Strongly Stable and Maximum Weakly Stable Noncrossing Matchings

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Abstract

In IWOCA 2019, Ruangwises and Itoh introduced *stable noncrossing matchings*, where participants of each side are aligned on each of two parallel lines, and no two matching edges are allowed to cross each other. They defined two stability notions, *strongly stable noncrossing matching* (*SSNM*) and *weakly stable noncrossing matching* (*WSNM*), depending on the strength of blocking pairs. They proved that a WSNM always exists and presented an $O(n^2)$-time algorithm to find one for an instance with *n* men and *n* women. They also posed open questions of the complexities of determining existence of an SSNM and finding a largest WSNM. In this paper, we show that both problems are solvable in polynomial time. Our algorithms are applicable to extensions where preference lists may include ties, except for one case which we show to be NP-complete. This NP-completeness holds even if each person's preference list is of length at most two and ties appear in only men's preference lists. To complement this intractability, we show that the problem is solvable in polynomial time if the length of preference lists of one side is bounded by one (but that of the other side is unbounded).

Keywords Stable marriage · Noncrossing matching · Polynomial-time algorithms · NP-completeness

1 Introduction

In the classical *stable marriage problem* [9], there are two sets of participants, traditionally illustrated as men and women, where each person has a *preference list* that strictly orders all the members of the opposite gender. A *matching* is a

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Table 1 Previous and our results (our results in bold)

		Exist_SSNM	Find_WSNM	Max_WSNM
SMI		$O(n^2)$ [**Proposition** 1]	$O(n^2)$ [26]	$O(n^4)$ [**Theorem** 3]
SMTI	Super-	$O(n^2)$ [**Corollary** 1]		$O(n^4)$ [**Corollary** 3]
	Strong	$O(n^3)$ [**Corollary** 2]		$O(n^4)$ [**Corollary** 3]
	Weak	**NPC**[*1] [**Theorem** 1]	$O(n^2)$ [26]	$O(n^4)$ [**Corollary** 3]
		$O(n)$[*2] [**Theorem** 2]		

[*1] even if each person's preference list contains at most two persons and ties appear in only men's preference lists.

[*2] if each man's preference list contains at most one woman

set of (man, woman)-pairs where no person appears more than once. A *blocking pair* for a matching M is (informally) a pair of a man and a woman who are not matched together in M but both of them become better off if they are matched together. A matching that admits no blocking pair is a *stable matching*. The stable marriage problem is one of the recently best-studied topics, with a lot of applications to matching and assignment systems, such as high-school match [1, 2] and medical resident assignment [23]. See some textbooks [11, 19, 21, 25] for more information.

Recently, Ruangwises and Itoh [26] incorporated the notion of noncrossing matchings [3, 7, 17, 20, 27] to the stable marriage problem *with incomplete lists* (denoted *SMI*), where a preference list may contain a *subset* of the members of the opposite side. In their model, there are two parallel lines where n men are aligned on one line and n women are aligned on the other line. A matching is *noncrossing* if no two edges of it cross each other. A *stable noncrossing matching* is a matching which is simultaneously stable and noncrossing. They defined two notions of stability: In a *strongly stable noncrossing matching* (*SSNM*), the definition of a blocking pair is the same as that of the standard stable marriage problem. Thus the set of SSNMs is exactly the intersection of the set of stable matchings and that of noncrossing matchings. In a *weakly stable noncrossing matching* (*WSNM*), a blocking pair has an additional condition that it must not cross matching edges. Ruangwises and Itoh [26] proved that a WSNM exists for any instance, and presented an $O(n^2)$-time algorithm for the problem of finding a WSNM (denoted Find_WSNM). They also showed that the same results hold for the weak stability when ties are present in preference lists. Furthermore, they demonstrated that an SSNM does not always exist, and that there can be WSNMs of different sizes. Concerning these observations, they posed open questions on the complexities of the problems of determining the existence of an SSNM (denoted Exist_SSNM) and finding a WSNM of maximum cardinality (denoted Max_WSNM).

Our Contributions. Table 1 summarizes previous and our results, where our results are described in bold. We first show that both the above mentioned open problems are solvable in polynomial time. Specifically, Exist_SSNM is solved in $O(n^2)$-time by exploiting the well-known Rural Hospitals theorem (Proposition 1)

and MAX_WSNM is solved in $O(n^4)$-time by an algorithm based on dynamic programming (Theorem 3).

We then consider extended problems where preference lists may include ties. When ties are allowed in preference lists, the problem is denoted *SMTI* and there are three stability notions, *super-*, *strong*, and *weak* stability [13]. We show that our algorithm for solving MAX_WSNM is applicable to all of the three stability notions with slight modifications (Corollary 3). We also show that our algorithm for solving EXIST_SSNM can be applied to super- and strong stabilities without any modification (Corollaries 1 and 2). In contrast, we show that EXIST_SSNM is NP-complete for the weak stability (Theorem 1).

This NP-completeness holds even for a restricted case where the length of each person's preference list is at most two and ties appear in only men's preference lists. To complement this intractability, we show that if each man's preference list contains at most one woman (but women's preference lists may be of unbounded length), the problem is solvable in $O(n)$-time (Theorem 2). If we parameterize this problem by two positive integers p and q that bound the lengths of preference lists of men and women, respectively, Theorem 1 shows that the problem is NP-complete even if $p \leq 2$ and $q \leq 2$, while Theorem 2 shows that the problem is solvable in polynomial time if $p = 1$ or $q = 1$ (by symmetry of men and women). Thus the computational complexity of the problem is completely solved in terms of the length of preference lists. We remark that this is a rare case since many NP-hard variants of the stable marriage problem can be solved in polynomial time if the length of preference lists of one side is bounded by two [5, 6, 14, 22].

Progress from the Conference Version. A preliminary version of this paper appeared in the proceedings of the 31st International Workshop on Combinatorial Algorithms (IWOCA 2020) [12]. In [12], only NP-completeness was shown for EXIST_SSNM in the weak stability in SMTI (of unbounded-length preference lists). In this journal version, we investigated the computational complexity of this problem in terms of the length of preference lists: We strengthened the reduction in the proof of Theorem 1 to show that NP-completeness holds even if the length of preference lists is at most two and ties appear in only men's preference lists. Moreover, we added Theorem 2 that shows that the problem can be solved in polynomial time if the length of preference lists of one-side is at most one. As mentioned previously, these two theorems solve the computational complexity of this problem in terms of the length of preference lists.

2 Preliminaries

In this section, we give necessary definitions and notations, some of which are taken from Ruangwises and Itoh [26]. An instance consists of n men $m_1, m_2, \ldots, m_n$ and n women $w_1, w_2, \ldots, w_n$. We assume that the men are lying on a vertical line in an increasing order of indices from top to bottom, and similarly the women are lying in the same manner on another vertical line parallel to the first one. Each person has a preference list over a subset of the members of the opposite gender. For now, assume

that preference lists are strict, i.e., do not contain ties. We call such an instance an *SMI-instance*. If a person q appears in a person p's preference list, we say that q is *acceptable* to p. If p and q are acceptable to each other, we say that (p, q) is an *acceptable pair*. We assume without loss of generality that acceptability is mutual, i.e., p is acceptable to q if and only if q is acceptable to p. If p prefers q_1 to q_2, then we write $q_1 >_p q_2$.

