Parameterized Complexity of (A, ℓ) -Path Packing

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Abstract

Given a graph $G = (V, E)$, $A \subseteq V$, and integers k and ℓ , the (A, ℓ) -PATH PACKING problem asks to find k vertex-disjoint paths of length exactly ℓ that have endpoints in A and internal points in $V \setminus A$. We study the parameterized complexity of this problem with parameters $|A|$, ℓ , k , treewidth, pathwidth, and their combinations. We present sharp complexity contrasts with respect to these parameters. Among other results, we show that the problem is polynomial-time solvable when $\ell \leq 3$, while it is NP-complete for constant $\ell \geq 4$. We also show that the problem is W[1]-hard parameterized by pathwidth $+ |A|$, while it is fixed-parameter tractable parameterized by treewidth $+ \ell$. Additionally, we study a variant called SHORT A -PATH PACKING that asks to find k vertex-disjoint paths of length *at most* ℓ . We show that all our positive results on the exact-length version can be translated to this version and show the hardness of the cases where $|A|$ or ℓ is a constant.

Keywords A -path packing · Fixed-parameter tractability · Treewidth

1 Introduction

Let $G = (V, E)$ be a graph and $A \subseteq V$. A path P in G with at least one edge is an A -path if the first and last vertices of P belong to A and all other vertices of P belong to $V \setminus A$. Given G and A , A -PATH PACKING is the problem of finding the maximum number of vertex-disjoint A -paths in G . The A -PATH PACKING problem is well studied and even some generalized versions are known to be polynomial-time

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solvable (see e.g., [7,8,17,26,31,32]). Note that A -PATH PACKING is a generalization of MAXIMUM MATCHING since they are equivalent when $A = V$.

In this paper, we study a variant of A -PATH PACKING that also generalizes MAXIMUM MATCHING. An A -path of length ℓ is an (A, ℓ) -path, where the length of a path is the number of edges in the path. Now our problem is defined as follows:

(A, ℓ) -PATH PACKING (ALPP)

Input: A tuple (G, A, k, ℓ) , where $G = (V, E)$ is a graph, $A \subseteq V$, and k and ℓ are positive integers.

Question: Does G contain k vertex-disjoint (A, ℓ) -paths?

To the best of our knowledge, this natural variant of A -PATH PACKING was not studied in the literature. Our main motivation of studying ALPP is to see theoretical differences from the original A -PATH PACKING, but practical motivations of having the length constraint may come from some physical restrictions or some fairness requirements. Note that if $\ell = 1$, then ALPP is equivalent to MAXIMUM MATCHING. Another related problem is ℓ -PATH PARTITION [29,33,34], which asks for vertex-disjoint paths of length ℓ (without specific endpoints).

In the rest of paper, we assume that $k \leq |A|/2$ in every instance as otherwise the instance is a trivial no-instance. The restricted version of the problem where the equality $k = |A|/2$ is forced is also of our interest as that version corresponds to a “full” packing of A -paths. We call this version FULL (A, ℓ) -PATH PACKING (Full-ALPP, for short). In this paper, all our positive results showing tractability of some cases will be on the general ALPP, while all our negative (or hardness) results will be on the possibly easier Full-ALPP.

We also study a natural variant of ALPP, which we call SHORT A -PATH PACKING. The problem is defined as follows, where an $A_{\leq \ell}$ -path is an A -path of length at most ℓ :

SHORT A -PATH PACKING (SAPP)

Input: A tuple (G, A, k, ℓ) , where $G = (V, E)$ is a graph, $A \subseteq V$, and k and ℓ are positive integers.

Question: Does G contain k vertex-disjoint $A_{\leq \ell}$ -paths?

Similarly to Full-ALPP, we define Full-SAPP as the restricted version of SAPP with $k = |A|/2$.

We assume that the reader is familiar with terminologies in the parameterized complexity theory. See the textbook by Cygan et al. [11] for standard definitions.

Our Results

In summary, we show that ALPP is intractable even on very restricted inputs, while it has some nontrivial cases that admit efficient algorithms. (See Fig. 1.)

We call $|A|$, k , and ℓ the *standard parameters* of ALPP as they naturally arise from the definition of the problem. We determine the complexity of ALPP with respect to all standard parameters and their combinations. We first observe that Full-ALPP is NP-complete for any constant $|A| \geq 2$ (Observation 3.1) and for any constant $\ell \geq 4$

Fig. 1 Summary of the results. An arrow $\alpha \rightarrow \beta$ indicates that there is a function f such that $\alpha \geq f(\beta)$ for every instance of ALPP. Some possible arrows are omitted to keep the figure readable. The results on the parameters marked with $*$ are explicitly shown in this paper, and the other results follow by the hierarchy of the parameters. We have a bidirectional arrow $\text{treedepth} \leftrightarrow \text{treedepth} + \ell$ because the maximum length of a path in a graph is bounded by a function of treedepth [30, Section 6.2]

(Observation 3.2), while it is polynomial-time solvable when $\ell \leq 3$ (Theorem 3.3). On the other hand, ALPP is fixed-parameter tractable when parameterized by $k + \ell$ and thus by $|A| + \ell$ as well (Theorem 3.5). We later strengthen Observation 3.2 by showing that the problem is NP-complete for every fixed $\ell \geq 4$ even on grid graphs (Theorem 5.3).

We then study structural parameters such as treewidth and pathwidth in combination with the standard parameters. We first observe that ALPP can be solved in time $n^{O(\text{tw})}$ (Observation 4.1), where n and tw are the number of vertices and the treewidth of the input graph, respectively. Furthermore, we show that ALPP parameterized by $\text{tw} + \ell$ is fixed-parameter tractable (Observation 4.2). We finally show that Full-ALPP parameterized by $\text{pw} + |A|$ is W[1]-hard (Theorem 4.5), where pw is the pathwidth of the input graph.

For SAPP, a simple reduction (Lemma 6.1) will show that all our positive results on ALPP hold also on this short-path version. Although negative results cannot be translated directly, we can show the hardness of the cases where $|A|$ or ℓ is a constant (Theorems 6.6 and 6.7). We leave the complexity of SAPP parameterized by tw unsettled.

2 Preliminaries

A graph $G = (V, E)$ is a *grid graph* if V is a finite subset of $\mathbb{Z}^2$ and $E = \{(r, c), (r', c') \mid |r - r'| + |c - c'| = 1\}$. From the definition, all grid graphs are planar, bipartite, and of maximum degree at most 4. To understand the intractability of a graph problem, it is preferable to show hardness on a very restricted graph class. The class of grid graphs is one of such target classes.

A *tree decomposition* of a graph $G = (V, E)$ is a pair $(\{X_i \mid i \in I\}, T = (I, F))$, where $X_i \subseteq V$ for each i and T is a tree such that

- for each vertex $v \in V$, there is $i \in I$ with $v \in X_i$;
- for each edge $\{u, v\} \in E$, there is $i \in I$ with $u, v \in X_i$;
- for each vertex $v \in V$, the induced subgraph $T[\{i \mid v \in X_i\}]$ is connected.

The *width* of a tree decomposition $(\{X_i \mid i \in I\}, T)$ is $\max_{i \in I} |X_i| - 1$, and the *treewidth* of a graph G , denoted $\text{tw}(G)$, is the minimum width over all tree decompositions of G .

The *pathwidth* of a graph G , denoted $\text{pw}(G)$, is defined by restricting the trees T in tree decompositions to be paths. We call such decompositions *path decompositions*. It is easy to see that pathwidth does not change significantly by subdividing some edges and attaching paths to some vertices. To be self-contained, we present a proof in the appendix (see Section A).

