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Singular dissipative stochastic equations in Hilbert spaces

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Abstract. Existence of solutions to martingale problems corresponding to singular dissipative stochastic equations in Hilbert spaces are proved for any initial condition. The solutions for the single starting points form a conservative diffusion process whose transition semigroup is shown to be strong Feller. Uniqueness in a generalized sense is proved also, and a number of applications is presented.

0. Introduction

The purpose of this paper is to construct weak solutions (i.e. solution of the corresponding martingale problem) to stochastic differential equations on a Hilbert space (norm $|\cdot|$, inner product $\langle\cdot,\cdot\rangle$) of type

$$\begin{cases} dX = (AX + F_0(X))dt + \sqrt{C}\, dW_t \\ X(0) = x \in H. \end{cases} \tag{0.1}$$

Here C is a positive definite bounded self–adjoint linear operator on H, $A : D(A) \subset H \to H$ the infinitesimal generator of a C_0 semigroup on H and

$$F_0(x) := y_0, \text{ where } y_0 \in F(x) \text{ such that } |y_0| = \min_{y\in F(x)} |y|, \; x \in D(F),$$

and

$$F : D(F) \subset H \to 2^H$$

is an m–dissipative map. We emphasize that the map $F_0 : D(F) \to H$ has no continuity properties in general.

Our strategy is based on first solving the Kolmogorov equations corresponding to (0.1) on an appropriate L^2–space, and then constructing a conservative diffusion process (i.e. a strong Markov process with continuous sample paths and infinite life time) having transition probabilities given by the solutions of the Kolmogorov equations.

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To be more precise, let us describe the three steps to implement this approach in more detail.

a) Solution of Kolmogorov equations on $L^2(H, \nu)$
(see Sections 1–4 below)

Let $\mathcal{E}_A(H)$ be the linear span of all (real parts of) functions of the form $\varphi = e^{i\langle h,\cdot\rangle}$, $h \in D(A^*)$, and define

$$N_0\varphi(x) := \frac{1}{2}\,\mathrm{Tr}\,[CD^2\varphi(x)] + \langle x, A^*D\varphi(x)\rangle + \langle F_0(x), D\varphi(x)\rangle, \quad \varphi \in \mathcal{E}_A(H).$$

Here D, D^2 denotes first, second Fréchet derivatives respectively.

Let ν be a probability measure on H such that

$$\int_H N_0\varphi d\nu = 0,\ \forall\, \varphi \in \mathcal{E}_A(H),$$

and set $H_0 := \mathrm{supp}\,\nu$. (We should note that recently there has been a lot of results on existence of such measures, called *infinitesimally invariant*, see e.g. V. Bogachev and M. Röckner [6]).

We prove that (under some conditions) the closure $(N_2, D(N_2))$ of $(N_0, \mathcal{E}_A(H))$ generates a Markovian C_0–semigroup

$$P_t = e^{tN_2},\ t \geq 0,$$

on $L^2(H, \nu)$, i.e. for all $\varphi \in L^2(H, \nu)$

$$\frac{d}{dt}\,P_t\varphi = N_2P_t\varphi,\ t \geq 0, \quad P_0\varphi = \varphi,$$

giving a solution to the Kolmogorov equations corresponding to (0.1) on $L^2(H, \nu)$. Furthermore, at least in the case where C^{-1} is bounded (but see also Remark 4.4 below), we have the following regularizing (in particular strong Feller) property: for all $\varphi \in L^\infty(H, \nu)$, $t > 0$, the $L^2(H, \nu)$–class $P_t\varphi$ has a Lipschitz–continuous ν–version.

b) Construction of corresponding strong Feller probability kernels
(see Section 5 below).

We show that there exist probability kernels $p_t,\ t > 0$, such that for all Borel measurable and bounded functions $\varphi : H \to \mathbb{R}$, $p_t\varphi$ is a Lipschitz continuous ν–version of $P_t\varphi$ on H_0. In particular, $(p_t)_{t>0}$ is strong Feller. Furthermore,

$$\lim_{t\to 0} p_t\varphi(x) = \varphi(x),\ \forall\, \varphi \in C_b^1(H),\ x \in H_0.$$

c) Construction of the diffusion weakly solving (0.1)
(see Sections 6,7 below).

As a consequence of b) there exists a canonical normal Markov process $\mathbb{M}^0 = (\Omega, \mathcal{F}^0, (\mathcal{F}_t^0)_{t\geq 0}, (X_t^0)_{t\geq 0}, (\mathbb{P}_x)_{x\in H_0})$ with $\Omega = H_0^{\mathbb{R}^+}$, $X_t^0 : \Omega \to H_0$ being the coordinate maps, $\mathcal{F}_t^0 := \sigma(X_s^0|\ s \leq t)$ and $\mathcal{F}^0 := \mathcal{F}_\infty^0$. We then prove that $\mathbb{M}^0$ has a modification with $\mathbb{P}_x$–a.s. continuous sample paths for all $x \in H_0$. This is done in two steps: first we show that for some modification the sample paths are continuous $\mathbb{P}_\nu$–a.s., where

$$\mathbb{P}_\nu := \int_H \mathbb{P}_x \nu(dx).$$

To this end we prove a general result verifying Kolmogorov's continuity criterion for $\mathbb{P}_\nu$ (see Theorem 6.3 below) based on the fact that N_0 is a diffusion operator (in the sense of e.g. Eberle [16, Appendix B]). Second, we employ a result due to J. Dohmann [15] that shows how one can use the strong Feller property to deduce continuity of sample paths $\mathbb{P}_x$–a.s. for all $x \in H_0$.

We want to really stress at this point that our situation is entirely different from the classical ones where the state space H is locally compact (i.e., in our case this is equivalent to $\dim H < \infty$). On locally compact spaces the standard process construction works if the semigroup maps C_∞ into C_∞, where C_∞ are the continuous functions vanishig at infinity. Only this way, one has control about right limits of sample paths and about what happens at infinity, i.e. outside any compact set. In our infinite dimensional situation, this notion makes no sense what so ever, and our transition semigroups map bounded functions into continuous functions which are merely bounded with no condition at "infinity", whatever the latter means.

It is well known that the diffusion, whose construction we have described above, constitutes a solution to the martingale problem given by (0.1) with test functions space

$$\{\varphi \in D(N_2) \cap C_b(H) |\ N_2\varphi \text{ bounded}\}.$$

(More precisely, it is a strong Markov selection of such solutions in the sense of Stroock and Varadhan, see [25, Section 12.2]).

So far, we have only discussed existence of a martingale solution of (0.1). However, our diffusion process is also unique in the sense that it is the (in distribution) unique conservative Feller diffusion, solving (0.1) in the above sense whose transition semigroup $(p_t)_{t>0}$ consists of continuous operators on $L^2(H, \nu)$. Details on this are contained in Section 8 below.

In Section 9 we discuss applications, in particular, the gradient case.

Finally, to recover a weak solution for (0.1) from the solution of the corresponding martingale problem is more or less standard provided $H = H_0$. With respect to the lenght of this paper we shall not give details here, but refer instead to the nice and coincise presentation in [24, Chapter 3.2] for the finite dimensional case and for the infinite dimensional case to [3, Section 6].

1. Notation and framework

Let H be a real separable Hilbert space (with norm $|\cdot|$ and inner product $\langle\cdot,\cdot\rangle$), and let $A : D(A) \subset H \to H$ and $C \in L(H)$ ([1]) be linear operators such that

Hypothesis 1.1. *(i) A is the infinitesimal generator of a strongly continuous semigroup e^{tA} in H. There exists $\omega > 0$ such that*

$$\langle Ax, x\rangle \le -\omega|x|^2, \quad \forall\, x \in H.$$

[1] $L(H)$ denotes the set of all bounded linear operators on H.

(ii) C is symmetric, nonnegative definite and such that $\mathrm{Tr}\, Q < +\infty$, where

$$Qx := \int_0^\infty e^{tA} C e^{tA^*} x dt, \quad x \in H$$

and A^* denotes the adjoint of A.

We denote by R_t the Ornstein–Uhlenbeck semigroup

$$R_t\varphi(x) := \int_H \varphi(e^{tA}x + y) N_{Q_t}(dy),$$

where

$$Q_t x := \int_0^t e^{sA} C e^{sA^*} x ds, \quad x \in H,$$

and N_{Q_t} is the Gaussian measure in H with mean 0 and covariance operator Q_t.

We shall denote by $C_{b,2}(H)$ the Banach space of all functions $\varphi : H \to \mathbb{R}$ having at most quadratic growth, that is $\frac{\varphi(\cdot)}{1+|\cdot|^2}$ is uniformly continuous and bounded. Endowed with the norm

$$\|\varphi\|_{b,2} := \sup_{x\in H} \frac{\varphi(x)}{1+|x|^2},$$

$C_{b,2}(H)$ is a Banach space. Moreover, $C^1_{b,2}(H)$ will represent the subspace of $C_{b,2}(H)$ of those functions φ that are continuously differentiable and such that

$$[\varphi]_{1,2} := \sup_{x\in H} \frac{|D\varphi(x)|}{1+|x|^2} < +\infty.$$

It is easy to see that R_t maps $C_{b,2}(H)$ (resp. $C^1_{b,2}(H)$) into itself for all $t \geq 0$.

Let us define the infinitesimal generator L of R_t through its resolvent by setting

$$R(\lambda, L)\varphi(x) = \int_0^{+\infty} e^{-\lambda t} R_t\varphi(x) dt, \; x \in H, \; \lambda > 0.$$

Then $R(\lambda, L)$ maps $C_{b,2}(H)$ (resp. $C^1_{b,2}(H)$) into itself for all $\lambda > 0$.