A *matching* is a set of acceptable pairs of a man and a woman in which each person appears at most once. If $(m, w) \in M$, we write $M(m) = w$ and $M(w) = m$. If a person p is not included in a matching M, we say that p is *single* in M and write $M(p) = \emptyset$. Every person prefers to be matched with an acceptable person rather than to be single, i.e., $q >_p \emptyset$ holds for any p and any q acceptable to p.

A pair in a matching can be seen as an edge on the plane, so we may use "pair" and "edge" interchangeably. Two edges (m_i, w_j) and (m_x, w_y) are said to *cross* each other if they share an interior point, or formally, if $(x - i)(y - j) < 0$ holds. A matching is *noncrossing* if it contains no pair of crossing edges.

For a matching M, an acceptable pair $(m, w) \notin M$ is called a *blocking pair* for M (or (m, w) *blocks* M) if both $w >_m M(m)$ and $m >_w M(w)$ hold. A *noncrossing blocking pair* for M is a blocking pair for M that does not cross any edge of M. A matching M is a *weakly stable noncrossing matching* (*WSNM*) if M is noncrossing and does not admit any noncrossing blocking pair. A matching M is a *strongly stable noncrossing matching* (*SSNM*) if M is noncrossing and does not admit any blocking pair. Note that an SSNM is always a WSNM by definition but the converse is not true.

We then extend the above definitions to the case where preference lists may contain ties. A *tie* of a person p's preference list is a set of one or more persons who are equally preferred by p, and p's preference list is a strict order of ties. We call such an instance an *SMTI-instance*. In a person p's preference list, suppose that a person q_1 is in tie T_1, q_2 is in tie T_2, and p prefers T_1 to T_2. Then we say that p strictly prefers q_1 to q_2 and write $q_1 >_p q_2$. If q_1 and q_2 are in the same tie (including the case that q_1 and q_2 are the same person), we write $q_1 =_p q_2$. If $q_1 >_p q_2$ or $q_1 =_p q_2$ holds, we write $q_1 \geq_p q_2$ and say that p weakly prefers q_1 to q_2.

When ties are present, there are three possible definitions of blocking pairs, and accordingly, there are three stability notions, *super-stability*, *strong stability*, and *weak stability* [13]:

- In the super-stability, a blocking pair for a matching M is an acceptable pair $(m, w) \notin M$ such that $w \geq_m M(m)$ and $m \geq_w M(w)$.
- In the strong stability, a blocking pair for a matching M is an acceptable pair $(p, q) \notin M$ such that $q \geq_p M(p)$ and $p >_q M(q)$. Note that the person q, who strictly prefers the counterpart p of the blocking pair, may be either a man or a woman.
- In the weak stability, a blocking pair for a matching M is an acceptable pair $(m, w) \notin M$ such that $w >_m M(m)$ and $m >_w M(w)$.

With these definitions of blocking pairs, the terms "noncrossing blocking pair", "WSNM", and "SSNM" for each stability notion can be defined analogously. In the

SMTI case, we extend the names of stable noncrossing matchings using the type of stability as a prefix. For example, a WSNM in the super-stability is denoted *super-WSNM*.

Note that, in this paper, the terms "weak" and "strong" are used in two different meanings. This might be confusing but we decided not to change these terms, respecting previous literature.

For implementation of our algorithms, we use ranking arrays described in Sect. 1.2.3 of [11]. Although in [11] ranking arrays are defined for complete preference lists without ties, they can easily be modified for incomplete lists and/or with ties. Then, by the aid of ranking arrays, we can determine, given persons p, q_1, and q_2, whether $q_1 \succ_p q_2$ or $q_2 \succ_p q_1$ or $q_1 =_p q_2$ in constant time. Also we can determine, given m and w, if (m, w) is an acceptable pair or not in constant time.

3 Strongly Stable Noncrossing Matchings

3.1 SMI

In SMI, an easy observation shows that existence of an SSNM can be determined in $O(n^2)$ time:

Proposition 1 *There exists an $O(n^2)$-time algorithm to find an SSNM or to report that none exists, given an SMI-instance.*

Proof Note that an SSNM is a stable matching in the original sense. In SMI, there always exists at least one stable matching [11], and due to the Rural Hospitals theorem [10, 23, 24], the set of matched agents is the same in any stable matching. These agents can be determined in $O(n^2)$ time by using the Gale-Shapley algorithm [9]. There is only one way of matching them in a noncrossing manner. Hence the matching constructed in this way is the unique candidate for an SSNM. All we have to do is to check if it is stable, which can be done in $O(n^2)$ time.   □

3.2 SMTI

In the presence of ties, super-stable and strongly stable matchings do not always exist. However, there is an $O(n^2)$-time ($O(n^3)$-time, respectively) algorithm that finds a super-stable (strongly stable, respectively) matching or reports that none exists [13, 18]. Also, the Rural Hospitals theorem takes over to the super-stability [15] and strong stability [16]. Therefore, the same algorithm as in Sect. 3.1 applies for these cases, implying the following corollaries:

Corollary 1 *There exists an $O(n^2)$-time algorithm to find a super-SSNM or to report that none exists, given an SMTI-instance.*

Corollary 2 *There exists an $O(n^3)$-time algorithm to find a strong-SSNM or to report that none exists, given an SMTI-instance.*

In contrast, the problem becomes NP-complete for the weak stability even for a highly restricted case:

Theorem 1 *The problem of determining if a weak-SSNM exists, given an SMTI-instance, is NP-complete, even if each person's preference list contains at most two persons and ties appear in only men's preference lists.*

Proof Membership in NP is obvious. We show NP-hardness by a reduction from 3SAT [8]. An instance of 3SAT consists of a set of variables and a set of clauses. Each variable takes either true (1) or false (0). A *literal* is a variable or its negation. A *clause* is a disjunction of at most three literals. A clause is *satisfied* if at least one of its literals takes the value 1. A 0/1 assignment to variables that satisfies all the clauses is called a *satisfying assignment*. An instance f of 3SAT is *satisfiable* if it has at least one satisfying assignment. 3SAT asks if there exists a satisfying assignment. 3SAT is NP-complete even if each variable appears exactly four times, exactly twice positively and exactly twice negatively, and each clause contains exactly three literals [4]. We use 3SAT instances restricted in this way.

Now we show the reduction. Let f be an instance of 3SAT having n variables $x_i (1 \le i \le n)$ and m clauses $C_j (1 \le j \le m)$. For each variable x_i, we construct a *variable gadget*. It consists of six men $p_{i,1}$, $p_{i,2}$, $p_{i,3}$, $p_{i,4}$, $a_{i,1}$, and $a_{i,2}$, and four women $q_{i,1}$, $q_{i,2}$, $q_{i,3}$, and $q_{i,4}$. A variable gadget corresponding to x_i is called an x_i-*gadget*. For each clause C_j, we construct a *clause gadget*. It consists of seven men $y_{j,k} (1 \le k \le 7)$ and nine women $v_{j,k} (1 \le k \le 6)$ and $z_{j,k} (1 \le k \le 3)$. A clause gadget corresponding to C_j is called a C_j-*gadget*. Additionally, we create a man s and a woman t, who constitute a gadget called the *separator*.