Lemma 2.1 *If G' is a graph obtained from a graph $G = (V, E)$ by subdividing a set of edges $F \subseteq E$ an arbitrary number of times, and attaching a path of arbitrary length to each vertex in a set $U \subseteq V$, then $\text{pw}(G') \leq \text{pw}(G) + 2$.*

3 Standard Parameterizations of ALPP

In this section, we completely determine the complexity of ALPP with respect to the standard parameters $|A|, k, \ell$, and their combinations. (Recall that $k \leq |A|/2$.) We first observe that using one of them as a parameter does not make the problem tractable. That is, the problem remains NP-complete even if one of $|A|, k, \ell$ is a constant. We then show that the problem is tractable when $\ell \leq 3$ or when $k + \ell$ is the parameter.

3.1 Intractable Cases

The first observation is that Full-ALPP is NP-complete on grid graphs even if $|A| = 2$ (and thus $k = 1$). This can be shown by an easy reduction from HAMILTONIAN CYCLE on grid graphs [20]. This observation is easily extended to every fixed even $|A|$ by adding dummy paths of length ℓ as connected components into the graph and their endpoints into A .

Observation 3.1 For every even constant $\alpha \geq 2$, Full-ALPP with $|A| = \alpha$ is NP-complete on grid graphs.

The NP-hardness of Full-ALPP for fixed $\ell \geq 4$ can be shown by an easy reduction from PATH PARTITION [21].

Observation 3.2 For every constant $\ell \geq 4$, Full-ALPP is NP-complete.

We omit the proof as we later strengthen Observation 3.2 to hold on grid graphs (see Sect. 5). Note that the condition $\ell \geq 4$ in the statement above is tight as the problem is polynomial-time solvable when $\ell \leq 3$ (see Theorem 3.3).

3.2 Tractable Cases

Theorem 3.3 *If $\ell \leq 3$, then ALPP can be solved in time $O(\sqrt{n} \cdot m)$, where n and m are the numbers of vertices and edges, respectively.*

Fig. 2 The construction of G' (right) from G (left)

Proof Let (G, A, k, ℓ) with $G = (V, E)$ be an instance of ALPP with $\ell \leq 3$. We reduce, in linear time, the problem to the maximum cardinality matching problem on graphs with $O(|V|)$ vertices and $O(|E|)$ edges. This proves the theorem since the maximum cardinality matching problem can be solved in time $O(\sqrt{n} \cdot m)$ [28].

If $\ell = 1$, then the problem is equivalent to the maximum cardinality matching problem on $G[A]$. Assume that $\ell = 2$. We reduce this case to the case of $\ell = 3$. We can assume that $G[A]$ and $G[V \setminus A]$ do not contain any edges as such edges are not included in any $(A, 2)$ -path. The new instance $(G', A, k, 3)$ is constructed by adding a true twin v' to each vertex $v \in V \setminus A$; i.e., $V(G') = V \cup \{v' \mid v \in V \setminus A\}$ and $E(G') = E \cup \{\{v, v'\} \mid v \in V \setminus A\} \cup \{\{u, v'\} \mid u \in A, v \in V \setminus A, \{u, v\} \in E\}$. Clearly, $(G, A, k, 2)$ is a yes-instance if and only if so is $(G', A, k, 3)$.

For the case of $\ell = 3$, we construct an auxiliary graph $G' = (A \cup V_1 \cup V_2, E_{A,1} \cup E_{1,2} \cup E_{2,2})$ as follows (see Fig. 2):

$$\begin{aligned} V_i &= \{v_i \mid v \in V \setminus A\} \text{ for } i \in \{1, 2\}, \\ E_{A,1} &= \{\{u, v_1\} \mid u \in A, v \in V \setminus A, \{u, v\} \in E\}, \\ E_{1,2} &= \{\{v_1, v_2\} \mid v \in V\}, \\ E_{2,2} &= \{\{u_2, v_2\} \mid u, v \in V \setminus A, \{u, v\} \in E\}. \end{aligned}$$

We show that $(G, A, k, 3)$ is a yes-instance if and only if G' has a matching of cardinality $k + |V \setminus A|$.

To prove the only-if direction, let $P_1, \dots, P_k$ be k vertex-disjoint $(A, 3)$ -path in G . We set $M = M_{A,1} \cup M_{1,2} \cup M_{2,2}$, where

$$\begin{aligned} M_{A,1} &= \{\{u, v_1\} \in E_{A,1} \mid \text{edge } \{u, v\} \text{ appears in some } P_i\}, \\ M_{1,2} &= \{\{v_1, v_2\} \in E_{1,2} \mid \text{vertex } v \text{ does not appear in any } P_i\}, \\ M_{2,2} &= \{\{u_2, v_2\} \in E_{2,2} \mid \text{edge } \{u, v\} \text{ appears in some } P_i\}. \end{aligned}$$

Since the $(A, 3)$ -paths $P_1, \dots, P_k$ are pairwise vertex-disjoint, M is a matching. We can see that $|M| = k + |V \setminus A|$ as $|M_{2,2}| = k$ and $|M_{A,1}| + |M_{1,2}| = |V_1| = |V \setminus A|$.

To prove the if direction, assume that G' has a matching of size $k + |V \setminus A|$. Let M be a maximum matching of G' that includes the maximum number of vertices in $V_1 \cup V_2$ among all maximum matchings of G' . We claim that M actually includes all

vertices in $V_1 \cup V_2$. Suppose to the contrary that v_1 or v_2 is not included in M for some $v \in V \setminus A$. Now, since M is maximum, exactly one of v_1 and v_2 is included in M .

Case 1: $v_1 \in V(M)$ and $v_2 \notin V(M)$. There is a vertex $u \in A$ such that $\{u, v_1\} \in M$. The set $M - \{u, v_1\} + \{v_1, v_2\}$ is a maximum matching that uses more vertices in $V_1 \cup V_2$ than M . This contradicts how M was selected.

Case 2: $v_1 \notin V(M)$ and $v_2 \in V(M)$. There is a vertex $w_2 \in V_2$ such that $\{v_2, w_2\} \in M$. The edge set $M' := M - \{v_2, w_2\} + \{v_1, v_2\}$ is a maximum matching that uses the same number of vertices in $V_1 \cup V_2$ as M . Since M' is maximum and w_2 is not included in M' , the vertex w_1 has to be included in M' , but such a case leads to a contradiction as we saw in Case 1.

Now we construct k vertex-disjoint $(A, 3)$ -paths from M as follows. Let $\{u_2, v_2\} \in M \cap E_{2,2}$. Since M includes all vertices in V_1 , it includes edges $\{u_1, x\}$ and $\{v_1, y\}$ for some $x, y \in A$. This implies that G has an $(A, 3)$ -path (x, u, v, y) . Let (x', u', v', y') be the $(A, 3)$ -path constructed in the same way from a different edge in $M \cap E_{2,2}$. Since M is a matching, these eight vertices are pairwise distinct, and thus (x, u, v, y) and (x', u', v', y') are vertex-disjoint $(A, 3)$ -paths. Since $|M| \geq k + |V \setminus A|$ and each edge in $E_{A,1} \cup E_{1,2}$ uses one vertex of V_1 , M includes at least k edges in $E_{2,2}$. By constructing an $(A, 3)$ -path for each edge in $M \cap E_{2,2}$, we obtain a desired set of k vertex-disjoint $(A, 3)$ -paths. $\square$

In their celebrated paper on *Color-Coding* [1], Alon, Yuster, and Zwick showed the following result.

Proposition 3.4 ([1, Theorem 6.3]) *Let H be a graph on h vertices with treewidth t . Let G be a graph on n vertices. A subgraph of G isomorphic to H , if one exists, can be found in time $O(2^{O(h)} \cdot n^{t+1} \log n)$.*

By using Proposition 3.4 as a black box, we can show that ALPP parameterized by $k + \ell$ is fixed-parameter tractable.