We set

$$D(L, C_{b,2}(H)) = R(\lambda, L)(C_{b,2}(H)),$$

and

$$D(L, C^1_{b,2}(H)) = R(\lambda, L)(C^1_{b,2}(H)).$$

One can easily show that

$$L\varphi = \frac{1}{2}\,\mathrm{Tr}\,[CD^2\varphi] + \langle x, A^* D\varphi\rangle, \forall\, \varphi \in \mathcal{E}_A(H),$$

where $\mathcal{E}_A(H)$ is the linear span of all (real parts of) functions of the form $\varphi(x) = e^{i\langle h,x\rangle}$ with $h \in D(A^*)$. Note that $\mathcal{E}_A(H) \subset D(L, C_{b,2}(H))$.

We are also given an m–dissipative mapping

$$F : D(F) \subset H \to 2^H.$$

This means that $D(F)$ is a Borel set in H and

$$\langle u - v, x - y\rangle \le 0, \quad \forall\, x, y \in D(F),\ u \in F(x),\ v \in F(y),$$

and Range $(I - F) := \bigcup_{x\in D(F)}(x - F(x)) = H$ (where obviously this union consists of disjoint sets). For any $x \in D(F)$ the set $F(x)$ is closed, non empty, and convex; we set

$$F_0(x) := y_0, \text{ where } y_0 \in F(x) \text{ such that } |y_0| = \min_{y\in F(x)} |y|,\ x \in D(F).$$

We are concerned with the differential operator

$$N_0\varphi := L\varphi + \langle F_0, D\varphi\rangle, \quad \varphi \in \mathcal{E}_A(H).$$

Our goal in the following section is to prove that the closure of N_0 is m–dissipative in $L^2(H, \nu)$, where ν is a suitable Borel measure on H such that ν is infinitesimally invariant, i.e.,

$$\int_H N_0\varphi d\nu = 0,\ \forall\, \varphi \in \mathcal{E}_A(H).$$

We note that, since N_0 is a diffusion operator, the latter always implies that (N_0, $\mathcal{E}_A(H)$) is dissipative on every $L^p(H, \nu)$, (see A. Eberle [16], Lemma 1.8, page 36, and also Proposition 2.1 below in the case $p = 2$). Hence it is, in particular, closable in $L^2(H, \nu)$.

Our main assumptions are the following.

Hypothesis 1.2. *There is a Borel probability measure ν on H such that*

(i) $\displaystyle\int_{D(F)} (|x|^{12} + |F_0(x)|^2 + |x|^4|F_0(x)|^2)\nu(dx) < +\infty.$

(ii) For all $\varphi \in \mathcal{E}_A(H)$ we have $N_0\varphi \in L^2(H, \nu)$ and

$$\int_H N_0\varphi\ d\nu = 0.$$

(iii) $\nu(D(F)) = 1.$

Remark 1.3. (i). For sufficient conditions of existence of infinitesimally invariant measures as in Hypothesis 1.2 we refer e.g. to [6, Sections 5 and 7] and also to Section 3 below.

(ii). We emphasize that $\int_{D(F)} |x|^{12}\nu(dx) < +\infty$ is only needed below in the proof of Theorem 6.3. Up to and including Section 5, $\int_{D(F)} |x|^4\nu(dx) < +\infty$ will be sufficient (see however Remark 7.5 below). In particular, our result on m–dissipativity of N_0 in $L^2(H, \nu)$ holds under this weaker assumption. We could study

m–dissipativity of N_0 in $L^p(H,\nu)$, $p \geq 1$. We should only change Hypothesis 1.2–(i) by assuming

$$\int_H (|x|^{2p} + |F_0(x)|^p + |x|^{2p}|F_0(x)|^p)\nu(dx) < +\infty.$$

(iii). In many cases (cfr. [4, the main result]) Hypothesis 1.2 implies that $\nu << N_Q$. For conditions implying supp $\nu = H$ see [2].

We finish this section by giving some preliminaries. We first recall that when $F : H \to H$ is dissipative and Lipschitz continuous, then the following result holds, see [10, Propositions 1.3 and 3.3]

Proposition 1.4. *Assume that $F : H \to H$ is dissipative and Lipschitz continuous. Then there is a unique Borel probability measure ν on H such that N_0 is dissipative in $L^2(H,\nu)$ and its closure N_2 is m–dissipative. If $C^{-1} \in L(H)$ then $\nu << N_Q$.*

Moreover the semigroup P_t generated by N_2 is given by

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t,x))],$$

where $X(t,x)$ is the solution of the stochastic differential equation

$$\begin{cases} dX = (AX + F(X))dt + \sqrt{C}dW_t \\ \\ X(0) = x \in H, \end{cases} \tag{1.1}$$

and W_t is a cylindrical Wiener process in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Let us introduce the Yosida approximations of F. For any $\alpha > 0$ we set

$$F_\alpha(x) := \frac{1}{\alpha}\,(J_\alpha(x) - x), \quad x \in H,$$

where

$$J_\alpha(x) := (I - \alpha F)^{-1}(x), \quad x \in H,\ \alpha > 0.$$

It is well known that

$$\begin{aligned} &\lim_{\alpha\to 0} F_\alpha(x) = F_0(x), \quad \forall\, x \in D(F). \\ \\ &|F_\alpha(x)| \leq |F_0(x)|, \quad \forall\, x \in D(F). \end{aligned} \tag{1.2}$$

Moreover, F_α is Lipschitz continuous (but not differentiable in general), so F_0 is Borel measurable. Therefore, we introduce a further regularization by setting

$$F_{\alpha,\beta}(x) = \int_H e^{\beta B} F_\alpha(e^{\beta B}x + y) N_{\frac{1}{2}\,B^{-1}(e^{2\beta B}-1)}(dy), \quad \alpha, \beta > 0, \tag{1.3}$$

where $B : D(B) \subset H \to H$ is a self–adjoint negative definite operator such that B^{-1} is of trace class.

$F_{\alpha,\beta}$ is dissipative, of class C^∞, and has bounded derivatives of all orders, and $F_{\alpha,\beta} \to F_\alpha$ pointwise, see [13, Theorem 9.19].

2. *m*–dissipativity of N_0

We assume here that Hypotheses 1.1 and 1.2 hold.

Proposition 2.1. *For all* $\varphi \in \mathcal{E}_A(H)$ *we have*

$$\int_H N_0\varphi\, \varphi\, d\nu = -\frac{1}{2}\int_H |C^{1/2}D\varphi|^2\, d\nu. \tag{2.1}$$

Consequently, N_0 *is dissipative in* $L^2(H,\nu)$*.*

Proof. Since

$$N_0(\varphi^2) = 2\varphi N_0\varphi + |C^{1/2}D\varphi|^2, \quad \forall\, \varphi \in \mathcal{E}_A(H),$$

the conclusion follows integrating with respect to ν and using Hypothesis 1.2–(ii). □

Since N_0 is dissipative, it is closable in $L^2(H,\nu)$. (Here we recall that since obviously $\mathcal{E}_A(H)$ contains a countable subset separating the points of H, $\mathcal{E}_A(H)$ is dense in $L^2(H,\nu)$ by a monotone class argument.) We shall denote by N_2 its closure and by $D(N_2)$ its domain. We are going to show that N_2 is *m*–dissipative.

Lemma 2.2. *Let* $\varphi \in D(L, C^1_{b,2}(H))$*. Then there exists* $\varphi_{\overline{n}} \in \mathcal{E}_A(H)$*,* $\overline{n} \in \mathbb{N}^4$*, such that for some* $c_1 \in (0,\infty)$

$$|\varphi_{\overline{n}}(x)| + |D\varphi_{\overline{n}}(x)| \le c_1(1+|x|^2), \ \forall\, \overline{n} \in \mathbb{N}^4$$

and $\varphi_{\overline{n}}(x) \to \varphi(x)$*,* $D\varphi_{\overline{n}}(x) \to D\varphi(x)$ *for all* $x \in H$ *and* $\varphi_{\overline{n}} \to \varphi$ *in* N_2*–graph norm* ([2])*. Consequently*

$$D(L, C^1_{b,2}(H)) \subset D(N_2).$$

Furthermore, for all $\varphi \in D(L, C^1_{b,2}(H))$ *we have*

$$N_2\varphi = L\varphi + \langle F_0(x), D\varphi\rangle. \tag{2.2}$$

Proof. Let $\varphi \in D(L, C^1_{b,2}(H))$. Then, by [12, Proposition 2.5], there exists a sequence $\{\varphi_{\overline{n}}\} = \{\varphi_{n_1,n_2,n_3,n_4}\} \subset \mathcal{E}_A(H)$ such that, for some constant $c_1 > 0$,

$$\varphi_{\overline{n}}(x) \to \varphi(x),\ \ L\varphi_{\overline{n}}(x) \to L\varphi(x),\ \ D\varphi_{\overline{n}}(x) \to D\varphi(x),\ \forall\, x \in H.$$

$$|\varphi_{\overline{n}}(x)| + |L\varphi_{\overline{n}}(x)| + |D\varphi_{\overline{n}}(x)| \le c_1(1+|x|^2),\ \forall\, x \in H,\ \overline{n} \in \mathbb{N}^4.$$