Thus, there are $6n + 7m + 1$ men and $4n + 9m + 1$ women in the created SMTI-instance, denoted $I(f)$. Finally, we add dummy persons who have empty preference lists to make the numbers of men and women equal. They do not play any role in the following arguments, so we omit them.

Suppose that x_i's kth positive occurrence $(k = 1, 2)$ is in the $d_{i,k}$th clause $C_{d_{i,k}}$ as the $e_{i,k}$th literal $(1 \le e_{i,k} \le 3)$. Similarly, suppose that x_i's kth negative occurrence $(k = 1, 2)$ is in the $g_{i,k}$th clause $C_{g_{i,k}}$ as the $h_{i,k}$th literal $(1 \le h_{i,k} \le 3)$. The preference lists of ten persons in the x_i-gadget are constructed as shown in Fig. 1. Here, each preference list is described as a sequence from left to right according to preference, i.e., the leftmost person is the most preferred and the rightmost person is the least preferred. Tied persons (i.e., persons with the equal preference) are included in parentheses. Men are aligned in the order of $p_{i,1}$, $p_{i,3}$, $a_{i,1}$, $a_{i,2}$, $p_{i,2}$, and $p_{i,4}$ from top to bottom, and women are aligned in the order of $q_{i,1}$, $q_{i,3}$, $q_{i,2}$, and $q_{i,4}$. (See Fig. 2. Edges depicted in the figure are those within the variable gadget.)

It might be helpful to explain here intuition behind a variable gadget. People there are partitioned into two groups, $\{p_{i,1}, a_{i,1}, p_{i,2}, q_{i,1}, q_{i,2}\}$ and

$$
\begin{array}{lll}
p_{i,1}: & q_{i,1} & z_{g_{i,1},h_{i,1}} \\
a_{i,1}: & (q_{i,1} & q_{i,2}) \\
p_{i,2}: & q_{i,2} & z_{d_{i,1},e_{i,1}} \\
p_{i,3}: & q_{i,3} & z_{g_{i,2},h_{i,2}} \\
a_{i,2}: & (q_{i,3} & q_{i,4}) \\
p_{i,4}: & q_{i,4} & z_{d_{i,2},e_{i,2}}
\end{array}
\qquad
\begin{array}{lll}
q_{i,1}: & a_{i,1} & p_{i,1} \\
q_{i,2}: & a_{i,1} & p_{i,2} \\
\\
q_{i,3}: & a_{i,2} & p_{i,3} \\
q_{i,4}: & a_{i,2} & p_{i,4}
\end{array}
$$

Fig. 1 Preference lists of persons in x_i-gadget. Tied persons are included in parentheses

Fig. 2 Alignment of agents in a variable gadget. This gadget admits two noncrossing stable matchings highlighted in blue and red, associated with assignment $x_i = 0$ and $x_i = 1$, respectively
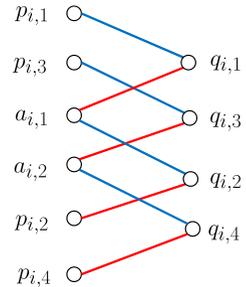
$\{p_{i,3}, a_{i,2}, p_{i,4}, q_{i,3}, q_{i,4}\}$. The first group corresponds to the first positive occurrence and the first negative occurrence of x_i. It has two stable matchings $\{(p_{i,1}, q_{i,1}), (a_{i,1}, q_{i,2})\}$ (blue in Fig. 2) and $\{(a_{i,1}, q_{i,1}), (p_{i,2}, q_{i,2})\}$ (red). We associate the former with the assignment $x_i = 0$ and the latter with the assignment $x_i = 1$. The second group corresponds to the second positive occurrence and the second negative occurrence of x_i. It has two stable matchings $\{(p_{i,3}, q_{i,3}), (a_{i,2}, q_{i,4})\}$ (blue) and $\{(a_{i,2}, q_{i,3}), (p_{i,4}, q_{i,4})\}$ (red). We associate the former with $x_i = 0$ and the latter with $x_i = 1$. Entanglement of two groups as in Fig. 2 plays a role of ensuring consistency of assignments between the first and the second group. Depending on the choice of the matching in the first group, edges with the same color must be chosen from the second group to avoid edge-crossing.

Let us continue the reduction. We then construct preference lists of clause gadgets. Consider a clause C_j, and suppose that its kth literal is of a variable x_{j_k}. Define $\ell_{j,k}$ as

$$
\ell_{j,k} = \begin{cases}
1 & \text{if this is the 1st negative occurrence of } x_{j_k} \\
2 & \text{if this is the 1st positive occurrence of } x_{j_k} \\
3 & \text{if this is the 2nd negative occurrence of } x_{j_k} \\
4 & \text{if this is the 2nd positive occurrence of } x_{j_k}.
\end{cases}
$$

The preference lists of persons in the C_j-gadget are as shown in Fig. 3. The alignment order of persons in each clause gadget is the same as in Fig. 3. Since a clause gadget is complicated, we show a structure in the leftmost figure of Fig. 4 (three matchings $N_{j,1}$, $N_{j,2}$, and $N_{j,3}$ will be used later).

$$
\begin{array}{llll}
y_{j,1}: & (v_{j,1} & v_{j,3}) & \qquad v_{j,1}: \quad y_{j,1}\\
y_{j,2}: & (v_{j,2} & z_{j,1}) & \qquad v_{j,2}: \quad y_{j,2}\\
y_{j,3}: & (v_{j,3} & v_{j,4}) & \qquad v_{j,3}: \quad y_{j,1} \quad y_{j,3}\\
y_{j,4}: & (z_{j,2} & v_{j,5}) & \qquad z_{j,1}: \quad y_{j,2} \quad p_{j_1,\ell_{j,1}}\\
y_{j,5}: & (v_{j,4} & v_{j,6}) & \qquad z_{j,2}: \quad y_{j,4} \quad p_{j_2,\ell_{j,2}}\\
y_{j,6}: & (v_{j,5} & z_{j,3}) & \qquad v_{j,4}: \quad y_{j,5} \quad y_{j,3}\\
y_{j,7}: & v_{j,6} & & \qquad v_{j,5}: \quad y_{j,6} \quad y_{j,4}\\
 & & & \qquad v_{j,6}: \quad y_{j,5} \quad y_{j,7}\\
 & & & \qquad z_{j,3}: \quad y_{j,6} \quad p_{j_3,\ell_{j,3}}
\end{array}
$$