Theorem 3.5 *ALPP on n -vertex graphs can be solved in $O(2^{O(k\ell)} n^6 \log n)$ time.*

Proof Let (G, A, k, ℓ) be an instance of ALPP. Observe that the problem ALPP can be seen as a variant of the SUBGRAPH ISOMORPHISM problem as we search for $H = kP_{\ell+1}$ in G as a subgraph with the restriction that each endpoint of $P_{\ell+1}$ in H has to be mapped to a vertex in A , where $P_{\ell+1}$ denotes an $(\ell + 1)$ -vertex path (which has length ℓ) and $kP_{\ell+1}$ denotes the disjoint union of k copies of $P_{\ell+1}$. We reduce this problem to the standard SUBGRAPH ISOMORPHISM problem [18].

Let G' and H' be the graphs obtained from G and H , respectively, by subdividing each edge once. The graphs G' and $H' = kP_{2\ell+1}$ are bipartite. We then construct G'' from G' by attaching a triangle to each vertex in A ; that is, for each vertex $u \in A$ we add two new vertices v, w and edges $\{u, v\}$, $\{v, w\}$, and $\{w, u\}$. Similarly, we construct H'' from H' by attaching a triangle to each endpoint of each $P_{2\ell+1}$. Note that $|V(G'')| \in O(n^2)$, $|V(H'')| = k(2\ell + 1)$, and $\text{tw}(H'') = 2$. Thus, by Proposition 3.4, it suffices to show that (G, A, k, ℓ) is a yes-instance of ALPP if and only if G'' has a subgraph isomorphic to H'' .

To show the only-if direction, assume that G has k vertex-disjoint (A, ℓ) -paths $P_1, \dots, P_k$. In G'' , for each P_i , there is a unique path Q_i of length 2ℓ plus triangles

attached to the endpoints; that is, Q_i consists of the vertices of P_i , the new vertices and edges introduced by subdividing the edges in P_i , and the triangles attached to the endpoints of the subdivided path. Furthermore, since the paths P_i are pairwise vertex-disjoint, the subgraphs Q_i of G'' are pairwise vertex-disjoint. Thus, G'' has a subgraph isomorphic to $H'' = \bigcup_{1 \leq i \leq k} Q_i$.

To prove the if direction, assume that G'' has a subgraph X isomorphic to H'' . Let $R_1, \dots, R_k$ be the connected components of X . Each R_i is isomorphic to a path of length 2ℓ with a triangle attached to each endpoint. Let $u, v \in V(R_i)$ be the degree-3 vertices of R_i . Since G'' is obtained from the triangle-free graph G' by attaching triangles at the vertices in A , we have $u, v \in A$. Since the u - v path of length 2ℓ in R_i is obtained from a u - v path of length ℓ in G by subdividing each edge once, the graph $G[V(R_i) \cap V(G)]$ contains an (A, ℓ) -path. Since $V(R_1), \dots, V(R_k)$ are pairwise disjoint, G contains k vertex-disjoint (A, ℓ) -paths. $\square$

4 Structural Parameterizations

In this section, we study structural parameterizations of ALPP. First we present XP and FPT algorithms parameterized by tw and $\text{tw} + \ell$, respectively. As the techniques are rather standard in the field, we only sketch the high-level ideas of the algorithms.

The XP-time algorithm parameterized by tw is a standard dynamic-programming over a tree decomposition. We first compute a nice tree decomposition of width $O(\text{tw})$ using known algorithms [4,6,23] and then solve the problem in a bottom-up manner. The DP table for each bag stores the existence of a partial solution for all possible ways of the intersection between the bag and the partial solution. Since a bag intersects at most $O(\text{tw})$ vertex-disjoint (A, ℓ) -paths, it is straightforward to see that the table size can be bounded by $(|A| \cdot \ell)^{O(\text{tw})}$ and that we can compute all entries of the tables for all bags in time polynomial in the total table size. As $|A|$ and ℓ are at most n , the total running time is $n^{O(\text{tw})}$.

Observation 4.1 ALPP can be solved in time $n^{O(\text{tw})}$, where n is the number of vertices.

The FPT algorithm parameterized $\text{tw} + \ell$ is achieved by expressing the problem in the *monadic second-order logic* (MSO_2) on graphs. It is known that given an n -vertex graph of treewidth w and assignments to some free variables of an MSO_2 formula φ , one can find in time $O(f(|\varphi| + w) \cdot n)$, where f is some computable function, an assignment to the rest of free variables that satisfies φ and maximizes a given linear function in the sizes of the free variables of φ [2,4,10]. It is straightforward to express in MSO_2 a formula $\varphi_\ell(X, Y)$ with two free variables X and Y such that

- the length of $\varphi_\ell(X, Y)$ depends only on ℓ , and
- when we assign a vertex set A to X and an edge set F to Y , G models $\varphi_\ell(X, Y)$ if and only if G has vertex-disjoint (A, ℓ) -paths formed by F .

We assign the terminal vertex set A in the input of ALPP to the variable X , and maximize the size of the edge set F assigned to Y . As mentioned above, this can be done in time $O(f(|\varphi_\ell| + w) \cdot n)$ for n -vertex graphs of treewidth at most w , where f is some computable function.

Observation 4.2 ALPP parameterized by $\text{tw} + \ell$ is fixed-parameter tractable.

Now we show that Full-ALPP is $W[1]$ -hard parameterized by pathwidth (and hence also by treewidth), even if we also consider $|A|$ as an additional parameter. We present a reduction from a $W[1]$ -complete problem k -MULTI-COLORED CLIQUE (k -MCC) [15], which goes through an intermediate version of our problem. Specifically, we will consider a version of Full-ALPP with the following modifications: the graph has (positive integer) edge weights, and the length of a path is the sum of the weights of its edges; the set A is given to us partitioned into pairs indicating the endpoints of the sought A -paths; for each such pair the value of ℓ may be different.

More formally, Extended-ALPP is the following problem: we are given a graph $G = (V, E)$, a weight function $w: E \rightarrow \mathbb{Z}^+$, and r triples $(s_1, t_1, \ell_1), \dots, (s_r, t_r, \ell_r) \in V \times V \times \mathbb{Z}^+$, where all $s_i, t_i \in V$ are distinct. We are asked if there exists a set of r vertex-disjoint paths in G such that for each $i \in [r]$ ¹, the i th path in this set has s_i and t_i as its endpoints and the sum of the weights of its edges is ℓ_i . We first show that establishing that this variation of the problem is hard implies also the hardness of Full-ALPP.

Lemma 4.3 *There exists an algorithm which, given an instance of Extended-ALPP on an n -vertex graph G with r triples and maximum edge weight W , constructs in time polynomial in $n + W$ an equivalent instance $(G', A, |A|/2, \ell)$ of Full-ALPP with the properties: (i) $|A| = 2r$, (ii) $\text{pw}(G') \leq \text{pw}(G) + 2$.*

Proof First, we simplify the given instance of Extended-ALPP by removing edge weights: for every edge $e = \{u, v\} \in E(G)$ with $w(e) > 1$, we remove this edge and replace it with a path from u to v with length $w(e)$ going through new vertices (in other words, e has been subdivided $w(e) - 1$ times). It is not hard to see that we have an equivalent instance of Extended-ALPP on the new graph, which we call G_1 , where the weight of all edges is 1 and $|V(G_1)| \leq n^2 \cdot W$. We now give a polynomial-time reduction from this new instance of Extended-ALPP to Full-ALPP.

Let $p = |V(G_1)|$ and $\ell = 2p^2$. For each $i \in [r]$ we do the following: we construct a new vertex s'_i and connect it to s_i using a path of length $p^2 + ip$ going through new vertices; we construct a new vertex t'_i and connect it to t_i using a path of length $p^2 - ip - \ell_i$ through new vertices. (Note that $p^2 - ip - \ell_i > 0$ since $p \geq n \geq 2$, $i \leq n/2$, and $\ell_i < n$.) We set A to contain all the vertices s'_i, t'_i for $i \in [r]$. This completes the construction and it is clear that $|A| = 2r$, the new graph G' has order at most $2p^3 \leq 2n^6 \cdot W^3$ and can be constructed in time polynomial in $n + W$.