It follows that

$$N_0\varphi_{\overline{n}}(x) = L\varphi_{\overline{n}}(x) + \langle F_0(x), D\varphi_{\overline{n}}(x)\rangle$$

$$\to L\varphi(x) + \langle F_0(x), D\varphi(x)\rangle,\ \forall\, x \in D(F).$$

[2] We set $\overline{n} = (n_1,n_2,n_3,n_4)$ and $\displaystyle\lim_{\overline{n}\to\infty} = \lim_{n_1\to\infty}\lim_{n_2\to\infty}\lim_{n_3\to\infty}\lim_{n_4\to\infty}$

There is $c_2 > 0$ such that for all $x \in D(F)$

$$|N_0\varphi_{\overline{n}}(x)| \le c_2(1+|x|^2+|F_0(x)|+|F_0(x)||x|^2),\ \forall\, \overline{n} \in \mathbb{N}^4.$$

By Hypothesis 1.2–(i) it follows that the right hand side is in $L^2(H,\nu)$. Consequently,

$$N_0\varphi_{\overline{n}} \to L\varphi(x) + \langle F_0(x), D\varphi\rangle \text{ in } L^2(H,\nu),$$

and $\varphi \in D(N_2)$ as claimed. □

Let us consider the approximating equation

$$\lambda\varphi_{\alpha,\beta} - L\varphi_{\alpha,\beta} - \langle F_{\alpha,\beta}, D\varphi_{\alpha,\beta}\rangle = f,\quad \alpha,\beta > 0. \tag{2.3}$$

where $\lambda > 0$ and $f \in C_b^2(H)$. ([3])

It is not difficult to see that equation (2.3) has a unique solution $\varphi_{\alpha,\beta} \in D(L, C^1_{b,2}(H)) \cap C_b^2(H)$ given by

$$\varphi_{\alpha,\beta}(x) = \int_0^{+\infty} e^{-\lambda t}\mathbb{E}[f(X_{\alpha,\beta}(t,x))]dt, \tag{2.4}$$

where $X_{\alpha,\beta}(\cdot,x)$ is the solution to problem (1.1) with F replaced by $F_{\alpha,\beta}$. We have moreover for all $h \in H$,

$$\langle D\varphi_{\alpha,\beta}(x), h\rangle = \int_0^{+\infty} e^{-\lambda t}\mathbb{E}[\langle Df(X_{\alpha,\beta}(t,x)), D_xX_{\alpha,\beta}(t,x)h\rangle]dt. \tag{2.5}$$

For any $h \in H$ we set $\eta^h_{\alpha,\beta} := D_xX_{\alpha,\beta}(t,x)$. Then we have (in the mild sense)

$$\begin{cases} \dfrac{d}{dt}\,\eta^h_{\alpha,\beta}(t,x) = A\eta^h_{\alpha,\beta}(t,x) + DF_{\alpha,\beta}(X_{\alpha,\beta}(t,x))\eta^h_{\alpha,\beta}(t,x) \\[2mm] \eta^h_{\alpha,\beta}(0,x) = h. \end{cases} \tag{2.6}$$

Multiplying both sides of equation (2.6) by $\eta^h_{\alpha,\beta}(t,x)$, integrating with respect to t and taking into account the dissipativity of $DF_{\alpha,\beta}$, we find

$$|\eta^h_{\alpha,\beta}(t,x)|^2 \le 2\int_0^t \langle A\eta^h_{\alpha,\beta}(s,x), \eta^h_{\alpha,\beta}(s,x)\rangle ds + |h|^2. \tag{2.7}$$

This argument is a bit informal (realize that in general $\eta^h_{\alpha,\beta}(t,x) \notin D(A)$), but it can be made rigorous by using the Yosida approximation, see e.g. [7, Proof of Proposition 6.2.2]. Now, recalling Hypothesis 1.1–(i), we have

$$\|D_xX_{\alpha,\beta}(t,x)\| \le e^{-\omega t},\ t \ge 0. \tag{2.8}$$

Consequently by (2.5) it follows that

$$|D\varphi_{\alpha,\beta}(x)| \le \frac{1}{\lambda}\,\|f\|_1,\ x \in H. \tag{2.9}$$

Now we can prove the following result.

[3] $C_b^2(H)$ is the space of all functions $\varphi : H \to \mathbb{R}$ that are uniformly continuous and bounded together with their first and second derivatives.

Theorem 2.3. *Under Hypotheses 1.1 and 1.2, N_2 is m–dissipative in $L^2(H, \nu)$.*

Proof. Let $f \in C_b^2(H)$ and let $\varphi_{\alpha,\beta}$ be the solution to equation (2.3). Then by Lemma 2.2 we know that $\varphi_{\alpha,\beta} \in D(N_2)$ and we have

$$\lambda\varphi_{\alpha,\beta} - N_2\varphi_{\alpha,\beta} = f + \langle F_{\alpha,\beta} - F_0, D\varphi_{\alpha,\beta}\rangle. \tag{2.10}$$

We claim that

$$\lim_{\alpha\to 0}\lim_{\beta\to 0}\langle F_{\alpha,\beta} - F_0, D\varphi_{\alpha,\beta}\rangle = 0 \ \text{ in } L^2(H, \nu).$$

In fact by (2.9) it follows that

$$\begin{aligned} I_{\alpha,\beta} &:= \int_H |\langle F_{\alpha,\beta} - F_0, D\varphi_{\alpha,\beta}\rangle|^2 d\nu \\ &\le \frac{1}{\lambda^2}\ \|f\|_1^2 \int_H |F_{\alpha,\beta} - F_0|^2 d\nu. \end{aligned} \tag{2.11}$$

Now, since for fixed $\alpha > 0$, $F_{\alpha,\beta}$ is Lipschitz continuous with a Lipschitz constant that can be choosen independent of β, we see that for any $\alpha > 0$ there is $c_\alpha > 0$ such that

$$|F_{\alpha,\beta}(x)| \le c_\alpha(1 + |x|),\ \ x \in H,$$

and so

$$\limsup_{\beta\to 0} I_{\alpha,\beta} \le \frac{1}{\lambda^2}\ \|f\|_1^2 \int_H |F_\alpha - F_0|^2 d\nu.$$

Now the claim follows, in view of the dominated convergence theorem, from (1.2) and Hypothesis 1.2–(iii).

In conclusion we have proved that

$$\lim_{\alpha\to 0}\lim_{\beta\to 0}(\lambda - N_2)\varphi_{\alpha,\beta} = f \ \text{ in } L^2(H, \nu).$$

Therefore the closure of the range of $\lambda - N_2$ includes $C_b^2(H)$ which is dense in $L^2(H, \nu)$. By the Lumer–Phillips theorem it follows that N_2 is m–dissipative as required. □

As a consequence of the proof of Theorem 2.3 we have:

Corollary 2.4. *Let $f \in C_b^2(H)$, $\lambda > 0$. Then there exist $\varphi_n \in D(L, C_{b,2}^1(H)) \cap C_b^2(H)$, $n \in \mathbb{N}$, such that $\varphi_n \to R(\lambda, N_2)f$ as $n \to \infty$ in $L^2(H, \nu)$ and*

$$\sup_n \int_H |N_2\varphi_n|^2 d\nu < +\infty$$

and

$$\sup_n \sup_{x\in H} (|D\varphi_n(x)| + |\varphi_n(x)|) < \infty.$$

Here $R(\lambda, N_2) := (\lambda - N_2)^{-1}$.

Let

$$P_t = e^{tN_2}, \quad t > 0,$$

be the C_0–semigroup generated by N_2 on $L^2(H, \nu)$ (which exists by Theorem 2.3).

Corollary 2.5. *$(P_t)_{t \geq 0}$ is Markovian, i.e. $P_t 1 = 1$ and $P_t f \geq 0$ for all nonnegative $f \in L^2(H, \nu)$ and all $t > 0$.*

Proof. By A. Eberle [16, Appendix B, Lemma 1.9] P_t is positivity preserving. Since $1 \in \mathcal{E}_A(H)$ and $N_0 1 = 0$, it follows that $P_t 1 = 1$. □

3. Construction of an infinitesimally invariant measure ν

We assume here that Hypothesis 1.1 holds, and consider an m–dissipative mapping $F : D(F) \subset H \to 2^H$.

For any $\alpha > 0$ we consider the Kolmogorov operator ([4])

$$N_\alpha \varphi := L\varphi + \langle F_\alpha, D\varphi \rangle, \quad \varphi \in \mathcal{E}_A(H). \tag{3.1}$$

By Proposition 1.4 we know that there exists a unique probability measure ν_α on H such that N_α is dissipative in $L^2(H, \nu_\alpha)$ and its closure is m–dissipative.

Moreover, the corresponding semigroup P_t^α is given by

$$P_t^\alpha \varphi(x) = \mathbb{E}[\varphi(X_\alpha(t, x))],$$

where $X_\alpha(t, \cdot)$ is the solution of the equation

$$X_\alpha(t, x) = e^{tA}x + \int_0^t e^{(t-s)A} F_\alpha(X_\alpha(s, x))ds + W_A(t), \tag{3.2}$$

and

$$W_A(t) = \int_0^t e^{(t-s)A} \sqrt{C}\, dW(s). \tag{3.3}$$

Our goal is to show that, under additional assumptions, the sequence ν_α is tight and that any weak limit ν fulfills Hypothesis 1.2.

We start with an a–priori estimate.