Fig. 3 Preference lists of persons in C_j-gadget. Tied persons are included in parentheses
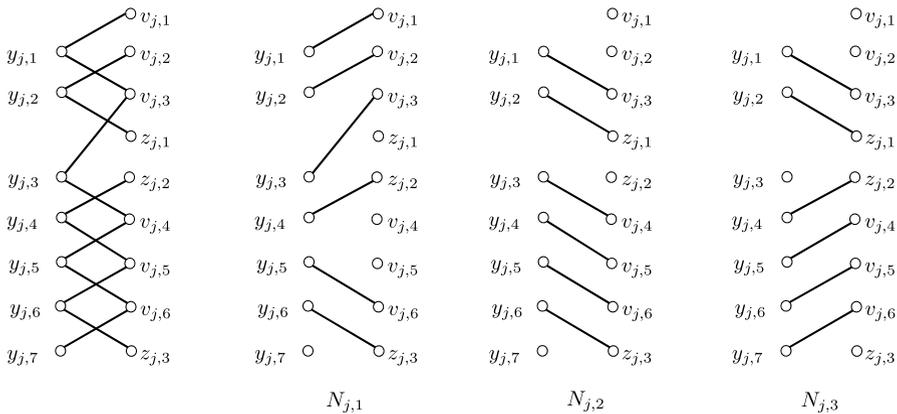
Fig. 4 Acceptability graph of a clause gadget C_j and its matchings $N_{j,1}$, $N_{j,2}$, and $N_{j,3}$

$$s: \quad t \qquad\qquad t: \quad s$$
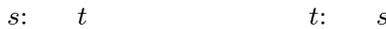
Fig. 5 Preference lists of the man and the woman in the separator

Finally, each of the man and the woman in the separator includes only the other in the list (Fig. 5). They are guaranteed to be matched together in any stable matching.

Alignment of the whole instance is depicted in Fig. 6. Variable gadgets are placed top, then followed by the separator, clause gadgets come bottom. The separator plays a role of prohibiting a person of a variable gadget to be matched with a person of a clause gadget; if they are matched, then the corresponding edge crosses the separator.

Now the reduction is completed. It is not hard to see that the reduction can be performed in polynomial time and the conditions on the preference lists stated in the theorem are satisfied.
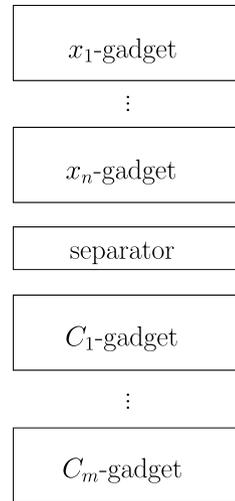
We then show the correctness. First, suppose that f is satisfiable and let A be a satisfying assignment. We construct a weak-SSNM M of $I(f)$ from A. For an x_i-gadget, define two matchings

- $M_{i,0} = \{(p_{i,1}, q_{i,1}), (a_{i,1}, q_{i,2}), (p_{i,3}, q_{i,3}), (a_{i,2}, q_{i,4})\}$ (blue in Fig. 2) and
- $M_{i,1} = \{(a_{i,1}, q_{i,1}), (p_{i,2}, q_{i,2}), (a_{i,2}, q_{i,3}), (p_{i,4}, q_{i,4})\}$ (red in Fig. 2).

If $x_i = 0$ under A, then add $M_{i,0}$ to M; otherwise, add $M_{i,1}$ to M. For a C_j-gadget, we define three matchings

- $N_{j,1} = \{(y_{j,1}, v_{j,1}), (y_{j,2}, v_{j,2}), (y_{j,3}, v_{j,3}), (y_{j,4}, z_{j,2}), (y_{j,5}, v_{j,6}), (y_{j,6}, z_{j,3})\}$,
- $N_{j,2} = \{(y_{j,1}, v_{j,3}), (y_{j,2}, z_{j,1}), (y_{j,3}, v_{j,4}), (y_{j,4}, v_{j,5}), (y_{j,5}, v_{j,6}), (y_{j,6}, z_{j,3})\}$, and
- $N_{j,3} = \{(y_{j,1}, v_{j,3}), (y_{j,2}, z_{j,1}), (y_{j,4}, z_{j,2}), (y_{j,5}, v_{j,4}), (y_{j,6}, v_{j,5}), (y_{j,7}, v_{j,6})\}$,

that are depicted in Fig. 4. Note that, for each $k \in \{1, 2, 3\}$, only $z_{j,k}$ (among $z_{j,1}$, $z_{j,2}$, and $z_{j,3}$) is single in $N_{j,k}$. If C_j is satisfied by the kth literal ($k \in \{1, 2, 3\}$), then add $N_{j,k}$ to M. (If C_j is satisfied by more than one literal, then choose one arbitrarily.) Finally add the pair (s, t) to M.

It is not hard to see that M is noncrossing. We show that it is weakly stable. Clearly, neither s nor t in the separator forms a blocking pair. Next, consider the x_i-gadget. In $M_{i,0}$, women $q_{i,2}$ and $q_{i,4}$ are matched with the first-choice man. The woman $q_{i,1}$ is matched with the second-choice man $p_{i,1}$ but her first-choice man $a_{i,1}$ is matched with a first-choice woman $q_{i,2}$. Similarly, $q_{i,3}$'s first-choice man $a_{i,2}$ is matched with a first-choice woman $q_{i,4}$. Men $p_{i,1}$, $a_{i,1}$, $p_{i,3}$, and $a_{i,2}$ are matched with a first-choice woman. Hence these persons cannot be a part of a blocking pair; only $p_{i,2}$ and $p_{i,4}$ may participate in a blocking pair. Similarly, we can argue that, in $M_{i,1}$, only $p_{i,1}$ and $p_{i,3}$ may participate in a blocking pair.

Consider a C_j-gadget. In $N_{j,1}$, all the men except for $y_{j,7}$ are matched with a first-choice woman. $y_{j,7}$'s unique choice $v_{j,6}$ is matched with the first-choice man $y_{j,5}$. Hence no man in this gadget can participate in a blocking pair, and so no blocking pair exists within this gadget. Since $z_{j,2}$ and $z_{j,3}$ are matched with their respective first-choice woman, only the possibility is that $z_{j,1}$ forms a blocking pair with $p_{j_1,\ell_{j,1}}$ of a variable gadget. The same observation applies for $N_{j,2}$ and $N_{j,3}$ and we can see that for each $k \in \{1, 2, 3\}$ only $z_{j,k}$ can participate in a blocking pair in $N_{j,k}$.

To summarize, if there exists a blocking pair, it must be of the form $(p_{i,\ell}, z_{j,k})$ for some i, ℓ, j, and k, and both $p_{i,\ell}$ and $z_{j,k}$ are single in M. Suppose that $\ell = 1$. The reason for $(p_{i,1}, z_{j,k})$ being an acceptable pair is that C_j's kth literal is $\neg x_i$, a negative occurrence of x_i. Since $p_{i,1}$ is single, $M_{i,1} \subset M$ and hence $x_i = 1$ under A. Since $z_{j,k}$ is single, $N_{j,k} \subset M$ and hence C_j is satisfied by its kth literal $\neg x_i$, but this is a contradiction. The other cases $\ell = 2, 3, 4$ can be argued in the same manner, and we can conclude that M is stable.

Conversely, suppose that $I(f)$ admits a weak-SSNM M. We construct a satisfying assignment A of f. Before giving construction, we observe structural properties of M in two lemmas:

Lemma 1 *For each i ($1 \le i \le n$), either $M_{i,0} \subset M$ or $M_{i,1} \subset M$.*

Proof Note that preference lists of the ten persons of the x_i-gadget include persons of the same x_i-gadget or some persons from clause gadgets. Hence, due to the separator, persons of the x_i-gadget can only be matched within this gadget to avoid edge-crossings.