We claim that the new graph G' has $|A|/2$ vertex-disjoint (A, ℓ) -paths if and only if the Extended-ALPP instance of G_1 has a positive answer. Indeed, if there exists a collection of r vertex-disjoint paths in G_1 such that the i -th path has endpoints s_i, t_i and length ℓ_i , we add to this path the paths from s'_i to s_i and from t_i to t'_i and this gives a path of length $\ell = 2p^2$ with endpoints in A . Observe that all these paths are vertex-disjoint, so we obtain a yes-certificate of Full-ALPP. For the converse direction, suppose that G' has a set $\mathcal{A}$ of $|A|/2$ vertex-disjoint (A, ℓ) -paths. The set $\mathcal{A}$ does not contain a path with endpoints s'_i and s'_j since such a path has length at least

¹ For a positive integer n , we denote the set $\{1, 2, \dots, n\}$ by $[n]$.

Fig. 3 An example of the vertex-selection gadget for $n = 9$, $k = 4$, and $i = 2$

$2p^2 + (i + j)p + 1 > \ell$. Furthermore, a path in $\mathcal{A}$ cannot connect some s'_i and t'_j with $i > j$ since the length of such a path is at least $2p^2 + (i - j)p - \ell_j + 1 > \ell$. Since $A = \{s'_i \mid i \in [r]\} \cup \{t'_i \mid i \in [r]\}$, we can conclude that each path P in $\mathcal{A}$ connects s'_i and t'_i for some i , and the subpath of P connecting s_i and t_i has length $2p^2 - (p^2 + ip) - (p^2 - ip - \ell_i) = \ell_i$. We therefore obtain a solution to the Extended-ALPP instance.

Finally, observe that the only modifications we have done on G is to subdivide some edges and to attach paths to some vertices. By Lemma 2.1, the pathwidth is increased only by at most 2. $\square$

We can now reduce the k -MCC problem to Extended-ALPP.

Lemma 4.4 *There exists a polynomial-time algorithm which, given an instance of k -MCC on a graph G with n vertices, produces an equivalent instance of Extended-ALPP on a graph G' , with $r \in O(k^2)$ triples, $\text{pw}(G') \in O(k^2)$, and maximum edge weight $W \in n^{O(1)}$.*

Proof We are given a graph $G = (V, E)$ with V partitioned into k sets $V_1, \dots, V_k$, and are asked for a clique of size k that contains one vertex from each set. To ease notation, we will assume that n is odd and $|V_i| = n$ for $i \in [k]$ (so the graph has kn vertices in total) and that the vertices of V_i are numbered $1, \dots, n$. We define two lengths $L_1 = (k + 1)(n - 1)$ and $L_2 = 60n^6$.

For $i \in [k]$ we construct a vertex-selection gadget as follows (see Fig. 3): we make n paths of length k , call them $P_{i,j}$, where $j \in [n]$. Let $a_{i,j}$ and $b_{i,j}$ be the first and last vertices of path $P_{i,j}$, respectively. We label the remaining vertices of the path $P_{i,j}$ as $x_{i,j,i'}$ for $i' \in \{1, \dots, k\} \setminus \{i\}$ in some arbitrary order. Then for each $j \in [n - 1]$ we connect $a_{i,j}$ to $a_{i,j+1}$ and $b_{i,j}$ to $b_{i,j+1}$. All edges constructed so far have weight 1. We set $s_i = a_{i,1}$ and $t_i = a_{i,n}$. We add to the instance the triple (s_i, t_i, L_1) .

We now need to construct an edge-verification gadget as follows (see Fig. 4): for each $i_1, i_2 \in [k]$ with $i_1 < i_2$, we construct three vertices s_{i_1,i_2} , t_{i_1,i_2} , p_{i_1,i_2} . For each edge e of G between V_{i_1} and V_{i_2} , we do the following: suppose e connects vertex j_1 of V_{i_1} to vertex j_2 of V_{i_2} . We add the following four edges:

Fig. 4 An example of the edge-verification gadget for V_{i_1} and V_{i_2} ($i_1 < i_2$). In this example, there are exactly three edges between V_{i_1} and V_{i_2}

1. An edge from s_{i_1,i_2} to x_{i_1,j_1,i_2} . This edge has weight $L_2/4 + j_1 n^4 + j_2 n^2$.
2. An edge from x_{i_1,j_1,i_2} to p_{i_1,i_2} . This edge has weight $L_2/4$.
3. An edge from p_{i_1,i_2} to x_{i_2,j_2,i_1} . This edge has weight $L_2/4$.
4. An edge from x_{i_2,j_2,i_1} to t_{i_1,i_2} . This edge has weight $L_2/4 - j_1 n^4 - j_2 n^2$.

We call the edges constructed in the above step heavy edges, since their weight is close to $L_2/4$. We add the $k(k-1)/2$ triples $(s_{i_1,i_2}, t_{i_1,i_2}, L_2)$ to the instance, for all $i_1, i_2 \in [k]$, with $i_1 < i_2$.

Note that in the above description we have created some parallel edges, for example from s_{i_1,i_2} to $x_{i_1,2j_1,i_2}$ (if the vertex j_1 of V_{i_1} has several neighbors in V_{i_2}). This can be avoided by subdividing such edges once and assigning weights to the new edges so that the total weight stays the same. For simplicity we ignore this detail in the remainder since it does not significantly affect the pathwidth of the graph (see Lemma 2.1). This completes the construction.

Let us now prove correctness. First assume that we have a k -multicolored clique in G , encoded by a function $\sigma: [k] \rightarrow [n]$, that is, $\sigma(i)$ is the vertex of the clique that belongs in V_i . For the i -th vertex-selection gadget we have the triple (s_i, t_i, L_1) . We construct a path from s_i to t_i by traversing the paths $P_{i,j}$ for $j \in [n] \setminus \{\sigma(i)\}$ in the increasing order of j and by appropriately using the “horizontal” edges connecting adjacent paths. See Fig. 5. The path has length L_1 : we have traversed $n-1$ paths $P_{i,j}$ with $j \neq \sigma(i)$, each of which has k edges; we have also traversed $n-1$ horizontal edges connecting adjacent paths. The total length is therefore, $(n-1)k + n-1 = L_1$. In this way we have satisfied all the k triples (s_i, t_i, L_1) and have not used the vertices $x_{i,\sigma(i),i'}$ for any $i' \neq i$.

Consider now a triple $(s_{i_1,i_2}, t_{i_1,i_2}, L_2)$, for $i_1 < i_2$. Because we have selected a clique, there exists an edge between vertex $\sigma(i_1)$ of V_{i_1} and $\sigma(i_2)$ of V_{i_2} . For this edge we have constructed four edges in our new instance, linking s_{i_1,i_2} to t_{i_1,i_2} with a total weight of L_2 . We use these paths to satisfy the $\binom{k}{2}$ triples $(s_{i_1,i_2}, t_{i_1,i_2}, L_2)$. These paths are disjoint from each other: when $i_1 < i_2$, $x_{i_1,\sigma(i_1),i_2}$ is only used in the path from s_{i_1,i_2} to t_{i_1,i_2} and when $i_1 > i_2$, $x_{i_1,\sigma(i_1),i_2}$ is only used in the path from s_{i_2,i_1} to t_{i_2,i_1} . Furthermore, these paths are disjoint from the paths in the vertex-selection gadgets,

Fig. 5 Construction of paths from σ

as we observed that $x_{i,\sigma(i),i'}$ are not used by the path connecting s_i to t_i . We thus have a valid solution. See Fig. 5.