Lemma 3.1. *Assume, besides Hypothesis 1.1, that for some $m \in \mathbb{N}$ there is $k(m) \geq m$ and $c_m > 0$ such that for any $\alpha > 0$*

$$\mathbb{E}|F_\alpha(W_A(t))|^{2m} \leq c_m t^{k(m)}, \ t \geq 0. \tag{3.4}$$

Then there is $c_{1,m} > 0$ and an integer $h(m)$ such that

$$\mathbb{E}|X_\alpha(t, x)|^{2m} \leq c_{1,m} t^{h(m)} (1 + e^{-m\omega t}|x|^{2m}). \tag{3.5}$$

[4] Here we could consider instead $N_{\alpha,\beta}$, but this does not seem to be necessary.

Proof. Setting $Y(t) = X_\alpha(t,x) - W_A(t)$, $Y(t)$ is the solution to

$$\begin{cases} Y'(t) = AY(t) + F_\alpha(Y(t) + W_A(t)) \\ \\ Y(0) = x. \end{cases} \tag{3.6}$$

Multiplying the first equation by $|Y(t)|^{2m-2}Y(t)$ and taking into account Hypothesis 1.1–(i) and the dissipativity of F_α, for a suitable constant $c_{2,m}$ we obtain

$$\frac{1}{2m}\,\frac{d}{dt}\,|Y(t)|^{2m} \le -\omega|Y(t)|^{2m} + \langle F_\alpha(W_A(t)), Y(t)\rangle|Y(t)|^{2m-2}$$

$$+ \langle F_\alpha(Y(t) + W_A(t)) - F_\alpha(W_A(t)), Y(t)\rangle|Y(t)|^{2m-2}$$

$$\le -\omega|Y(t)|^{2m} + |F_\alpha(W_A(t))|\;|Y(t)|^{2m-2}$$

$$\le -\frac{\omega}{2}\,|Y(t)|^{2m} + c_{2,m}F_\alpha(W_A)|^{2m}.$$

By the Gronwall lemma it follows that

$$|Y(t)|^{2m} \le e^{-m\omega t}|x|^{2m} + 2mc_{2,m}\int_0^t e^{-m\omega(t-s)}|F_\alpha(W_A(s))|^{2m}ds,$$

and finally, for some $c_{3,m}$

$$|X_\alpha(t,x)|^{2m} \le c_{3,m}e^{-m\omega t}|x|^{2m}$$

$$+ c_{3,m}\left(\int_0^t e^{-m\omega(t-s)}|F_\alpha(W_A(s))|^{2m}ds + |W_A(t)|^{2m}\right).$$

Now the conclusion follows taking expectation since $\mathbb{E}|W_A(t)|^{2m} \le ct^{\tilde{k}(m)}$ for some integer $\tilde{k}(m)$. □

Corollary 3.2. *Under the assumptions of Lemma 3.1 there is* $k_{1,m} > 0$ *such that*

$$\int_H |x|^{2m}\nu_\alpha(dx) \le k_{1,m}. \tag{3.7}$$

Proof. Integrating (3.5) with respect to ν_α and taking into account the invariance of ν_α gives

$$\int_H |x|^{2m}\nu_\alpha(dx) \le c_{1,m}t^{k(m)}(1 + e^{-m\omega t}\int_H |x|^{2m}\nu_\alpha(dx)). \tag{3.8}$$

Choose $t_0 > 0$ such that

$$c_{1,m}t_0^{k(m)}e^{-m\omega t_0} < 1,$$

then, setting in (3.8) $t = t_0$ yields (3.7). □

To prove tightness of ν_α we shall assume that A is a variational operator $A : V \to V'$ with $V \subset H \subset V'$ with a compact embedding $V \subset H$, and that there exists $\kappa > 0$ such that

$$\langle Ax, x\rangle \le -\kappa \|x\|_V^2,\ x \in D(A). \tag{3.9}$$

Proposition 3.3. *Assume that the assumptions of Lemma 3.1 hold, that A is variational as above and, in addition, that there is* $\delta \in (0, 1/2)$ *and* $c_\delta > 0$ *such that*

$$\mathbb{E}|W_A(t)|^2_{D(-A)^\delta} \le c_\delta t^\delta,\ t \ge 0. \tag{3.10}$$

Then there is $c_{1,\delta} > 0$ *such that*

$$\int_H |x|^2_{D(-A)^\delta} \nu_\alpha(dx) \le c_{1,\delta}. \tag{3.11}$$

Therefore, ν_α *are tight.*

Proof. Proceeding as in the proof of Lemma 3.1 we obtain

$$\frac{1}{2}\,\frac{d}{dt}\,|Y_\alpha(t)|^2 + \kappa\|Y_\alpha(t)\|_V^2 \le |Y_\alpha(t)||F_\alpha(W_\alpha(t))|.$$

Let $\lambda_0 > 0$ be such that $|x| \le \lambda_0\|x\|_V$. Then we have

$$\frac{1}{2}\,\frac{d}{dt}\,|Y_\alpha(t)|^2 + \frac{\kappa}{2}\,\|Y_\alpha(t)\|_V^2 \le \frac{\lambda_0^2}{2\kappa}\,|F_\alpha(W_\alpha(t))|^2.$$

It follows that

$$|Y_\alpha(t)|^2 + \kappa\int_0^t \|Y_\alpha(s)\|_V^2 ds \le |x|^2 + \frac{\lambda_0^2}{\kappa}\int_0^t |F_\alpha(W_\alpha(s))|^2 ds,$$

and so there is $c_1 > 0$ such that

$$\int_0^t |Y_\alpha(s)|^2_{D(-A)^\delta} ds \le c_1\left(|x|^2 + \frac{\lambda_0^2}{\kappa}\int_0^t |F_\alpha(W_\alpha(s))|^2 ds\right).$$

Consequently, there exists $c(t) > 0$ such that

$$\int_0^t \mathbb{E}|X_\alpha(s)|^2_{D(-A)^\delta} ds \le c(t)(1 + |x|^2).$$

Now we fix $t_0 > 0$ and by the invariance of ν_α we find for a constant c'

$$\int_H |x|^2_{D(-A)^\delta} \nu_\alpha(dx) \le c'\left(1 + \int_H |x|^2 \nu_\alpha(dx)\right),$$

and the conclusion follows. □

Remark 3.4. Let ν be a cluster point of ν_α. To check Hypothesis 1.2 it remains to show that

(i) There exists $a > 0$ such that

$$\int_H |F_0(x)|^{2+a} \nu(dx) < +\infty. \tag{3.12}$$

(ii) We have

$$\lim_{\alpha\to 0}\int_H \langle F_\alpha, D\varphi\rangle d\nu_\alpha = \int_H \langle F_0, D\varphi\rangle d\nu,\ \forall\, \varphi\in \mathcal{E}_A(H). \tag{3.13}$$

In fact by (3.7), (3.12) and the Hölder inequality it follows that Hypothesis 1.2–(i) is fulfilled. Moreover by (3.13) it easily follows that $\int_H N_0\varphi d\nu = 0$ for all $\varphi\in\mathcal{E}_A(H)$.

A sufficient condition (fulfilled for reaction–diffusion equations) for (3.13) is the following

$$\begin{aligned}&x\to\langle h, F_0(x)\rangle \text{ is continuous } \forall\, h\in D(A^*) \text{ and}\\ &|F_0(x)-F_\alpha(x)|\le \alpha|G(x)|,\end{aligned}$$

with $G : H\to\mathbb{R}$ Borel measurable such that $\sup_{\alpha>0}\int_H |G(x)|d\nu_\alpha\le c$.

4. Strong Feller properties for the operator resolvent

We assume here that Hypotheses 1.1 and 1.2 are fulfilled. We denote by $X_{\alpha,\beta}$ the solution of the following stochastic differential equation,

$$\begin{cases} dX_{\alpha,\beta} = (AX_{\alpha,\beta} + F_{\alpha,\beta}(X_{\alpha,\beta}))dt + \sqrt{C}dW_t\\[2ex] X_{\alpha,\beta}(0) = x\in H,\end{cases} \tag{4.1}$$

and by $P_t^{\alpha,\beta}$ the transition semigroup

$$P_t^{\alpha,\beta}\varphi(x) = \mathbb{E}[\varphi(X_{\alpha,\beta}(t,x))].$$

Then $P_t^{\alpha,\beta}$ is strong Feller (see the proof of Proposition 4.3 below). We set moreover

$$N_0^{\alpha,\beta}\varphi = L\varphi + \langle F_{\alpha,\beta}(x), D\varphi\rangle,\quad \varphi\in\mathcal{E}_A(H).$$

By Proposition 1.4 there exists a unique invariant probability measure $\nu_{\alpha,\beta}$ for $P_t^{\alpha,\beta}$, so that we can extend the semigroup $P_t^{\alpha,\beta}$ to $L^2(H,\nu_{\alpha,\beta})$. Moreover its infinitesimal generator $N_2^{\alpha,\beta}$ is precisely the closure of $N_0^{\alpha,\beta}$ in $L^2(H,\nu_{\alpha,\beta})$.

We denote the set of bounded Lipschitz functions on H by $Lip_b(H)$ and $\|\cdot\|_{Lip}$ denotes the Lipschitz norm.

Below we need a particular $\nu_{\alpha,\beta}$–version of $R(\lambda, N_2^{\alpha,\beta})f$, namely

$$\int_0^{+\infty} e^{-\lambda t}P_t^{\alpha,\beta}f(x)dt,\quad x\in H,$$

which we denote again by $R(\lambda, N_2^{\alpha,\beta})f$.