Note that a stable matching is a maximal matching. With regard to $p_{i,1}$, $a_{i,1}$, $p_{i,2}$, $q_{i,1}$, and $q_{i,2}$, there are three maximal matchings $\{(p_{i,1}, q_{i,1}), (a_{i,1}, q_{i,2})\}$, $\{(a_{i,1}, q_{i,1}), (p_{i,2}, q_{i,2})\}$, and $\{(p_{i,1}, q_{i,1}), (p_{i,2}, q_{i,2})\}$, but the last one is blocked by $(a_{i,1}, q_{i,1})$ and $(a_{i,1}, q_{i,2})$. Hence either the first or the second one must be in M. With regard to $p_{i,3}$, $a_{i,2}$, $p_{i,4}$, $q_{i,3}$, and $q_{i,4}$, there are three maximal matchings $\{(p_{i,3}, q_{i,3}), (a_{i,2}, q_{i,4})\}$, $\{(a_{i,2}, q_{i,3}), (p_{i,4}, q_{i,4})\}$, and $\{(p_{i,3}, q_{i,3}), (p_{i,4}, q_{i,4})\}$, but the last one is blocked by $(a_{i,2}, q_{i,3})$ and $(a_{i,2}, q_{i,4})$. Hence either the first or the second one must be in M.

If we choose $\{(p_{i,1}, q_{i,1}), (a_{i,1}, q_{i,2})\}$, then we must choose $\{(p_{i,3}, q_{i,3}), (a_{i,2}, q_{i,4})\}$ to avoid edge-crossing, which constitute $M_{i,0}$. If we choose $\{(a_{i,1}, q_{i,1}), (p_{i,2}, q_{i,2})\}$, then we must choose $\{(a_{i,2}, q_{i,3}), (p_{i,4}, q_{i,4})\}$, which constitute $M_{i,1}$. Hence either $M_{i,0}$ or $M_{i,1}$ must be a part of M. $\qquad\square$

Lemma 2 *For a C_j-gadget, at least one of $z_{j,1}$, $z_{j,2}$, and $z_{j,3}$ is unmatched in M.*

Proof Note that preference lists of the persons of the C_j-gadget include persons of the same C_j-gadget or some persons from variable gadgets. To avoid edge-crossing, persons must be matched within the same C_j-gadget.

For contradiction, suppose that all $z_{j,1}$, $z_{j,2}$, and $z_{j,3}$ are matched in M. Then $(y_{j,2}, z_{j,1})$, $(y_{j,4}, z_{j,2})$, and $(y_{j,6}, z_{j,3})$ are in M (Fig. 7(1)). To avoid edge-crossing, $(y_{j,3}, v_{j,3})$, $(y_{j,3}, v_{j,4})$, and $(y_{j,7}, v_{j,6})$ must not be in M (Fig. 7(2)). The pair $(y_{j,5}, v_{j,4})$
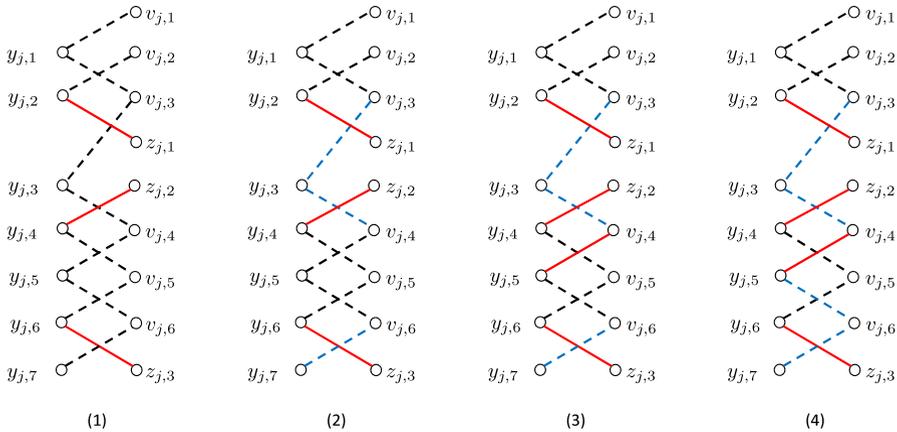
Fig. 7 Situation in the proof of Lemma 2. Red solid edges are those confirmed to be in M, blue dashed edges are those confirmed not to be in M, and black dashed edges are uncertain

must be in M as otherwise $(y_{j,3}, v_{j,4})$ is a blocking pair (Fig. 7(3)). For M to be a matching, $(y_{j,5}, v_{j,6})$ must not be in M (Fig. 7(4)). Then $(y_{j,7}, v_{j,6})$ is a blocking pair, a contradiction. □

For each i, either $M_{i,0} \subset M$ or $M_{i,1} \subset M$ holds by Lemma 1. If $M_{i,0} \subset M$ holds then we set $x_i = 0$ in A, and if $M_{i,1} \subset M$ holds then we set $x_i = 1$ in A. We show that A satisfies f. Let C_j be an arbitrary clause. By Lemma 2, at least one of $z_{j,1}$, $z_{j,2}$, and $z_{j,3}$ is unmatched in M. If there are two or more unmatched women, then choose one arbitrarily and let this woman be $z_{j,k}$. We show that C_j is satisfied by its kth literal. Suppose not.

First suppose that the kth literal of C_j is the first positive occurrence of x_i. Then, by construction of preference lists, $(p_{i,2}, z_{j,k})$ is an acceptable pair. If $x_i = 0$ under A, then $M_{i,0} \subset M$ by construction of A, and hence $p_{i,2}$ is single in M. Thus $(p_{i,2}, z_{j,k})$ is a blocking pair, which contradicts stability of M. Hence $x_i = 1$ under A and C_j is satisfied by x_i. When the kth literal of C_j is the second positive occurrence of x_i, the same argument holds if we replace $p_{i,2}$ by $p_{i,4}$.

Next suppose that the kth literal of C_j is the first negative occurrence of x_i. Then, by construction of preference lists, $(p_{i,1}, z_{j,k})$ is an acceptable pair. If $x_i = 1$ under A, then $M_{i,1} \subset M$ by construction of A, and hence $p_{i,1}$ is single in M. Thus $(p_{i,1}, z_{j,k})$ is a blocking pair, which contradicts stability of M. Hence $x_i = 0$ under A and C_j is satisfied by $\neg x_i$. If the kth literal of C_j is the second negative occurrence of x_i, the same argument holds if we replace $p_{i,1}$ by $p_{i,3}$. Thus A is a satisfying assignment of f and the proof is completed. □

Next we give a positive result.

Theorem 2 The problem of determining if a weak-SSNM exists, given an SMTI-instance, is solvable in $O(n)$-time if each man's preference list contains at most one woman.

