in linear time (Theorem 9). \square

The existence of an eccentricity 2-approximating spanning tree in an (α_1, Δ) -metric graph is known already from [32]. The second part of Theorem 11 provides an answer to an open question from [25].

4 Concluding Remarks

We conclude the paper with some immediate questions building off our results.

1. Can our (approximation) bounds on eccentricities in general α_i -metric graphs be improved? In particular,
 - (i) is our bound on the eccentricity of a middle vertex of a shortest path between two mutually distant vertices best possible?
 - (ii) is our bound on the eccentricity of a vertex furthest from an arbitrary vertex sharp?
 - (iii) can our bound $3i + 2$ on the diameter of the center be improved to $2i + 1$?
2. What best approximations of the radius and of the diameter of an α_1 -metric graph G can be achieved in linear time? In particular,
 - (iv) does there exist a linear-time algorithm for finding a central vertex (and, hence, the exact radius) of an α_1 -metric graph?

- (v) is it possible to show that the eccentricity of a vertex furthest from an arbitrary vertex is at least $diam(G) - 2$?
- (vi) can a vertex with eccentricity at least $diam(G) - 1$ be found in linear time?

Recall that in chordal graphs (a subclass of α_1 -metric graphs) a central vertex can be found in linear time and the eccentricity of a vertex furthest from an arbitrary vertex is at least $diam(G) - 2$ [16]. Furthermore, a vertex with eccentricity at least $diam(G) - 1$ can be found in linear time by a LexBFS [34]. On the other hand, computing the exact diameter of a chordal graph in subquadratic time is impossible unless the well known Strong Exponential Time Hypothesis (SETH) is false [6].

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Declaration

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article.

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References

1. Abboud, A., Vassilevska Williams, V., Wang, J.: Approximation and fixed parameter subquadratic algorithms for radius and diameter in sparse graphs. In: SODA, pp. 377–391, SIAM, (2016)
2. Alon, N., Yuster, R., Zwick, U.: Finding and counting given length cycles. *Algorithmica* **17**, 209–223 (1997)
3. Backurs, A., Roditty, L., Segal, G., Vassilevska Williams, V., Wein, N.: Towards tight approximation bounds for graph diameter and eccentricities. In: STOC, pp. 267–280 (2018)
4. Bandelt, H.-J., Chepoi, V.: 1-Hyperbolic graphs. *SIAM J. Discr. Math.* **16**, 323–334 (2003)
5. Boltyanskii, V. G., Soltan, P. S.: *Combinatorial Geometry of Various Classes of Convex Sets [in Russian]*, Știința, Kishinev (1978)
6. Borassi, M., Crescenzi, P., Habib, M.: Into the square: on the complexity of some quadratic-time solvable problems. *Electron. Notes TCS* **322**, 51–67 (2016)
7. Brandstädt, A., Chepoi, V., Dragan, F.F.: The algorithmic use of hypertree structure and maximum neighbourhood orderings. *DAM* **82**, 43–77 (1998)
8. Bringmann, K., Husfeldt, T., Magnusson, M.: Multivariate analysis of orthogonal range searching and graph distances parameterized by treewidth. In: IPEC, (2018)
9. Cabello, S.: Subquadratic algorithms for the diameter and the sum of pairwise distances in planar graphs. *ACM TALG* **15**(2), 21 (2018)
10. Corneil, D., Dragan, F.F., Habib, M., Paul, C.: Diameter determination on restricted graph families. *DAM* **113**, 143–166 (2001)
11. Corneil, D.G., Dragan, F.F., Köhler, E.: On the power of BFS to determine a graph's diameter. *Networks* **42**, 209–222 (2003)