Proposition 4.1. *Let $\lambda>0$ and $f\in Lip_b(H)$. Then*

$$\|R(\lambda,N_2)f - R(\lambda,N_2^{\alpha,\beta})f\|_{L^2(H,\nu)}\le \frac{1}{\lambda}\ \|f\|_{Lip}\|\ |F_{\alpha,\beta}-F_0|\ \|_{L^2(H,\nu)}. \tag{4.2}$$

In particular,

$$\lim_{\alpha\to 0}\lim_{\beta\to 0} R(\lambda, N_2^{\alpha,\beta})f = R(\lambda, N_2)f \quad \text{in } L^2(H,\nu).$$

Proof. Since f can be approximated pointwise by uniformly bounded functions $f_n \in C_b^\infty(H)$ such that their first derivatives are bounded by $\|f\|_{Lip}$ we may assume that $f \in C_b^2(H)$.

Let $\varphi_{\alpha,\beta}$ be the solution of the equation

$$\lambda\varphi_{\alpha,\beta} - L\varphi_{\alpha,\beta} - \langle F_{\alpha,\beta}, D\varphi_{\alpha,\beta}\rangle = f. \tag{4.3}$$

By Lemma 2.2 we can write

$$\lambda\varphi_{\alpha,\beta} - N_2\varphi_{\alpha,\beta} = f + \langle F_{\alpha,\beta} - F_0, D\varphi_{\alpha,\beta}\rangle.$$

Consequently,

$$\varphi_{\alpha,\beta} = R(\lambda, N_2)[f + \langle F_{\alpha,\beta} - F_0, D\varphi_{\alpha,\beta}\rangle].$$

Now the assertion follows by (2.9), (2.11) and the proof of Theorem 2.3. □

Remark 4.2. Since $P_t^{\alpha,\beta}$ are only bounded on $L^2(H,\nu_{\alpha,\beta})$ and not in $L^2(H,\nu)$, it is not clear to us whether they converge to P_t in the sense of Proposition 4.1.

Proposition 4.3. *Assume that $C^{-1} \in L(H)$ and let $\lambda > 0$. Then $R(\lambda, N_2)$ is strong Feller. More precisely, let $f : H \to \mathbb{R}$ be bounded and Borel measurable, then for ν–a.e. $x, y \in H$*

$$|R(\lambda, N_2)f(x) - R(\lambda, N_2)f(y)| \le (\lambda/\pi)^{-1/2}\|C^{-1}\|^{1/2}\ \|f\|_0|x-y|, \tag{4.4}$$

where $\|\cdot\|_0$ denotes the supremum norm.

Proof. Let us first recall the Bismut–Elworthy formula,

$$\langle DP_t^{\alpha,\beta}f(x), h\rangle = \frac{1}{t}\,\mathbb{E}\left[f(X_{\alpha,\beta}(t,x))\int_0^t \langle C^{-1/2}\eta^h_{\alpha,\beta}(s,x), dW(s)\rangle\right], \tag{4.5}$$

where $h \in H$ and $\eta^h_{\alpha,\beta} = DX_{\alpha,\beta}\cdot h$ is the solution to (2.6).

By using the Hölder inequality we find

$$|\langle DP_t^{\alpha,\beta}f(x), h\rangle|^2 = \frac{1}{t^2}\ \|f\|_0^2\mathbb{E}\left[\int_0^t |C^{-1/2}\eta^h_{\alpha,\beta}(s,x)|^2 ds\right]. \tag{4.6}$$

Now by (2.8) and Hypothesis 1.1–(i), we have

$$|\eta^h_{\alpha,\beta}(t,x)|^2 \le |h|^2.$$

We deduce from (4.6) that

$$|\langle DP_t^{\alpha,\beta}f(x), h\rangle|^2 \le \frac{1}{t}\ \|C^{-1}\|\ \|f\|_0^2|h|^2,$$

that yields

$$|P_t^{\alpha,\beta} f(x) - P_t^{\alpha,\beta} f(y)| \le t^{-1/2} \|C^{-1}\|^{1/2} \|f\|_0 |x-y|, \quad x, y \in H. \tag{4.7}$$

Multiplying with $e^{-t\lambda}$ and integrating with respect to t we obtain the assertion for $R(\lambda, N_2^{\alpha,\beta})$ replacing $R(\lambda, N_2)$. Hence, if $f \in Lip_b(H)$, Proposition 4.1 implies (4.4). Since every bounded, Borel measurable $f : H \to \mathbb{R}$ can be approximated in $L^2(H, \nu)$ by $f_n \in Lip_b(H)$ such that $\|f_n\|_0 \le \|f\|_0 + \varepsilon$ for any $\varepsilon > 0$, we obtain the result. □

Remark 4.4. As will become clear later, (4.7) is crucial in subsequent sections. This is the main reason why $C^{-1} \in L(H)$ is assumed in subsequent sections. In fact, except forTheorem 7.4 (where $C^{-1} \in L(H)$ is used for other reasons), it would be sufficient to assume (4.7) with $\|C^{-1}\|$ replaced by any positive constant, to hold in all those places. We therefore emphasize that, following S. Cerrai [7, Proposition 8.3.3], we can prove such an inequality also in some cases when $C^{-1} \notin L(H)$.

Assume for instance that A is self–adjoint and that

$$C = (-A)^{-\gamma}, \ \text{for some } \gamma \in (0, 1].$$

Then by (2.7) we deduce that

$$\int_0^t |(-A)^{1/2} \eta_{\alpha,\beta}^h(s,x)|^2 ds \le |h|^2.$$

Since

$$C^{-1/2} = (-A)^{-(1-\gamma)/2}(-A)^{1/2},$$

we deduce that

$$\int_0^t |(-C)^{-1/2} \eta_{\alpha,\beta}^h(s,x)|^2 ds \le \|(-A)^{-(1-\gamma)/2}\|^2 |h|^2.$$

Consequently

$$|\langle DP_t^{\alpha,\beta} f(x), h\rangle| = \frac{1}{\sqrt{t}} \ \|f\|_0 \|(-A)^{-(1-\gamma)/2}\| |h|,$$

which still yields (4.7), with $\|(-A)^{-(1-\gamma)/2}\|$ replacing $\|C^{-1}\|^{1/2}$.

Proposition 4.5. *Let $\varphi \in Lip_b(H)$, $\lambda > 0$. Then for ν–a.e. $x, y \in H$*

$$|R(\lambda, N_2)\varphi(x) - R(\lambda, N_2)\varphi(y)| \le \lambda^{-1} \|\varphi\|_{Lip} |x-y|.$$

Proof. By the same argument as in the proof of Proposition 4.1 we may assume that $\varphi \in C_b^1(H)$. Let us prove that

$$|P_t^{\alpha,\beta}\varphi(x) - P_t^{\alpha,\beta}\varphi(y)| \le \|\varphi\|_1 |x-y|, \ \forall\, \varphi \in C_b^1(H). \tag{4.8}$$

But

$$P_t^{\alpha,\beta}\varphi(x) = \mathbb{E}\left[\varphi(X_{\alpha,\beta}(t,x))\right],$$

and for any $h \in H$,

$$\langle DP_t^{\alpha,\beta}\varphi(x), h\rangle = \mathbb{E}\left[\langle D\varphi(X_{\alpha,\beta}(t,x)), DX_{\alpha,\beta}(t,x)\cdot h\rangle\right].$$

Since

$$\|DX_{\alpha,\beta}(t,x)\| \le e^{-\omega t},\ t \ge 0,$$

we find

$$|\langle DP_t^{\alpha,\beta}\varphi(x), h\rangle| \le e^{-\omega t}\|\varphi\|_{Lip}|h|,$$

that yields (4.8) since $\omega > 0$.

Multiplying (4.8) by $e^{-t\lambda}$, integrating over to t, and letting $\beta \to 0$ and then $\alpha \to 0$ we obtain the assertion. □

5. Strong Feller probability kernels

Assume throughout this section that $C^{-1} \in L(H)$ (or more generally that (4.7) holds, see Remark 4.4) and that Hypotheses 1.1 and 1.2 are fulfilled.

5.1. Resolvents

For a topological space X we denote its Borel σ–algebra by $\mathcal{B}(X)$ and by $B_b(X)$ the set of all $f : X \to \mathbb{R}$, which are Borel measurable and bounded.

Define $H_0 := \text{supp}\ \nu$.

Lemma 5.1. *Let $\lambda > 0$ and $f \in B_b(H)$. Then $R(\lambda, N_2)f$ has a ν– version $\widetilde{R(\lambda, N_2)}f$, unique on H_0, such that for all $x, y \in H_0$*

$$|\widetilde{R(\lambda, N_2)}f(x) - \widetilde{R(\lambda, N_2)}f(y)| \le (\lambda/\pi)^{-1/2}\|C^{-1}\|^{1/2}\ \|f\|_0|x-y|. \quad (5.1)$$

Furthermore, if $g \in B_b(H)$ is such that $f = g$ ν–a.e., then

$$\widetilde{R(\lambda, N_2)}f(x) = \widetilde{R(\lambda, N_2)}g(x),\ \forall\, x \in H.$$

Proof. By Proposition 4.3, $R(\lambda, N_2)f$ has a ν–version satisfying the estimate in Proposition 4.3 for all x, y in a dense subset of H_0. Defining $\widetilde{R(\lambda, N_2)}f$ as the continuous extension to all of H_0 of this version we obtain the desired function satisfying (5.1).

Since any other ν–version of $R(\lambda, N_2)f$ satisfying (5.1) coincides with the one just constructed ν–a.s., hence on a dense subset of H_0, we have uniqueness of such a version.