Proof Let I be an input SMTI-instance. First, we construct the bipartite graph $G_I = (U_I, V_I, E_I)$, where U_I and V_I correspond to the sets of men and women in I, respectively, and $(m, w) \in E_I$ if and only if m is a first-choice of w. For a vertex $v \in V_I$, let $d(v)$ denote its degree in G_I. Since acceptability is mutual, if a woman w's preference list in I is nonempty, $d(w) \geq 1$ holds. Note that it can happen that $d(w) \geq 2$ because preference lists may contain ties. In the following lemma, we characterize (not necessarily noncrossing) stable matchings of I.

Lemma 3 *M is a stable matching of I if and only if $M \subseteq E_I$ and each woman $w \in V_I$ such that $d(w) \geq 1$ is matched in M.*

Proof Suppose that M is stable. If $M \nsubseteq E_I$, there is an edge $(m, w) \in M \setminus E_I$. The fact $(m, w) \notin E_I$ means that m is not w's first-choice so there is an edge $(m', w) \in E_I$ such that $m' >_w m$. Since $(m, w) \in M$, m' is single in M. Therefore, (m', w) is a blocking pair for M, a contradiction. If there is a woman $w \in V_I$ such that $d(w) \geq 1$ but w is single in M, then any man m such that (m, w) is an acceptable pair is a blocking pair because m is also single in M, a contradiction.

Conversely, suppose that M satisfies the conditions of the right hand side. Then each woman who has a nonempty list is matched with a first-choice man, so there cannot be a blocking pair. $\square$

By Lemma 3, our task is to select from E_I one edge per woman w such that $d(w) \geq 1$, in such a way that the resulting matching is noncrossing. We do this greedily. M is initially empty, and we add edges to M by processing vertices of V_I from top to bottom. At w_i's turn, if $d(w_i) \geq 1$, then choose the topmost edge that does not cross any edge in M, and add it to M. If there is no such edge, then we immediately conclude that I admits no weak-SSNM. If we can successfully process all the women, we output the final matching M, which is a weak-SSNM.

In the following, we formalize the above idea. A pseudo-code of the whole algorithm WEAK-SSNM-1 is given in Algorithm 1.

Algorithm 1 WEAK-SSNM-1

Input: An SMTI-instance I.
Output: A weak-SSNM M or "No" if none exists.
1: Construct the bipartite graph $G_I = (U_I, V_I, E_I)$.
2: Let $M := \emptyset$.
3: **for** $i = 1$ to n **do**
4:     **if** $d(w_i) \geq 1$ **then**
5:         Let j^* (if any) be the smallest j such that $(m_j, w_i) \in E_I$ and $M \cup \{(m_j, w_i)\}$ is a noncrossing matching.
6:         Let $M := M \cup \{(m_{j^*}, w_i)\}$.
7:         **if** no such j^* exists **then**
8:             Output "No" and halt.
9:         **end if**
10:    **end if**
11: **end for**
12: Output M.

We show the correctness. Suppose that WEAK-SSNM-1 outputs a matching M. M is noncrossing by the condition of line 5, and M is stable because the construction of M follows the condition of Lemma 3.

Conversely, suppose that I admits a weak-SSNM M^*. We show that WEAK-SSNM-1 outputs a matching. Suppose not, and suppose that WEAK-SSNM-1 failed when processing woman (vertex) w_k. Let $\bar{M}$ be the matching constructed so far by WEAK-SSNM-1. Then for each i ($1 \le i \le k - 1$), w_i is single in M^* if and only if she is single in $\bar{M}$. Also, since $M^* \subseteq E_I$ by Lemma 3, we can show by a simple induction that for each i ($1 \le i \le k - 1$), if $M^*(w_i) = m_p$ and $\bar{M}(w_i) = m_q$, then $q \le p$. Then, at line 5, we could have chosen $(M^*(w_k), w_k)$ to add to $\bar{M}$, a contradiction.

Finally, we consider time-complexity. Since the preference list of each man contains at most one woman, the graph G_I at line 1 can be constructed in $O(n)$-time and contains at most n edges. The **for**-loop can be executed in $O(n)$-time because each edge is scanned at most once in the loop; whether or not an edge crosses edges of M at line 5 can be done in constant time by keeping the maximum index of the matched men in M at any stage. □

4 Maximum Cardinality Weakly Stable Noncrossing Matchings

In this section, we present an algorithm to find a maximum cardinality WSNM. For an instance I, let $opt(I)$ denote the size of the maximum cardinality WSNM.

4.1 SMI

Let I' be a given instance with men $m_1, \ldots, m_n$ and women $w_1, \ldots, w_n$. To simplify the description of the algorithm, we translate I' to an instance I by adding a man m_0 and a woman w_0, each of whom includes only the other in the preference list, and similarly a man m_{n+1} and a woman w_{n+1}, each of whom includes only the other in the preference list. It is easy to see that, for a WSNM M' of I', $M = M' \cup \{(m_0, w_0), (m_{n+1}, w_{n+1})\}$ is a WSNM of I. Conversely, any WSNM M of I includes the pairs (m_0, w_0) and (m_{n+1}, w_{n+1}), and $M' = M \setminus \{(m_0, w_0), (m_{n+1}, w_{n+1})\}$ is a WSNM of I'. Thus we have that $opt(I) = opt(I') + 2$. Hence, without loss of generality, we assume that a given instance I has $n + 2$ men and $n + 2$ women, with m_0, w_0, m_{n+1}, and w_{n+1} having the above mentioned preference lists.

Let $M = \{(m_{i_1}, w_{j_1}), (m_{i_2}, w_{j_2}), \ldots, (m_{i_k}, w_{j_k})\}$ be a noncrossing matching of I such that $i_1 < i_2 \cdots < i_k$ and $j_1 < j_2 \cdots < j_k$. We call (m_{i_k}, w_{j_k}) the *maximum pair* of M. Suppose that (m_x, w_y) is the maximum pair of a noncrossing matching M. We call M a *semi-WSNM* if each of its noncrossing blocking pairs (m_i, w_j) (if any) satisfies $x \le i \le n + 1$ and $y \le j \le n + 1$. Intuitively, a semi-WSNM is a WSNM up to its maximum pair. Note that any semi-WSNM must contain (m_0, w_0), as otherwise it is a noncrossing blocking pair. For $0 \le i \le n + 1$ and $0 \le j \le n + 1$, we define $X(i, j)$ as the maximum size of a semi-WSNM of I whose maximum pair is (m_i, w_j); if I does not admit a semi-WSNM with the maximum pair (m_i, w_j), $X(i, j)$ is defined to be $-\infty$.

Lemma 4 $opt(I) = X(n + 1, n + 1)$.

Proof Note that any WSNM of I includes (m_{n+1}, w_{n+1}), as otherwise it is a noncrossing blocking pair. Hence it is a semi-WSNM with the maximum pair (m_{n+1}, w_{n+1}). Conversely, any semi-WSNM with the maximum pair (m_{n+1}, w_{n+1}) does not include a noncrossing blocking pair and hence is also a WSNM. Therefore, the set of WSNMs is equivalent to the set of semi-WSNMs with the maximum pair (m_{n+1}, w_{n+1}). This completes the proof. □

To compute $X(n + 1, n + 1)$, we shortly define quantity $Y(i, j)$ ($0 \le i \le n + 1$, $0 \le j \le n + 1$) using recursive formulas, and show that $Y(i, j) = X(i, j)$ for all i and j. We then show that these recursive formulas allow us to compute $Y(i, j)$ in polynomial time using dynamic programming.