12. Coudert, D., Ducoffe, G., Popa, A.: Fully polynomial FPT algorithms for some classes of bounded clique-width graphs. *ACM TALG*, *15*(3), (2019)
13. Chechik, S., Larkin, D.H., Roditty, L., Schoenebeck, G., Tarjan, R.E., Vassilevska Williams, V.: Better Approximation Algorithms for the Graph Diameter. In: *SODA*, pp. 1041–1052 (2014)
14. Chepoi, V.: Some d -convexity properties in triangulated graphs. In: *Mathematical Research*, vol. 87, *Știința, Chișinău*, (1986), pp. 164–177 (Russian)
15. Chepoi, V.: Centers of triangulated graphs. *Math. Notes* **43**, 143–151 (1988)
16. Chepoi, V., Dragan, F.F.: A linear-time algorithm for finding a central vertex of a chordal graph. In: *Proceedings of the Second European Symposium on Algorithms—ESA’94*, pp. 159–170
17. Chepoi, V., Dragan, F. F.: Disjoint sets problem, (1992)
18. Chepoi, V., Dragan, F.F.: Finding a central vertex in an HHD-free graph. *DAM* **131**(1), 93–111 (2003)
19. Chepoi, V.D., Dragan, F.F., Estellon, B., Habib, M., Vaxès, Y.: Diameters, centers, and approximating trees of δ -hyperbolic geodesic spaces and graphs. In: *Proceedings of the 24th Annual ACM Symposium on Computational Geometry (SoCG 2008)*, June 9–11, (2008), College Park, Maryland, USA, pp. 59–68
20. Chepoi, V., Dragan, F.F., Habib, M., Vaxès, Y., Alrasheed, H.: Fast approximation of eccentricities and distances in hyperbolic graphs. *J. Graph Algorithms Appl.* **23**, 393–433 (2019)
21. Dragan, F. F.: Centers of graphs and the Helly property (in russian), PhD thesis, Moldova State University, Chișinău, (1989)
22. Dragan, F.F.: HT-graphs: centers, connected r -domination and Steiner trees. *Comput. Sci. J. Moldova (Kishinev)* **1**, 64–83 (1993)
23. Dragan, F.F.: Dominating cliques in distance-hereditary graphs. In: *SWAT ’94*, pp. 370–381, Springer, (1994)
24. Dragan, F.F.: Almost diameter of a house-hole-free graph in linear time via LexBFS. *Discret. Appl. Math.* **95**, 223–239 (1999)
25. Dragan, F.F.: An eccentricity 2-approximating spanning tree of a chordal graph is computable in linear time. *Inf. Process. Lett.* **154**, 105873 (2020)
26. Dragan, F.F., Ducoffe, G.: α_i -Metric graphs: radius, diameter and all eccentricities. *Lect. Notes Comput. Sci.* **14093**, 276–290 (2023)
27. Dragan, F.F., Ducoffe, G.: α_i -metric graphs: hyperbolicity. In: *Preparation*, 2022–2023
28. Dragan, F.F., Ducoffe, G., Guarnera, H.M.: Fast deterministic algorithms for computing all eccentricities in (Hyperbolic) Helly Graphs. In: *WADS*, pp. 300–314 (2021)
29. Dragan, F.F., Guarnera, H.M.: Eccentricity function in distance-hereditary graphs. *Theor. Comput. Sci.* **833**, 26–40 (2020)
30. Dragan, F.F., Guarnera, H.M.: Eccentricity terrain of δ -hyperbolic graphs. *J. Comput. Syst. Sci.* **112**, 50–65 (2020)
31. Dragan, F.F., Habib, M., Viennot, L.: Revisiting radius, diameter, and all eccentricity computation in graphs through certificates, *CoRR*, [arXiv:1803.04660](https://arxiv.org/abs/1803.04660), (2018)
32. Dragan, F.F., Köhler, E., Alrasheed, H.: Eccentricity approximating trees. *Discrete Appl. Math.* **232**, 142–156 (2017)
33. Dragan, F.F., Nicolai, F.: LexBFS-orderings of distance-hereditary graphs with application to the diametral pair problem. *DAM* **98**, 191–207 (2000)
34. Dragan, F.F., Nicolai, F., Brandstädt, A.: LexBFS-orderings and powers of graphs. In: *Proceedings of the WG’96, Lecture Notes in Computer Science*, Vol. 1197, Springer, Berlin, (1997), pp. 166–180
35. Ducoffe, G.: Easy computation of eccentricity approximating trees. *DAM* **260**, 267–271 (2019)
36. Ducoffe, G.: Around the diameter of AT-free graphs. *JGT* **99**(4), 594–614 (2022)
37. Ducoffe, G.: Beyond Helly graphs: the diameter problem on absolute retracts. In: *WG*, pp. 321–335, (2021)
38. Ducoffe, G.: Optimal diameter computation within bounded clique-width graphs. *Algorithmica* **84**, 3192–3222 (2022)
39. Ducoffe, G.: Distance problems within Helly graphs and k -Helly graphs. *Theor. Comput. Sci.* **946**, 113690 (2023)
40. Ducoffe, G., Dragan, F.F.: A story of diameter, radius, and (almost) Helly property. *Networks* **77**, 435–453 (2021)
41. Ducoffe, G., Habib, M., Viennot, L.: Diameter computation on H -minor free graphs and graphs of bounded (distance) VC-dimension, In *SODA*, pp. 1905–1922, SIAM, (2020)
42. Farley, A., Proskurowski, A.: Computation of the center and diameter of outerplanar graphs. *DAM* **2**, 185–191 (1980)

43. Gawrychowski, P., Kaplan, H., Mozes, S., Sharir, M., Weimann, O.: Voronoi diagrams on planar graphs, and computing the diameter in deterministic $O(n^{5/3})$ time. In: SODA, pp. 495–514, SIAM, (2018)
44. Gromov, M.: Hyperbolic Groups, pp. 75–263. Springer, New York, NY (1987)
45. Howorka, E.: A characterization of distance-hereditary graphs. *Quart. J. Math. Oxford Ser.* **28**, 417–420 (1977)
46. Koschützki, D., Lehmann, K., Peeters, L., Richter, S., Tenfelde-Podehl, D., Zlotowski, O.: Centrality indices. In: *Network Analysis*, pp. 16–61. Springer, Berlin (2005)
47. Olariu, S.: A simple linear-time algorithm for computing the center of an interval graph. *Int. J. Comput. Math.* **34**, 121–128 (1990)
48. Prisner, E.: Eccentricity-approximating trees in chordal graphs. *Discrete Math.* **220**, 263–269 (2000)
49. Roditty, L., Vassilevska Williams, V.: Fast approximation algorithms for the diameter and radius of sparse graphs. In: *STOC*, pp. 515–524. ACM, (2013)
50. Soltan, V.P., Chepoi, V.D.: Conditions for invariance of set diameters under d -convexification in a graph. *Cybernetics* **19**, 750–756 (1983). **(Russian, English transl.)**
51. Weimann, O., Yuster, R.: Approximating the diameter of planar graphs in near linear time. *ACM Trans. Algorithms* **12**(1), 12:1–12:13 (2016)
52. Wu, Y., Zhang, C.: Hyperbolicity and chordality of a graph. *Electr. J. Comb.* **18**(1), 43 (2011)
53. Yushmanov, S.V., Chepoi, V.: A general method of investigation of metric graph properties related to the eccentricity. *Math. Probl. Cybernet.* **3**, 217–232 (1991). **(Russian)**

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