For the converse direction, suppose we have a valid solution for the Extended-ALPP instance. First, consider the path connecting s_i to t_i . This path has length L_1 , therefore it cannot be using any heavy edges, since these edges have cost at least $L_2/4 - n^5 - n^3 > L_1$. Inside the vertex-selection gadget, the path may use either all of the edges of a path $P_{i,j}$ or none. Let us now see how many $P_{i,j}$ are unused. First, a simple parity argument shows that the number of paths traversed in the $a_{i,j} \rightarrow b_{i,j}$ direction is equal to those traversed in the opposite direction, so the total number of used paths is even. Since we have an odd number of paths in total (as n is odd), at least one path is not used. We conclude that exactly one $P_{i,j}$ is not used, otherwise the path from s_i to t_i would be too short. Let $\sigma(i)$ be defined as the index j such that the internal vertices of $P_{i,j}$ are not used in the $s_i \rightarrow t_i$ path of the solution. We define a clique in G by selecting for each i the vertex $\sigma(i)$.

Let us argue why this set induces a clique. Let j_1, j_2 be the vertices selected in V_{i_1}, V_{i_2} respectively, with $i_1 < i_2$, and consider the triple $(s_{i_1,i_2}, t_{i_1,i_2}, L_2)$. This triple must be satisfied by a path that uses exactly four heavy edges, since each heavy edge has weight at least $L_2/5 + n^6$ and at most $L_2/3 - 3n^6$, and all other edges together have weight smaller than n^3 . Hence, every such path is using at least two internal vertices of some $P_{i,j}$ because every heavy edge is incident on such a vertex. By our previous reasoning, the paths that satisfy the (s_i, t_i, L_1) triples have used all such vertices except for one path $P_{i,j}$ for each i . There exist therefore exactly $k(k-1)$ such vertices available, so each of the $k(k-1)/2$ triples $(s_{i_1,i_2}, t_{i_1,i_2}, L_2)$ has a path using exactly two of these vertices. Hence, each such path consists of four heavy edges and no other edges.

Such a path must therefore be using one edge incident on s_{i_1,i_2} , one edge incident on t_{i_1,i_2} and two edges incident on p_{i_1,i_2} . The used edge incident on s_{i_1,i_2} must have as other endpoint $x_{i_1,2j_1,i_2}$, which implies that its weight is $L_2/4 + j_1n^4 + j_2'n^2$, for some j_2' . Similarly, the edge incident on t_{i_1,i_2} must have weight $L_2/4 - j_1'n^4 - j_2n^2$, as its other endpoint is necessarily $x_{i_2,2j_2,i_1}$. We conclude that the only way that the length of this path is L_2 is if $j_1 = j_1'$ and $j_2 = j_2'$. Therefore, we have an edge between the two selected vertices, and as a result a k -clique.

To conclude we observe that deleting the $3 \cdot \binom{k}{2}$ vertices $s_{i_1, i_2}, p_{i_1, i_2}, t_{i_1, i_2}$ disconnects the graph into components that correspond to the vertex gadgets with some paths attached. By Lemma 2.1, each such component has pathwidth at most 4 as it can be seen as a graph obtained from a subdivision of the $2 \times n$ grid by attaching paths to some vertices. As a result the whole graph has pathwidth $3 \cdot \binom{k}{2} + 4$. $\square$

Theorem 4.5 *Full-ALPP is $W[1]$ -hard parameterized by $pw + |A|$.*

Proof We compose the reductions of Lemmas 4.3 and 4.4. Starting with an instance of k -MCC with n vertices this gives an instance of Full-ALPP with $n^{O(1)}$ vertices, $|A| = O(k^2)$, and pathwidth $O(k^2)$. $\square$

5 Hardness on Grid Graphs

In this section, we show that for every constant $\ell \geq 4$, Full-ALPP is NP-complete on grid graphs. We first reduce PLANAR CIRCUIT SAT to Full-ALPP on planar bipartite graphs of maximum degree at most 4. We then modify the instance by subdividing edges and adding terminal vertices in an appropriate way, and have an equivalent instance on grid graphs.

The input of CIRCUIT SAT is a Boolean circuit with a number of inputs and one output. The question is whether the circuit can output true by appropriately setting its inputs. CIRCUIT SAT is NP-complete since CNF SAT [9] can be seen as a special case. When the underlying graph of the circuit is planar, the problem is called PLANAR CIRCUIT SAT. Using planar crossover gadgets [27], we can show that PLANAR CIRCUIT SAT is NP-complete. Furthermore, since NOR gates can replace other gates such as AND, OR, NOT, NAND, and XOR without introducing any new crossing, we can conclude that PLANAR CIRCUIT SAT having NOR gates only is NP-complete.

Let $I = (G, A, \ell)$ be an instance of Full-ALPP with $G = (V, E)$. Let ψ be a mapping that assigns each $e \in E$ an (A, ℓ) -path in G , and $\psi(E) = \{\psi(e) \mid e \in E\}$. We say that ψ is a *guide* to I if every set of $|A|/2$ vertex-disjoint (A, ℓ) -paths, if any exists, is a subset of $\psi(E)$. When a guide is given additionally to an instance of Full-ALPP, we call the problem Guided Full-ALPP. Observe that a guide to an instance is not a restriction but just additional information.

Lemma 5.1 *For every fixed $\ell \geq 4$, Guided Full-ALPP is NP-complete on planar bipartite graphs of maximum degree at most 4.*

Proof Given a planar circuit with only NOR gates, we construct an equivalent instance of Guided Full-ALPP with the fixed ℓ . We only need input gadget, output gadget, split gadget, NOR gadget, and a way to connect the gadgets. See Fig. 6 for the high-level idea of the reduction.

The general ideas of our gadgets are as follows. The key ingredient is a cycle of length $2p\ell$ for some p in which every ℓ th vertex belongs to A . We call such a cycle a *basic cycle*. There are exactly two ways to pack p vertex-disjoint (A, ℓ) -paths into a basic cycle of length $2p\ell$, and each of the packings corresponds to the true or false

Fig. 6 A planar circuit and the corresponding ALPP instance (simplified). The vertices in $V \setminus A$ are omitted. The connection pairs are marked with dashed rectangles

Fig. 7 (A, ℓ) -paths at a connection pair. We draw an (A, ℓ) -path as a gray bar

Fig. 8 The input gadgets. The black vertices belong to A and the white vertices belong to $V \setminus A$

signal passing through the corresponding part of the circuit. Our gadgets mostly consist of basic cycles and they intersect in such a way that an (A, ℓ) -path using an edge in a basic cycle cannot use any edge in another basic cycle. This can be guaranteed if two basic cycles intersect at a vertex whose distances from A in the two basic cycles are of different parities. In this way, the choice for one basic cycle propagates to other basic cycles. The NOR gadgets are little more complicated as they involve self-intersecting cycles, but the general idea is the same.

We explicitly present the gadgets for the cases $\ell = 4$ and $\ell = 5$. For even (resp. odd) $\ell > 5$, the gadgets can be obtained from the one for $\ell = 4$ (resp. $\ell = 5$) by subdividing $\lfloor \ell/2 \rfloor - 2$ times each edge incident to a vertex in A .

Connections between gadgets We first explain how the gadgets are connected. Each gadget has one or three pairs of vertices that are shared with other gadgets. We call them *connection pairs*. All those vertices belong to the terminal set A . In the figures, we draw each connection pair so that the two vertices are next to each other vertically and mark them with a dashed rectangle. If the (A, ℓ) -paths using the vertices of a connection pair are going to the positive direction, then we interpret it as that a true signal is sent via the connection pair. If the paths are going to the negative direction, then the connection pair is carrying a false signal. (See Fig. 7.) Note that our reduction below forces the paths at each connection pair to proceed in the same direction.

Fig. 9 The output gadgets

Fig. 10 Splitting a wire in a circuit

Fig. 11 The split gadgets

Input gadgets The input gadget is simply a path of length 3ℓ , where the endpoints form its unique connection pair. See Fig. 8. For a full (A, ℓ) -path packing, we only have two options. One corresponds to true input (Fig. 8c) and the other to false input (Fig. 8d). **Output gadgets** The output gadget consists of two paths of length ℓ , where its unique connection pair includes one endpoint from each path. See Fig. 9. To have a full packing, the input to this gadget has to be true.