We say that two noncrossing edges (m_i, w_j) and (m_x, w_y) ($i < x, j < y$) are *conflicting* if they admit a noncrossing blocking pair between them; precisely speaking, (m_i, w_j) and (m_x, w_y) are conflicting if the matching $\{(m_i, w_j), (m_x, w_y)\}$ admits a blocking pair (m_s, w_t) such that $i \le s \le x$ and $j \le t \le y$. Otherwise, they are *nonconflicting*. Intuitively, two conflicting edges cannot be consecutive elements of a semi-WSNM.

Now $Y(i, j)$ is defined in the Eqs. (1)–(4). For convenience, we assume that $-\infty + 1 = -\infty$. In Eq. (4), $Y(i', j')$ in max{} is taken among all (i', j') such that (i) $0 \le i' \le i - 1$, (ii) $0 \le j' \le j - 1$, (iii) $(m_{i'}, w_{j'})$ is an acceptable pair, and (iv) (m_i, w_j) and $(m_{i'}, w_{j'})$ are nonconflicting. If no such (i', j') exists, max$\{Y(i', j')\}$ is defined as $-\infty$ and as a result $Y(i, j)$ is also computed as $-\infty$.

$$Y(0, 0) = 1 \tag{1}$$

$$Y(0, j) = -\infty \ (1 \le j \le n + 1) \tag{2}$$

$$Y(i, 0) = -\infty \ (1 \le i \le n + 1) \tag{3}$$

$$Y(i, j) = \begin{cases} 1+ \ \max\{Y(i', j')\} \ \text{(if } (m_i, w_j) \text{ is an acceptable pair)} \\ -\infty \ \text{(otherwise)} \end{cases}$$

$$(1 \le i \le n + 1, 1 \le j \le n + 1) \tag{4}$$

Lemma 5 $Y(i, j) = X(i, j)$ for any i and j such that $0 \le i \le n + 1$ and $0 \le j \le n + 1$.

Proof We prove the claim by induction. We first show that $Y(0, 0) = X(0, 0)$. The matching $\{(m_0, w_0)\}$ is the unique semi-WSNM with the maximum pair (m_0, w_0), so $X(0, 0) = 1$ by definition. Also, $Y(0, 0) = 1$ by Eq. (1). Hence we are done. We then show that $Y(0, j) = X(0, j)$ for $1 \le j \le n + 1$. Since (m_0, w_j) is an unacceptable pair, there is no semi-WSNM with the maximum pair (m_0, w_j), so $X(0, j) = -\infty$

by definition. Also, $Y(0, j) = -\infty$ by Eq. (2). We can show that $Y(i, 0) = X(i, 0)$ for $1 \le i \le n + 1$ by a similar argument.

Next we show that $Y(i, j) = X(i, j)$ holds for $1 \le i \le n + 1$ and $1 \le j \le n + 1$. As an induction hypothesis, we assume that $Y(a, b) = X(a, b)$ holds for $0 \le a \le i - 1$ and $0 \le b \le j - 1$. First, observe that if $X(i, j) \ne -\infty$, then $X(i, j) \ge 2$. This is because two pairs (m_0, w_0) and (m_i, w_j) must be present in any semi-WSNM having the maximum pair (m_i, w_j).

We first consider the case that $X(i, j) \ge 2$. Let $X(i, j) = k$. Then, there is a semi-WSNM M with the maximum pair (m_i, w_j) such that $|M| = k$. Let $M' = M \setminus \{(m_i, w_j)\}$ and (m_x, w_y) be the maximum pair of M'. It is not hard to see that M' is a semi-WSNM with the maximum pair (m_x, w_y) and that $|M'| = k - 1$. Therefore, $X(x, y) \ge k - 1$ by the definition of X, and $Y(x, y) = X(x, y) \ge k - 1$ by the induction hypothesis. Since M is a semi-WSNM, (m_i, w_j) and (m_x, w_y) are nonconflicting, so (x, y) satisfies the condition for (i', j') in Eq. (4). Hence $Y(i, j) \ge 1 + Y(x, y) \ge k$. Suppose that $Y(i, j) \ge k + 1$. By the definition of Y, this means that there is (i', j') that satisfies conditions (i)–(iv) for Eq. (4), and $Y(i', j') \ge k$. By the induction hypothesis, $X(i', j') = Y(i', j') \ge k$. Then there is a semi-WSNM M' with the maximum pair $(m_{i'}, w_{j'})$ such that $|M'| \ge k$. Since M' is a semi-WSNM, and $(m_{i'}, w_{j'})$ and (m_i, w_j) are nonconflicting, $M = M' \cup \{(m_i, w_j)\}$ is a semi-WSNM with the maximum pair (m_i, w_j) such that $|M| = |M'| + 1 \ge k + 1$. This contradicts the assumption that $X(i, j) = k$. Hence $Y(i, j) \le k$ and therefore $Y(i, j) = k$ as desired.

Finally, consider the case that $X(i, j) = -\infty$. If (m_i, w_j) is unacceptable, then the latter case of Eq. (4) is applied and $Y(i, j) = -\infty$. So assume that (m_i, w_j) is acceptable. Then the former case of Eq. (4) is applied. It suffices to show that for any (i', j') that satisfies conditions (i)–(iv) (if any), $Y(i', j') = -\infty$ holds. Assume on the contrary that there is such (i', j') with $Y(i', j') = k$. Then $X(i', j') = k$ by the induction hypothesis, and there is a semi-WSNM M' such that $|M'| = k$, $(m_{i'}, w_{j'})$ is the maximum pair of M', and $(m_{i'}, w_{j'})$ and (m_i, w_j) are nonconflicting. Then $M = M' \cup \{(m_i, w_j)\}$ is a semi-WSNM such that (m_i, w_j) is the maximum pair and $|M| = |M'| + 1 = k + 1$, implying that $X(i, j) = k + 1$. This contradicts the assumption that $X(i, j) = -\infty$ and the proof is completed. □

Now we analyze time-complexity of the algorithm. Computing each $Y(0, 0)$, $Y(0, j)$, and $Y(i, 0)$ can be done in constant time. For computing one $Y(i, j)$ according to Eq. (4), there are $O(n^2)$ candidates for (i', j'). For each (i', j'), checking if $(m_{i'}, w_{j'})$ and (m_i, w_j) are conflicting can be done in constant time with $O(n^4)$-time preprocessing described in subsequent paragraphs. Therefore one $Y(i, j)$ can be computed in time $O(n^2)$. Since there are $O(n^2)$ $Y(i, j)$s, the time-complexity for computing all $Y(i, j)$s is $O(n^4)$. Adding the $O(n^4)$-time for preprocessing mentioned above, the total time-complexity of the algorithm is $O(n^4)$.

In the preprocessing, we construct three tables S, A, and B.