Finally, if $f = g$ ν–a.e., then

$$\widetilde{R(\lambda, N_2)}f(x) = \widetilde{R(\lambda, N_2)}g(x),\ \text{for } \nu \text{ a.e. } x \in H,$$

hence as above for all $x \in H_0$. □

Define for $f \in B_b(H)$ and $\lambda > 0$,

$$R_\lambda f(x) := \widetilde{R(\lambda, N_2)}f(x),\ x \in H_0. \quad (5.2)$$

Proposition 5.2. $(R_\lambda)_{\lambda>0}$ *defined in* (5.2) *is a resolvent of kernels from* $(H_0, \mathcal{B}(H_0))$ *to* $(H, \mathcal{B}(H))$ *such that* $\lambda R_\lambda 1(x) = 1$ *for all* $x \in H_0$*. Furthermore, for all* $\varphi \in Lip_b(H)$, $\lambda > 0$,

$$|\lambda R_\lambda \varphi(x) - \lambda R_\lambda \varphi(y)| \leq \|\varphi\|_{Lip}|x-y|, \ \forall\, x, y \in H_0, \tag{5.3}$$

and hence

$$\lim_{\lambda\to\infty} \lambda R_\lambda \varphi(x) = \varphi(x), \ \forall\, x \in H_0.$$

Furthermore, each R_λ *satisfies* (5.1)*, so is in particular strong Feller.*

Proof. For two continuous functions $f, g : H_0 \to \mathbb{R}$, $f \leq g\nu$–a.e. implies that $f(x) \leq g(x)$ for all $x \in H_0$. Hence it follows that $f \to R_\lambda f(x)$ is linear and positive on $B_b(H)$ for all $x \in H_0$ because of the corresponding properties of $f \to R(\lambda, N_2) f$. By the same argument

$$R_\lambda - R_\alpha = (\alpha - \lambda) R_\lambda R_\alpha, \ \forall\, \alpha, \lambda > 0.$$

Now we want to show that for all $\lambda > 0$, and $f_n \in B_b(H)$, $n \in \mathbb{N}$, we have

$$f_n(x) \downarrow 0 \text{ as } n \to \infty \ \forall\, x \in H \quad \Rightarrow \lim_{n\to\infty} R_\lambda f_n(x) = 0 \ \forall\, x \in H_0.$$

Since $R_\lambda f_{n_k} \to 0$ ν–a.e. for some subsequence and $R_\lambda f_n(x)$ is decreasing for all $x \in H_0$, it follows that

$$A := \left\{x \in H_0 : \lim_{n\to\infty} R_\lambda f_n(x) = 0\right\}$$

has ν measure equal to 1. Hence A is dense in H_0. Since $\{R_\lambda f_n |\, n \in \mathbb{N}\}$ is by Lemma 5.1 equicontinuous it follows that

$$\lim_{n\to\infty} R_\lambda f_n(x) = 0 \ \forall\, x \in H_0.$$

Furthermore, $\lambda R_\lambda 1(x) = 1$ for ν–a.e. $x \in H$, hence as above for all $x \in H_0$. So, the first part of the assertion follows.

Furthermore, let $\varphi \in Lip_b(H)$. Then by Proposition 4.5

$$|\lambda R_\lambda \varphi(x) - \lambda R_\lambda \varphi(y)| \leq \|\varphi\|_{Lip}|x-y|$$

for ν–a.e. $x, y \in H_0$ and all $\lambda > 0$. Hence (5.3) follows. Consequently $\left\{\lambda R_\lambda \varphi |\, \lambda > 0\right\}$ is equicontinuous. Now assume $x_0 \in H_0$ and that for some sequence $\lambda_n \to 0$

$$\lim_{n\to\infty} \lambda_n R_{\lambda_n} \varphi(x_0) \neq \varphi(x_0).$$

Then there exists a subsequence such that $\lambda_{n_k} R_{\lambda_{n_k}} \varphi(x) \to \varphi(x)$ as $k \to \infty$ for ν–a.e. $x \in H$, (since $\lambda_n R_{\lambda_n} \varphi \to \varphi$ in $L^2(H, \nu)$). Hence by the same argument as above

$$\lambda_{n_k} R_{\lambda_{n_k}} \varphi(x) \to \varphi(x), \ \forall\, x \in H_0$$

which is a contradiction. □

Corollary 5.3. *For all* $f \in B_b(H_0)$, $\lambda > 0$

$$\int_H \lambda R_\lambda f d\nu = \int_H f d\nu. \tag{5.4}$$

Proof. Let $t > 0$, $\varphi \in \mathcal{E}_A(H)$. Then by Theorem 2.3 there exist $\varphi_n \in \mathcal{E}_A(H)$ such that $\varphi_n \to P_t\varphi$ and $N_0\varphi_n \to N_2 P_t\varphi$ in $L^2(H, \nu)$. Hence

$$\frac{d}{dt} \int_H P_t f d\nu = \int_H N_2 P_t f d\nu = \lim_{n\to\infty} \int_H N_0\varphi_n d\nu = 0,$$

so that

$$\int_H P_t f d\nu = \int_H f d\nu.$$

Multiplying by $\lambda e^{-\lambda t}$ and integrating we conclude that (5.4) holds with φ replacing f. But then (5.4) holds for all $f \in B_b(H)$ by a monotone class argument. □

Corollary 5.4. *For all* $\lambda > 0$ *there exists* $r_\lambda : H_0 \times H_0 \to \mathbb{R}_+$, $\mathcal{B}(H_0 \times H_0)$*–measurable such that for all* $f \in B_b(H)$

$$R_\lambda f(x) = \int_H f(y) r_\lambda(x, y)\nu(dy), \ \forall\, x \in H_0.$$

In particular, $\lambda R_\lambda(x, H_0) = 1$ *for all* $x \in H_0$.

Proof. Fix $\lambda > 0$. Let $N \in \mathcal{B}(H_0)$ such that $\nu(N) = 0$. Then by Corollary 5.3

$$0 = \int_H 1_N d\nu = \int_H \lambda R_\lambda 1_N d\nu,$$

so $R_\lambda 1_N = 0$ ν–a.e.; hence $R_\lambda 1_N(x) = 0\ \forall\, x \in H_0$. Consequently,

$$R_\lambda(x, dy) << \nu(dy) \ \ \forall\, x \in H_0.$$

That the density can be chosen jointly continuous is standard, since H_0 is polish. □

5.2. Semigroups

In contrast to the case of the resolvent we do not know whether

$$\lim_{\alpha\to 0, \beta\to 0} P_t^{\alpha,\beta} f = P_t f \text{ in } L^2(H, \nu)$$

for sufficiently many functions f. Therefore, the construction of strongly Feller probability kernels is much more difficult. Our aim is to establish properties (4.7) and (4.8) with P_t replacing $P_t^{\alpha,\beta}$, (cf. Proposition 5.7 below), then we can proceed as in the case of the resolvent. Though property (4.7) implies "a lot of tightness" for $P_t^{\alpha,\beta} f$, $f \in B_b(H)$, we cannot just consider limit points, since convergent subsequences would depend on (f and) t, so we cannot identify these to coincide with $P_t f$ using Proposition 4.1 and the uniqueness of the Laplace transform. To make this work neverthless, we need to find ν–versions $\widetilde{P_t f}$ of $P_t f$, continuous on H_0, so that $t \to \widetilde{P_t f}(x)$ is right continuous for all $x \in H_0$, and for all f in a large enough space S of functions on H_0. This is the content of Lemma 5.6 below.

First we define S. We introduce a countable set of smooth functions generating the topology of H which we shall use several times below.

Fix $h \in C_0^\infty(\mathbb{R})$ such that $0 \le h \le 1$, $h(r) = 1$ if $|r| \le 1$, $h(r) = 0$ if $|r| \ge 2$, and define

$$\psi(r) := \int_0^r h(s)ds.$$

Furthermore, fix $y_k \in H$, $k \in \mathbb{N}$, so that $\{y_k | k \in \mathbb{N}\}$ is dense in H and $\{y_k | k \in \mathbb{N}\} \cap H_0$ is dense in H_0. Define for $k \in \mathbb{N}$

$$f_k(x) := \psi(|x - y_k|^2), \; x \in H. \tag{5.5}$$

Then f_k, $k \in \mathbb{N}$, generate the topology of H and their restrictions to H_0 that of H_0. Consider the set

$$\mathcal{M} := \{m R_m f_k | \; m \in \mathbb{N}, k \in \mathbb{N}\} \tag{5.6}$$

where R_λ is as defined in (5.2), and recount to get

$$\mathcal{M} := \{g_n | \; n \in \mathbb{N}\}. \tag{5.7}$$

Lemma 5.5. *$\{g_n | \; n \in \mathbb{N}\}$ is a set of uniformly bounded, equi–Lipschitz continuous functions generating the topology of H_0.*

Proof. First note that as a consequence of Proposition 5.2, the functions g_n, $n \in \mathbb{N}$, are equi–Lipschitz continuous, since

$$\begin{aligned} \|f_k\|_1 &= \|\psi(|\cdot - x_k|^2)\|_0 + \|\psi'(|\cdot - x_k|^2)2(\cdot - x_k)\|_0 \\ &\le 2 + 1_{\left\{|\cdot - x_k| \le \sqrt{2}\right\}} \, 2\||\cdot - x_k|\|_0 \le 2 + 2\sqrt{2}. \end{aligned}$$