Split gadgets To simulate the split of a wire depicted in Fig. 10, the split gadget consists of three paths of length 3ℓ , each of which is identical to the input gadget, and a cycle of length 10ℓ that synchronizes the three paths. See Fig. 11. To have a full (A, ℓ) -path packing, there are only two ways to pack (A, ℓ) -paths into a split gadget. Figure 12 shows the two ways: one on the left corresponds to a split of a true signal, and the other a split of a false signal.

NOR gadgets Recall that NOR stands for “NOT OR” and that the output y of a NOR gate is true if and only if both inputs x_1 and x_2 are false. The NOR gadgets are given in Fig. 13. The structure of the gadget is rather involved. It has three connection pairs, two for the inputs and one for the output, and the endpoints of each pair are connected by a path of length 5ℓ . Additionally, there is a long self-intersecting cycle that somehow

Fig. 12 The possible (A, ℓ) -path packings of the split gadget

Fig. 13 The NOR gadgets

entangles the inputs and the output. There are only four ways to fully pack (A, ℓ) -paths into a NOR gadget, and each packing corresponds to a correct behavior of a NOR gate (see Fig. 14). To see the correctness of Fig. 14, it is important to observe that in the NOR gadgets for even ℓ , there are some (A, ℓ) -paths that are never used in a full (A, ℓ) -path packing. For example, the (A, ℓ) -path with endpoints v_1 and v_2 in Fig. 13a is such a path. In a full (A, ℓ) -path packing, w_2 has to be an endpoint of an (A, ℓ) -path either with w_1 or w_3 . Hence, if we use the (A, ℓ) -path with endpoints v_1 and v_2 , then one of u_1 and u_2 cannot belong to any (A, ℓ) -path in the packing.

Guides The guide $\psi(e)$ for each $e \in E$ can be easily set from Figs. 8c, d, 9c, 12a, b, 14a–d. For each edge e , the unique gray bar that includes the edge represents the (A, ℓ) -path $\psi(e)$.

Correctness The correctness of each gadget implies the correctness of the whole reduction. Thus, it suffices to show that the output of the reduction is planar bipartite graph of maximum degree at most 4. The resultant graph clearly has maximum degree 4 and is planar. To see that the graph is bipartite, consider a 2-coloring of a gadget, which is not the output gadget. If ℓ is even, then all vertices in the connection pairs have the same color. If ℓ is odd, then each upper vertex of a connection pair in the figures has the same color, and the other vertices in the connection pairs have the other color. Therefore, the entire graph is 2-colorable. $\square$

Fig. 14 The possible (A, ℓ) -path packings of the NOR gadget

Fig. 15 Subdividing e and introducing two new terminals ($\ell = 4$)

Let (G, A, ℓ, ψ) be an instance of Guided Full-ALPP with $G = (V, E)$. For $e = \{v, w\} \in E$, we denote by d_e the length of the subpath of $\psi(e)$ starting at v , passing w , and reaching an endpoint of $\psi(e)$. Let G_e be the graph obtained from G by subdividing e , 2ℓ times. Let P_e be the v - w path of length $2\ell + 1$ in G_e corresponding to e . We set $A_e = A \cup \{x_0, x_1\}$, where x_0 and x_1 are the vertices that have distance d_e and $d_e + \ell$ from v in P_e , respectively. (See Fig. 15.)

For each edge h of G_e , we set $\psi_e(h) = \psi(h)$ if h is not contained in the path $P_{\psi(e)}$ of length 3ℓ that corresponds to $\psi(e)$. If h is contained in $P_{\psi(e)}$, then we set $\psi_e(h)$ to the unique (A, ℓ) -path in $P_{\psi(e)}$ that contains h . Observe that ψ_e is a guide to (G_e, A_e, ℓ) . Furthermore, (G, A, ℓ, ψ) and (G_e, A_e, ℓ, ψ_e) are equivalent (see Fig. 16): if $\psi(e)$ is used in a full (A, ℓ) -path packing of G , then we use two (A, ℓ) -paths in $P_{\psi(e)}$; otherwise we use the middle (A, ℓ) -path in $P_{\psi(e)}$ connecting two new terminals.

Observation 5.2 (G, A, ℓ, ψ) and (G_e, A_e, ℓ, ψ_e) are equivalent instances of Guided Full-ALPP.

Fig. 16 Equivalence of (G, A, ℓ, ψ) and (G_e, A_e, ℓ, ψ_e)

Fig. 17 Local modification around v . The grid point of v is moved to $(x_v + 1, y_v)$

Now we are ready to prove the main theorem of this section.

Theorem 5.3 *For every constant $\ell \geq 4$, Full-ALPP is NP-complete on grid graphs.*

Proof We reduce Guided Full-ALPP on planar bipartite graphs of maximum degree at most 4 for fixed $\ell \geq 4$ (which is NP-complete by Lemma 5.1) to Full-ALPP on grid graphs for the same ℓ . Let (G, A, ℓ, ψ) be an instance of Guided Full-ALPP, where $G = (V, E)$ is a planar bipartite graph of maximum degree at most 4.

A *rectilinear embedding* of a graph is a planar embedding into the $\mathbb{Z}^2$ grid such that

- each vertex is mapped to a grid point;
- each edge $\{u, v\}$ is mapped to a rectilinear path between u and v consisting of vertical and horizontal segments connecting grid points;
- the rectilinear paths corresponding to two different edges may intersect only at their endpoints.

Every planar graph of maximum degree at most 4 has a rectilinear embedding, and a rectilinear embedding of area at most $(n + 1)^2$ can be computed in linear time [25], where n is the number of vertices.

Let R_1 be a rectilinear embedding of G with area at most $(n + 1)^2$. By multiplying each coordinate in the embedding by 2ℓ , we obtain an enlarged rectilinear embedding R_2 of G . Let U be one color class of a 2-coloring of G . Now, for each $v \in U$, we locally modify R_2 around the grid point (x_v, y_v) corresponding to v as illustrated in Fig. 17. We denote by R_3 the locally modified embedding.

From R_3 , we construct a new graph G' and its rectilinear embedding R' by inserting degree-2 vertices at each intersection point of a grid point and the inner part of a rectilinear path corresponding to an edge. Clearly, G' is a grid graph. Let $e \in E$ and λ_e be the (geometric) length of the rectilinear path in R_1 corresponding to e . Then the rectilinear path in R_3 corresponding to e has length $2\ell \cdot \lambda_e + 1$. Therefore, G' is the graph obtained from G by subdividing each edge e , $2\ell \cdot \lambda_e$ times. By Observation 5.2, we can easily compute A' and ψ' such that (G, A, ℓ, ψ) is equivalent to (G', A', ℓ, ψ') . Finally, from the definition of a guide to an instance of Full-ALPP, (G', A', ℓ, ψ') is equivalent to (G', A', ℓ) . As everything in this reduction can be done in time polynomial, the theorem holds. $\square$

Fig. 18 The construction of G' (right) from G (left) when $\ell = 3$

6 Short A-Paths

As mentioned before, the positive results on ALPP presented so far can be translated to the ones on SAPP by the following lemma.

Lemma 6.1 *Given an instance (G, A, k, ℓ) of SAPP where G has n vertices and m edges, one can compute an equivalent instance (G', A, k, ℓ) of ALPP in $O(mn^2)$ time, where G' has $O(mn^2)$ vertices and edges, and $\text{tw}(G') \leq \text{tw}(G) + 1$.*

Proof We construct $G' = (V', E')$ from $G = (V, E)$ by replacing each edge $\{u, v\} \in E$ with ℓ new u - v paths of lengths $1, 2, \dots, \ell$. (See Fig. 18.) We call these paths (including the one of length 1) the *detours* of $\{u, v\}$. Clearly, $|V'|, |E'| \in O(mn^2)$, and G' can be constructed in time linear in $|V'| + |E'|$.