- S is a $\Theta(n^4)$-sized four-dimensional table that takes logical values 0 and 1. For $0 \le i' \le i \le n + 1$ and $0 \le j' \le j \le n + 1$, $S(i', i, j', j) = 1$ if and only if there exists at least one acceptable pair (m, w) such that $m \in \{m_{i'}, m_{i'+1}, \dots, m_i\}$ and $w \in \{w_{j'}, w_{j'+1}, \dots, w_j\}$. Since $S(i, i, j, j) = 1$ if and only if (m_i, w_j) is an accept-

able pair, it can be computed in constant time. In general, $S(i', i, j', j)$ can be computed in constant time as follows.

$$S(i', i, j', j) = \begin{cases} 1 \text{ (if } (m_i, w_j) \text{ is an acceptable pair)} \\ S(i', i-1, j', j) \vee S(i', i, j', j-1) \text{ (otherwise)} \end{cases}$$

Hence S can be constructed in $O(n^4)$ time by a simple dynamic programming.

- A is a $\Theta(n^3)$-sized table. For convenience, we introduce an imaginary person λ who is acceptable to any person, where $q >_p \lambda$ holds for any person p and any person q acceptable to p. For $0 \leq i \leq n+1$ and $0 \leq j' \leq j \leq n+1$, $A(i, j', j)$ stores the woman whom m_i most prefers among $\{w_{j'}, \dots, w_j, \lambda\}$. Then, for i and j, $A(i, j, j) = w_j$ if (m_i, w_j) is an acceptable pair and $A(i, j, j) = \lambda$ otherwise. $A(i, j', j)$ can be computed as the better of $A(i, j', j-1)$ and $A(i, j, j)$ in m_i's list. By the above arguments, each element of A can be computed in constant time and hence A can be constructed in $O(n^3)$ time.

- B plays a symmetric role to A; for $0 \leq i' \leq i \leq n+1$ and $0 \leq j \leq n+1$, $B(i', i, j)$ stores the man whom w_j most prefers among $\{m_{i'}, \dots, m_i, \lambda\}$. B can also be constructed in $O(n^3)$ time.

It is easy to see that $(m_{i'}, w_{j'})$ and (m_i, w_j) are conflicting if and only if one of the following conditions hold. Condition 1 can be clearly checked in constant time. Thanks to the preprocessing, Conditions 2–4 can also be checked in constant time.

1. $(m_{i'}, w_j)$ or $(m_i, w_{j'})$ is a blocking pair for the matching $\{(m_{i'}, w_{j'}), (m_i, w_j)\}$.
2. $S(i'+1, i-1, j'+1, j-1) = 1$. (If this holds, there is a blocking pair (m, w) such that $m \in \{m_{i'+1}, m_{i'+2}, \dots, m_{i-1}\}$ and $w \in \{w_{j'+1}, w_{j'+2}, \dots, w_{j-1}\}$.)
3. m_i prefers $A(i, j'+1, j-1)$ to w_j or $m_{i'}$ prefers $A(i', j'+1, j-1)$ to $w_{j'}$. (If this holds, there exists a blocking pair (m, w) such that $m \in \{m_{i'}, m_i\}$ and $w \in \{w_{j'+1}, \dots, w_{j-1}\}$.)
4. w_j prefers $B(i'+1, i-1, j)$ to m_i or $w_{j'}$ prefers $B(i'+1, i-1, j')$ to $m_{i'}$. (If this holds, there exists a blocking pair (m, w) such that $m \in \{m_{i'+1}, \dots, m_{i-1}\}$ and $w \in \{w_{j'}, w_j\}$).

This completes the explanation on preprocessing, and from the discussion so far, we have the following theorem:

Theorem 3 *There exists an $O(n^4)$-time algorithm to find a maximum cardinality WSNM, given an SMI-instance.*

4.2 SMTI

Similarly to the SMI case, a weak-WSNM exists in any SMTI-instance, as remarked in page 415 of [26]: Given an SMTI-instance I, break all the ties arbitrarily and obtain an SMI-instance I'. Let M be a WSNM of I'. Then it is not hard to see that M is also a weak-WSNM of I. In contrast, there is a simple instance that admits neither a strong- nor a super-WSNM (Fig. 8). The empty matching is blocked by

$$m_1: \quad w_1 \qquad\qquad\qquad w_1: \quad (m_1 \quad m_2)$$
$$m_2: \quad w_1 \quad w_2 \qquad\qquad w_2: \quad m_2$$

Fig. 8 An instance that admits neither a strong-WSNM nor a super-WSNM

any acceptable pair. The matching $\{(m_1, w_1)\}$ is blocked by (m_2, w_2). The matching $\{(m_2, w_2)\}$ is blocked by (m_1, w_1). The matching $\{(m_1, w_1), (m_2, w_2)\}$ is blocked by (m_2, w_1). The matching $\{(m_2, w_1)\}$ is blocked by (m_1, w_1).

Nevertheless, the algorithm in Sect. 4.1 can be applied to SMTI straightforwardly. Necessary modifications are summarized as follows:

- As mentioned above, existence of a WSNM is not guaranteed. If our algorithm outputs $Y(n + 1, n + 1) = -\infty$, then it means that no solution exists.
- The definition of two edges (m_i, w_j) and (m_x, w_y) being conflicting must be modified depending on one of the three stability notions.
- The definitions of the tables A and B need to be modified as follows. $A(i, j', j)$ stores *one of* the women whom m_i most prefers among $\{w_{j'}, \dots, w_j, \lambda\}$. Similarly, $B(i', i, j)$ stores *one of* the men whom w_j most prefers among $\{m_{i'}, \dots, m_i, \lambda\}$.
- In the SMI case, $A(i, j', j)$ is computed as *the better* of $A(i, j', j - 1)$ and $A(i, j, j)$ in m_i's list. But now it can happen that $A(i, j', j - 1) =_{m_i} A(i, j, j)$, in which case $A(i, j', j)$ can be either $A(i, j', j - 1)$ or $A(i, j, j)$. (Strictly speaking, this treatment was needed already in the SMI case because there can be a case that $A(i, j', j - 1) = A(i, j, j) = \lambda$, but there we took simplicity.)
- Conditions 3 and 4 in checking confliction of two edges need to be modified as follows. In the super- and strong stabilities, "prefers" should be replaced by "weakly prefers". In the weak stability, "prefers" should be replaced by "strictly prefers".

With these modifications, whether two edges are conflicting or not can be checked in constant time. Therefore, we have the following corollary:

Corollary 3 *There exists an $O(n^4)$-time algorithm to find a maximum cardinality super-WSNM (strong-WSNM, weak-WSNM) or report that none exists, given an SMTI-instance.*

5 Conclusion

In this paper, we have shown algorithms and complexity results for the problems of determining existence of an SSNM and finding a maximum cardinality WSNM, in the settings both with and without ties.

One of interesting future works is to consider optimization problems. For example, in SMI we have shown that it is easy to determine if there exists an SSNM with zero-crossing. What is the complexity of the problem of finding an SSNM with the minimum number of crossings, and if it is NP-hard, is there a good approximation

algorithm for it? Also, it might be interesting to consider noncrossing stable matchings for other placements of agents, e.g., on a circle or on general position in 2-dimensional plane.

Availability of Data and Material Not applicable.

Declarations

Conflicts of interest The authors declare no conflicts of interest associated with this manuscript.

Code availability Not applicable.

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