Since each g_n is continuous, it remains to show that if $x_l, x \in H_0, l \in \mathbb{N}$, such that $g_n(x_l) \to g_n(x)$ for all $n \in \mathbb{N}$, then $x_l \to x$ in H_0. The latter is equivalent to $f_k(x_l) \to f_k(x)$ for all $k \in \mathbb{N}$. But this holds, since for $k \in \mathbb{N}$ fixed and all $n \in \mathbb{N}$

$$\begin{aligned} |f_k(x_n) - f_k(x)| &\le \limsup_{m \to \infty} |f_k(x_n) - m R_m f_k(x_n)| \\ &+ \sup_m |m R_m f_k(x_n) - m R_m f_k(x)| + \limsup_{m \to \infty} |m R_m f_k(x) - f_k(x)|, \end{aligned}$$

and since by Proposition 5.2 the two limsup's are zero while by equicontinuity the remaining term can be made arbitrarily small for large n. □

There exists a countable subset S_0 of $Lip_b(H_0)$ having the following property: for all $f \in Lip_b(H_0)$ there exists $\varphi_n \in S_0,\ n \in \mathbb{N}$, such that

$$\begin{aligned}&\lim_{n\to\infty} \varphi_n(x) = f(x),\ \forall\, x \in H,\\ &\|\varphi_n\|_0 \le \|f\|_0 + \frac{1}{n} \text{ and } \|\varphi_n\|_{Lip} \le \|f\|_{Lip} + \frac{1}{n}.\end{aligned} \tag{5.8}$$

The existence of such a set is easily proved by approximating by cylinder functions and applying the corresponding well known finite dimensional result. Define

$$S_0^{\pm} := \left\{f^+ |\ f \in S_0\right\} \cup \left\{f^- |\ f \in S_0\right\},$$

where $f^+ := \sup\{f, 0\}$, $f^- := -\inf\{f, 0\}$. Set

$$S_1 := \left\{R_m f |\ m \in \mathbb{N},\ f \in S_0^{\pm} \cup \{f_k,\ k \in \mathbb{N}\}\right\},$$

where f_k is as defined as in (5.5).

Recall that a function $f : H_0 \to \mathbb{R}_+$ is called α–supermedian for $(R_\lambda)_{\lambda>0}$ if

$$\lambda R_{\lambda+\alpha} f(x) \le f(x),\ \forall\, \lambda > 0\ \forall\, x \in H_0.$$

Clearly, by the resolvent equation any function in S_1 is m–supermedian for some $m \in \mathbb{N}$. Furthermore the α–supermedian functions form an inf stable convex cone, invariant under R_β for all $\beta > 0$ and, containing the positive constant functions. Hence we may consider the smallest set S_2 of bounded functions on H_0, α–supermedian for some $\alpha \in \mathbb{Q}_+^*$, having the following properties

$$S_1 \subset S_2,\ R_\alpha f, 1, f \wedge g, \alpha f + \beta g \in S_2 \text{ if } f, g \in S_2,\ \alpha, \beta \in \mathbb{Q}_+^*. \tag{5.9}$$

By [17, Lemma 6.1.1] S_2 is countable. Define the corresponding $\mathbb{Q}$–vector space. Define

$$S := S_2 - S_2. \tag{5.10}$$

Then S is countable and a vector lattice over $\mathbb{Q}$ containing $\mathcal{M}$, hence in particular S generates $\mathcal{B}(H_0)$.

Lemma 5.6. *Let $f \in S$. Then there exists a ν–version $\overline{p}_t f$ of $P_t f,\ t > 0$, such that for all $x \in H_0$*

$$t \to \overline{p}_t f(x) \text{ is right continuous on } [0, +\infty),$$

and for $t > 0$

$$x \to \overline{p}_t f(x) \text{ is continuous on } H_0.$$

Before we prove Lemma 5.6 we show that it implies the existence of strong Feller probability kernels for P_t:

Proposition 5.7. *(i) Let $f \in B_b(H), t > 0$. Then for ν–a.e. $x, y \in H$*

$$|P_t f(x) - P_t f(y)| \leq t^{-1/2} \|C^{-1}\|^{1/2} \|f\|_0 |x - y|. \tag{5.11}$$

(ii) Let $f \in Lip_b(H), t > 0$. Then for ν–a.e. $x, y \in H$

$$|P_t f(x) - P_t f(y)| \leq \|f\|_{Lip} |x - y|. \tag{5.12}$$

(iii) Let for $f \in B_b(H), t > 0$, $p_t f$ denote the unique Lipschitz continuous ν–version of $P_t f$ on H_0. Then $(p_t)_{t \geq 0}$ is a semigroup of strong Feller probability kernels satisfying (5.11) *and* (5.12) *with p_t replacing P_t. Furthermore, ν is an invariant measure for $(p_t)_{t \geq 0}$ and for all $f \in Lip_b(H)$*

$$\lim_{t \to 0} p_t f(x) = f(x), \ \forall\, x \in H_0, \tag{5.13}$$

and for all $\lambda > 0$ and all $f \in B_b(H)$

$$\int_0^\infty e^{-\lambda t} p_t f(x) dt = R_\lambda f(x), \ \forall\, x \in H_0.$$

(iv) For $t > 0$ there exists $p_t : H_0 \times H_0 \to \mathbb{R}_+$, $\mathcal{B}(H_0 \times H_0)$–measurable such that for all $f \in B_b(H)$

$$p_t f(x) = \int_H f(y) p_t(x, y) \nu(dy) \ \ \forall\, x \in H_0.$$

Proof. (iii) and (iv) follow from (i),(ii) by exactly the same arguments used in the proofs of Proposition 5.2 and Corollaries 5.3, 5.4. So, we only have to prove (i), (ii).

(i) Let $N \in \mathbb{N}$ and let Y_N denote the closed ball of radius $\sqrt{N}\ \|f\|_0$ in $L^2([0, N], ds)$ equipped with the weak topology. So, Y_N is compact. Let $\{l_n |\ n \in \mathbb{N}\}$ be a dense set in $L^2([0, N], ds)$ consisting of bounded functions. Then

$$d_{Y_N}(h_1, h_2) := \sum_{n=1}^\infty 2^{-n} \left(\|l_n\|_{L^\infty([0,N],ds)} + \|l_n\|_{L^2([0,N],ds)} + 1 \right)^{-1}$$

$$\inf \left(\left| \int_0^N l_n(s)(h_1(s) - h_2(s)) ds \right|, 1 \right), \ h_1 . h_2 \in Y_N,$$

defines a metric on Y_N generating its topology, which is complete, since Y_N is compact.

Now consider the maps $\Lambda_N^{\alpha,\beta} : H \to Y_N$ defined for $\alpha, \beta > 0$ by

$$\Lambda_N^{\alpha,\beta}(x) := \left(s \to P_s^{\alpha,\beta} f(x), \ s \in [0, N] \right), \ x \in H.$$

Then for all $x, y \in H$, $\alpha, \beta > 0$, by (4.7)

$$d_{Y_N}(\Lambda_N^{\alpha,\beta}(x), \Lambda_N^{\alpha,\beta}(y)) \leq \int_0^N s^{-1/2} ds \|C^{-1}\|^{1/2} \|f\|_0 |x - y|. \tag{5.14}$$

Since ν is a probability measure on a polish space there exist $\tilde{K}_n \subset H_0,\ n \in \mathbb{N}$, compact and increasing, such that

$$\lim_{n\to\infty} \nu(H_0 \backslash \tilde{K}_n) = 0$$

Defining

$$K_n := \text{supp}\,[1_{\tilde{K}_n}\nu],\ n \in \mathbb{N},$$

it is easy to check (cf. the proof of Z. M. Ma and M. Röckner [19], Chapter III, Proposition 3.8), that $K_n \subset \tilde{K}_n,\ n \in \mathbb{N}$, and still

$$\lim_{n\to\infty} \nu(H_0 \backslash K_n) = 0$$

and that, in addition,

$$K_n \cap U \neq \emptyset \Rightarrow \nu(K_n \cap U) > 0,\ \forall \text{ open sets } U \subset H_0,\ \forall\, n \in \mathbb{N}. \tag{5.15}$$

By Proposition 4.1 we can find $\alpha_n,\beta_n > 0$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} R(\lambda, N_2^{\alpha_n,\beta_n})f = R(\lambda, N_2)f,\ \forall\, \lambda > 0\ \text{ in } L^2(H,\nu) \text{ and } \nu-\text{a.e.}. \tag{5.16}$$

Applying the Ascoli theorem and a diagonal argument, selecting a subsequence if necessary, we obtain that there exists a map $\Lambda : \cup_n K_n \to L^\infty([0,N],ds)$ such that for all $N \in \mathbb{N}$

$$\Lambda(x)|_{[0,N]} = \lim_{n\to\infty} \Lambda_N^{\alpha_n,\beta_n}(x) \text{ uniformly for } x \in K_n,\ \forall\, n \in \mathbb{N}. \tag{5.17}$$

We show now that

$$\Lambda(\cdot)(s) \text{ is a } \nu-\text{version of } P_s f \text{ for a.e. } s \in (0,\infty). \tag{5.18}$$

To prove (5.18) let $\lambda > 0$. Then by (5.16), (5.17) and dominated convergence for all $g \in L^\infty(H,\nu)$

$$\begin{aligned}
\int_0^\infty e^{-\lambda s}\int_H g(x)P_s f(x)\nu(dx) &= \int_H g(x)R(\lambda,N_2)f(x)\nu(dx)ds\\
&= \int_H g(x) \lim_{n\to\infty}\lim_{N\to\infty}\int_0^N e^{-\lambda s}P_s^{\alpha_n,\beta_n} f(x)ds\ \nu(dx)\\
&= \int_H g(x) \lim_{N\to\infty}\int_0^N e^{-\lambda s}\Lambda(x)(s)ds\ \nu(dx)\\
&= \int_0^\infty e^{-\lambda s}\int_H g(x)\Lambda(x)(s)\ \nu(dx)ds,
\end{aligned}$$

where the interchange of limits is justified, since $|P_s^{\alpha_n,\beta_n} f(x)| \leq \|f\|_0$ and hence $|\Lambda(x)(s)| \leq \|f\|_0$ for ds-a.e. $s \in [0,\infty)$ and all $x \in \bigcup_n K_n$. So, (5.18) follows by the uniqueness of the Laplace transform.