To bound the treewidth of G' , observe that G' can be seen as a graph obtained from G by attaching triangles to edges, and then by subdividing some edges. Such operations preserve the treewidth unless G is a forest. (If G is a forest, then the treewidth increases by 1.) To see this, let $\{u, v\}$ be the target edge of one of such operations. Let w be the new vertex introduced by the operation. Every tree decomposition $\mathcal{T}$ of the original graph has a bag B including both u and v since $\{u, v\}$ is an edge of the original graph. We add to $\mathcal{T}$ a new bag $B' = \{u, v, w\}$ adjacent only to B . Clearly, the obtained decomposition is a tree decomposition of the graph obtained by the operation, and its width is the maximum of $|B'| - 1$ and the width of $\mathcal{T}$.

Now assume that (G, A, k, ℓ) is a yes instance of SAPP, and let $P_1, \dots, P_k$ be vertex-disjoint $A_{\leq \ell}$ -paths in G . For each $i \in [k]$, let ℓ_i be the length of P_i . We construct an (A, ℓ) -path P'_i in G from P_i by replacing an arbitrary edge $\{u, v\}$ in P_i with its detour of length $\ell - \ell_i + 1$. Since $P_1, \dots, P_k$ are vertex-disjoint and the detours are internally vertex-disjoint, the (A, ℓ) -paths $P'_1, \dots, P'_k$ are vertex-disjoint.

Conversely, assume that (G', A, k, ℓ) is a yes instance of ALPP, and let $P'_1, \dots, P'_k$ be vertex-disjoint (A, ℓ) -paths in G' . Note that each P'_i is a concatenation of some detours. We obtain an $A_{\leq \ell}$ -path P_i in G by replacing all detours in P'_i with the original edges in G . Since $P'_1, \dots, P'_k$ are vertex-disjoint and $V(P_i) \subseteq V(P'_i)$ for each $i \in [k]$, the $A_{\leq \ell}$ -paths $P_1, \dots, P_k$ are vertex-disjoint. $\square$

By Lemma 6.1, Theorems 3.3 and 3.5, Observations 4.1 and 4.2 imply the following positive results on SAPP.

Corollary 6.2 *If $\ell \leq 3$, then SAPP can be solved in polynomial time.*

Corollary 6.3 *SAPP parameterized by $k + \ell$ is fixed-parameter tractable.*

Corollary 6.4 *SAPP can be solved in time $n^{O(tw)}$.*

Corollary 6.5 *SAPP parameterized by $tw + \ell$ is fixed-parameter tractable.*

On the other hand, the negative results on ALPP cannot be directly translated to the one on SAPP. We here prove the hardness of the cases with constant $|A| \geq 4$ or with constant $\ell \geq 4$. Note that SAPP with $k = 1$ (or $|A| = 2$) is polynomial-time solvable because it reduces to the all-pairs shortest path problem. We leave the complexity of SAPP parameterized by tw unsettled.

Theorem 6.6 *For every even constant $\alpha \geq 4$, Full-SAPP with $|A| = \alpha$ is NP-complete.*

Proof Since the problem is clearly in NP, we present a reduction from the following NP-complete problem 2D1SP [13]. Given a graph $G = (V, E)$ and two terminal pairs (s_1, t_1) and (s_2, t_2) , the problem 2D1SP asks whether there exist two vertex-disjoint paths P_1 from s_1 to t_1 and P_2 from s_2 to t_2 , where P_1 is asked to be a shortest s_1 – t_1 path in G . We reduce 2D1SP to Full-SAPP with $|A| = 4$. (We can extend this to any even $\alpha = |A|$ by adding dummy components.)

Let $n = |V|$ and $\ell = 5n - 1$. Let ℓ_1 be the shortest path distance between s_1 and t_1 in G . We add four vertices s'_1, t'_1, s'_2 , and t'_2 to G . We add a path of length $3n$ between s_1 and s'_1 , a path of length $2n - \ell_1 - 1$ between t_1 and t'_1 , a path of length $2n$ between s_2 and s'_2 , and a path of length $2n$ between t_2 and t'_2 . We call the obtained graph G' and set $A = \{s'_1, t'_1, s'_2, t'_2\}$.

Assume that G has a shortest s_1 – t_1 path P_1 and a (not necessarily shortest) s_2 – t_2 path P_2 vertex-disjoint from P_1 . For each $i \in \{1, 2\}$, we extend P_i to a path P'_i between s'_i and t'_i by adding the unique paths between s'_i to s_i and t_i to t'_i . The length of P'_1 is $3n + \ell_1 + (2n - \ell_1 - 1) = \ell$ and the length of P'_2 is $2n + \|P_2\| + 2n \leq \ell$, where $\|P_2\|$ is the length of P_2 .

Conversely, assume that G' has two vertex-disjoint $A_{\leq \ell}$ -paths P'_1 and P'_2 . Without loss of generality, we can assume that one of the endpoints of P'_1 is s'_1 . Then we can see that the other endpoint of P'_1 is t'_1 since the distance between s'_1 and the vertices s'_2 and t'_2 is at least $3n + 2n > \ell$. Let P_1 be the subpath of P'_1 that connects s_1 and t_1 . Now the length of the $A_{\leq \ell}$ -path P'_1 is $3n + \|P_1\| + (2n - \ell_1 - 1) \leq \ell$, and thus $\|P_1\| \leq \ell_1$. Hence P_1 is a shortest path between s_1 and t_1 in G . Let P_2 be the subpath of P'_2 that connects s_2 and t_2 . Since P'_1 and P'_2 are vertex-disjoint, so are P_1 and P_2 . $\square$

Theorem 6.7 *For every constant $\ell \geq 4$, Full-SAPP is NP-complete.*

Proof We show the NP-hardness of Full-SAPP with constant $\ell \geq 4$ by a reduction from a variant of 3-SAT with the following restrictions: (1) each clause is a disjunction of two or three literals, and (2) each variable occurs exactly twice as a positive literal and exactly once as a negative literal. We call this variant 3-SAT(2, 1). It is known that 3-SAT(2, 1) is NP-complete [16].

Let $(U, \mathcal{C})$ be an instance of 3-SAT(2, 1) with the variables $U = \{u_1, \dots, u_n\}$ and the clauses $\mathcal{C} = \{C_1, \dots, C_m\}$. If the positive literal of u_i appears in C_p and C_q with $p < q$, then we say that the first occurrence of u_i is in C_p and the second is in C_q .

For each $i \in [n]$, we construct the variable gadget for u_i as follows (see Fig. 19 (left)). Take three paths of length ℓ from s_i^1 to t_i^1 , from s_i^2 to t_i^2 , and from $\bar{s}_i$ to $\bar{t}_i$. We

Fig. 19 The variable gadget (left) and the clause gadget (right)

call these paths *vertical*. Add three paths of length $\ell - 1$ from s_i^1 to the neighbor of $\bar{t}_1$, from s_i^2 to the neighbor of t_i^1 , and from $\bar{s}_i$ to the neighbor of t_i^2 . Now these paths together with the edges incident to t_i^1, t_i^2 , and $\bar{t}_i$ form three paths of length ℓ from s_i^1 to $\bar{t}_1$, from s_i^2 to t_i^1 , and from $\bar{s}_i$ to t_i^2 . We call these paths *slanted*. For $j \in \{1, 2\}$, we call the vertex of distance 2 from t_i^j on the corresponding vertical path x_i^j . We call the vertex of distance 2 from $\bar{t}_i$ on the corresponding slanted path $\bar{x}_i$.

For each $C \in \mathcal{C}$, we construct the clause gadget for C as follows (see Fig. 19 (right)). Assume that C includes c literals. Take two vertices s_C and t_C and then add c internally disjoint paths of length ℓ between s_C and t_C . We bijectively map the neighbors of t_C to the literals in C . For each neighbor v of t_C , if v is mapped to the j th positive occurrence of u_i , then we identify v with x_i^j in the variable gadget for u_i . Similarly, if v is mapped to the negative occurrence of u_i , then we identify v with $\bar{x}_i$.

We call the constructed graph G and set $A = \{s_i^1, s_i^2, \bar{s}_i, t_i^1, t_i^2, \bar{t}_i \mid i \in [n]\} \cup \{s_C, t_C \mid C \in \mathcal{C}\}$. This completes the construction. We show that $(U, \mathcal{C})$ is a yes instance of 3-SAT(2, 1) if and only if $(G, A, |A|/2, \ell)$ is a yes instance of Full-SAPP.

To show the only-if direction, assume that there is a truth assignment to the variables in U that satisfies $\mathcal{C}$. For each variable $u_i \in U$, if u_i is set to be true, then we take the slanted paths in the variable gadget for u_i ; otherwise, we take the vertical paths. Since the variable gadgets are vertex-disjoint, the paths taken so far are vertex-disjoint. Then for each clause $C \in \mathcal{C}$, let l be a literal in C that set to be true. Observe that the neighbor, say v_l , of t_C mapped to l is not used in the paths selected in the variable gadgets. Thus we can take the s_C – t_C path passing through v_l . Since all paths selected have length ℓ and all vertices in A are used as endpoints of the selected path, $(G, A, |A|/2, \ell)$ is a yes instance of Full-SAPP.

To prove the if direction, assume that there is a set of $|A|/2$ vertex-disjoint A_ℓ -paths $\mathcal{P}$ in G . First observe that for each $C \in \mathcal{C}$, t_C is the only vertex distance at most ℓ from s_C : for each neighbor v of t_C , the distance from s_C to v is $\ell - 1$, and the distance from v to any vertex in $A \setminus \{s_C, t_C\}$ is at least $\min\{2, \ell - 2\} \geq 2$. Thus, for each $C \in \mathcal{C}$,

there is a path in $\mathcal{P}$ that has s_C and t_C as its endpoints. Next we claim that for each variable $u_i \in U$, either all vertical paths or all slanted paths in the variable gadget for u_i are selected into $\mathcal{P}$. To see this, observe that $\{\bar{t}_i, t_i^1\}$ (resp. $\{t_i^1, t_i^2\}$, $\{t_i^2, \bar{t}_i\}$) is the set of vertices in $A \setminus \{s_C, t_C \mid C \in \mathcal{C}\}$ that are distance at most ℓ from s_i^1 (resp. $s_i^2, \bar{s}_i$). Therefore, if we pick a vertical (resp. slanted) path in a variable gadget, then we have to take all vertical (resp. slanted) paths in that variable gadget. We now construct a truth assignment to U by setting u_i true if and only if $\mathcal{P}$ includes the slanted paths in the variable gadget for u_i . For $C \in \mathcal{C}$, let $P \in \mathcal{P}$ be the path connecting s_C and t_C . Let l be the literal in C corresponding to the neighbor of t_C on P . If l is a positive literal of a variable u_i , then $\mathcal{P}$ includes the slanted paths in the variable gadget for u_i , and thus u_i is set to be true. If l is a negative literal of u_i , then $\mathcal{P}$ includes the vertical paths in the variable gadget for u_i , and thus u_i is set to be false. In both cases, l is true and C is satisfied. $\square$

7 Concluding Remarks

In this paper, we have introduced a new problem (A, ℓ) -PATH PACKING (ALPP) and showed tight complexity results. One possible future direction would be the parameterization by clique-width cw , a generalization of treewidth (see [19]). In particular, we ask the following two questions.

- Does ALPP admit an algorithm of running time $O(n^{cw})$?
- Is ALPP fixed-parameter tractable parameterized by $cw + \ell$?

We also considered a variant of the problem which we call $A_{\leq \ell}$ -PATH PACKING (SAPP). We showed that results similar to the ones on ALPP hold also on SAPP, but we were not able to determine the complexity parameterized by treewidth. We left the following question on SAPP.

- Is SAPP $W[1]$ -hard parameterized by tw , $tw + k$, or $tw + |A|$?

Another interesting direction on ALPP (and on SAPP) would be a systematic study on restricted graph classes. A special case of ALPP have been studied on important graph classes as the 2-fixed-endpoint Hamiltonian path problem (2HP), where we are given two vertices s and t and asked whether there is a Hamiltonian path from s to t . Clearly, 2HP is equivalent to ALPP with $A = \{s, t\}$ and $\ell = n - 1$. The complexity of 2HP on interval graphs was asked in 1993 [12] and is still open (see [24]). It is known that 2HP is polynomial-time solvable on proper interval graphs [3]. It is natural to expect that with its full generality, ALPP would be intractable on those graph classes.

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A Proof of Lemma 2.1

In this section, we prove Lemma 2.1 stating that: if G' is a graph obtained from a graph $G = (V, E)$ by subdividing a set of edges $F \subseteq E$ an arbitrary number of times, and attaching a path of arbitrary length to each vertex in a set $U \subseteq V$, then $\text{pw}(G') \leq \text{pw}(G) + 2$.

To this end, we use a characterization of pathwidth in terms of the following search game. We are given a graph $G = (V, E)$ with all edges *contaminated*. The goal in this game is to clear all edges. In each turn, we can place a *searcher* on a vertex or remove a searcher from a vertex. An edge is *cleared* by having searchers on both endpoints. A cleared edge is immediately *recontaminated* when a removal of a searcher results in a path not passing through any searchers from the edge to a contaminated edge. The minimum number of searchers needed to clear all edges of G is the *node search number*, and we denote it by $\text{ns}(G)$. It is known that $\text{ns}(G) = \text{pw}(G) + 1$ for every graph G [5,14,22].

Lemma A.1 (Folklore) *Let $G = (V, E)$ be a graph. If G' is a graph obtained from G by subdividing a set of edges $F \subseteq E$ an arbitrary number of times, then $\text{pw}(G') \leq \text{pw}(G) + 2$.*

Proof Let $p = \text{pw}(G) + 1$. Since $\text{ns}(G) \leq p$, there is a sequence S of placements and deletions of searchers to clear all edges of G using at most p searchers. To clear all edges of G' , we extend S as follows. For each placement of a searcher on a vertex $v \in V$, we insert, right after this placement, a subsequence that clears all paths corresponding to the edges between v and its neighbors having searchers on them at this point. This can be done with two extra searchers that clear the paths one-by-one. The extra searchers are deleted in the end of the subsequence, and thus we only need two extra searchers in total. This implies that $\text{pw}(G') = \text{ns}(G') - 1 \leq \text{ns}(G) + 1 = \text{pw}(G) + 2$. $\square$

Lemma A.2 (Folklore) *If G' is a graph obtained from a graph $G = (V, E)$ by attaching a path of arbitrary length to each vertex in a set $U \subseteq V$, then $\text{pw}(G') \leq \text{pw}(G) + 1$.*

Proof We assume that $E \neq \emptyset$ since otherwise the statement is clearly true. There is a sequence S of placements and deletions of searchers to clear all edges of G using at most $p = \text{pw}(G) + 1$ searchers. To clear all edges of G' , we extend S as follows. Using two searchers, we first clear the paths attached to the isolated vertices in U (if such exist). We then replace each placement of a searcher on a non-isolated vertex $u \in U$ with a subsequence that clears the path $P = (u, p_1, p_2, \dots, p_q)$ attached to u . This can be done with two searchers by first placing a searcher on p_q , then placing a searcher on p_{q-1} , deleting a searcher on p_q , placing a searcher on p_{q-2} , and so on. At the end of the subsequence, u has a searcher on it and all other vertices in P do not have searchers. We need only one extra searcher in total. Hence, $\text{pw}(G') = \text{ns}(G') - 1 \leq \text{ns}(G) = \text{pw}(G) + 1$. $\square$

In the proofs of Lemmas A.1 and A.2, we show that search sequences for the original graph can be “locally” extended for the new graph by using one or two temporal searchers. Thus we obtain Lemma 2.1 as the combined version of the lemmas.

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