Now we use Lemma 5.6 in a crucial way. If $f \in S$, then by (5.15), (5.17) and (5.18)

$$\Lambda(x)(t) = \overline{p}_t f(x) \text{ for a.e. } t \text{ and all } x \in \bigcup_{n\in\mathbb{N}} K_n. \tag{5.19}$$

(since $x \to \big(t \to \overline{p}_t f(x),\ t \in [0, N]\big)$ is continuous from K_n to Y_N for every $n \in \mathbb{N}$).

So, if $f \in S$, and $\delta_k \in C_0^\infty(\mathbb{R})$, $k \in \mathbb{N}$, approximate the identity, we obtain for all $x, y \in \bigcup_{n\in\mathbb{N}} K_n$, that for some subsequence $\{k_l\}$ and a.e. $t \in (0, N)$

$$\overline{p}_t f(x) - \overline{p}_t f(y) = \lim_{l\to\infty} \int_0^N \delta_{k_l}(t-s)(\overline{p}_s f(x) - \overline{p}_s f(y))ds. \tag{5.20}$$

But for $l \in \mathbb{N}$ the integral in (5.20) is by (5.19) and (5.17) equal to

$$\lim_{n\to\infty} \int_0^N \delta_{k_l}(t-s)(P_s^{\alpha_n,\beta_n} f(x) - P_s^{\alpha_n,\beta_n} f(y))ds,$$

which by (4.7) is dominated by

$$\int_0^N \delta_{k_l}(t-s)s^{-1/2}ds\ \|C^{-1}\|^{1/2}\|f\|_0|x-y| \to t^{-1/2}\ \|C^{-1}\|\|f\|_0|x-y|,$$

as $l \to \infty$.

Since $t \to \overline{p}_t f(x)$ is right continuous for all $x \in H_0$, (5.11) follows if $f \in S$. Since S is a vector lattice containing the constants and generating $\mathcal{B}(H_0)$, (5.11) follows for all $f \in B_b(H_0)$ and thus all $f \in B_b(H)$ by a monotone class argument.

(ii). Let $f \in S$. Then (5.12) follows by exactly the same arguments as above, but employing (4.8) instead of (4.7). If $f \in S_0$, then $mR_m f \in S$, $m \in \mathbb{N}$, $\|mR_m f\|_0 \le \|f\|_0$ and by Proposition 5.2, $\lim_{m\to\infty} mR_m f(x) = f(x)$ for all $x \in H_0$ and

$$\|mR_m f\|_{Lip} \le \|f\|_{Lip},\ \forall\, m \in \mathbb{N}.$$

Hence (5.12) follows by approximation for $f \in S_0$. Consequently, using (5.8) we can approximate again to obtain (5.12) for all $f \in Lip_b(H)$. □

So, it remains to prove Lemma 5.6. This is done using a modification of the classical compactification for Ray–resolvents (cf. R. Getoor [18] and also [19, Chapter 4]).

Proof of Lemma 5.6. Consider the injective map

$$i : x \to (f(x))_{f\in S}$$

from H_0 to $\prod_{f\in S}[-\|f\|_0, \|f\|_0]$ which is equipped with the product topology, hence is compact and metrizable because S is countable.

By Lemma 5.5, $i : H_0 \to i(H_0)$ is an homeomorphism where $i(H_0)$ is equipped with the trace topology. We consider the closure $\overline{H_0}$ of $H_0 = i(H_0)$ in $\prod_{f\in S}[-\|f\|_0, \|f\|_0]$. $\overline{H_0}$ is then a compact separable metric space, so that every

$f \in S$ has a unique continuous extension $\overline{f}$ to $\overline{H_0}$. By construction the space $\overline{S}$ of all such extensions separate the points of $\overline{H_0}$, hence the space $\overline{S_2}$ of all extensions of functions in S_2 separate the points of $\overline{H_0}$. For $\lambda \in \mathbb{Q}_+^*$ and $f \in \overline{S}$ we define

$$\overline{R}_\lambda f := \overline{R_\lambda (f_{|H_0})}. \tag{5.21}$$

which is possible, since $R_\lambda f_{|H_0} \in S$. Here $f_{|H_0}$ denotes the function f restricted to H_0. By the Stone–Weiestrass theorem $\overline{S}$ is dense in $C(\overline{H_0})$ with respect to the uniform norm $\|\cdot\|_0$ Therefore, each $\overline{R}_\lambda$ extends to a positive linear operator from $C(\overline{H_0})$ into $C(\overline{H_0})$. Clearly $(\overline{R}_\lambda)_{\lambda\in\mathbb{Q}_+^*}$ satisfies the resolvent equation, hence

$$\lambda \to \overline{R}_\lambda, \quad \lambda \in \mathbb{Q}_+^*,$$

is a Lipschitz continuous map into the space of bounded linear operators on $C(\overline{H_0})$, equipped with the usual operator norm. Consequently, it has a unique continuous extension $\lambda \to \overline{R}_\lambda$ for all $\lambda > 0$. By the Riesz–Markov theorem each $\lambda\overline{R}_\lambda,\ \lambda > 0$, is represented by a probability kernel (since $\lambda\overline{R}_\lambda 1 = 1$) on $\mathcal{B}(\overline{H_0})$, which we again denote by $\lambda\overline{R}_\lambda$. Then the following hold by construction:

$$(\overline{R}_\lambda)_{\lambda>0} \text{ satisfies the resolvent equation,} \tag{5.22}$$

$$\overline{R}_\lambda(C(\overline{H_0})) \subset C(\overline{H_0}),\ \forall\, \lambda > 0, \tag{5.23}$$

$\overline{S_2}$ separates the points and consists of functions which are supermedian with respect to $(\overline{R}_\lambda)_{\lambda>0}$, (5.24)

$$\lim_{\lambda\to\infty} \lambda\overline{R}_\lambda f(x) = f(x)\ \forall\, x \in H_0,\ \ f \in C(H_0). \tag{5.25}$$

Apart from (5.25) all other properties are obvious. To see (5.25) note that it is enough to prove this for $f \in \overline{S}_2$. But then f is α–supermedian for some $\alpha \in \mathbb{Q}_+^*$ and

$$\lim_{\lambda\to\infty,\ \lambda\in\mathbb{Q}_+^*} \lambda\overline{R}_{\lambda+\alpha} f(x) = \lim_{\lambda\to\infty,\ \lambda\in\mathbb{Q}_+^*} \lambda R_{\lambda+\alpha} f(x) = f(x)\ \forall\, x \in H_0,$$

by Proposition 5.2. This implies (5.25) since $\lambda \to \overline{R}_{\lambda+\alpha} f$ is increasing by the resolvent equation.

(5.22)–(5.25) imply that $(\overline{R}_\lambda)_{\lambda>0}$ is a Ray–resolvent on the compact separable metric space $\overline{H}_0$ with H_0 contained in the set of its non–branching points. Hence by [18, Theorem (3.6)], (see also [19, Chapter 4, Theorem 1.20]) there exists a unique semigroup $(\overline{p}_t)_{t\geq 0}$ of probability kernels on $\mathcal{B}(\overline{H_0})$ such that,

$$\overline{p}_0(x, dy) = \varepsilon_x(dy)\ \ \forall\, x \in H_0, \tag{5.26}$$

(where ε_x denotes the Dirac measure in x).

$$t \to \overline{p}_t f(x) \text{ is right continuous on } [0, \infty)\ \forall\, x \in \overline{H_0},\ \ f \in C(\overline{H_0}). \tag{5.27}$$

$$\overline{R}_\lambda f = \int_0^\infty e^{-\lambda t}\overline{p}_t f dt \ \forall\ \lambda > 0,\ \ f \in C(\overline{H_0}). \tag{5.28}$$

(5.28) implies that for $f \in S$, $\lambda > 0$,

$$R_\lambda f(x) = \int_0^\infty e^{-\lambda t}\overline{p}_t \overline{f}(x)dt \ \forall\ x \in H_0.$$

Hence for all $g \in L^\infty(H, \nu)$ by (5.2)

$$\int_0^\infty e^{-\lambda t}\int_H gP_t f d\nu dt = \int_H gR(\lambda, N_2) f d\nu$$

$$= \int_H g \int_0^\infty e^{-\lambda t}(\overline{p}_t \overline{f})|_{H_0} dt d\nu = \int_0^\infty e^{-\lambda t}\int_H g(\overline{p}_t \overline{f})|_{H_0} d\nu dt.$$

Hence by the uniqueness of the Laplace transform and right continuity we can take

$$\overline{p}_t f = (\overline{p}_t \overline{f})|_{H_0},\ \ t > 0,\ \ f \in S,$$

as the desired versions. □

6. Kolmogorov's continuity criterion and diffusion operators on $L^2(H, \nu)$

Assume again in this section that $C^{-1} \in L(H)$ (or more generally that (4.7) holds, see Remark 4.4) and that Hypotheses 1.1 and 1.2 hold. Let (p_t) be as constructed in the previous section and $H_0 = \ \text{supp}\ \ \nu$. By Kolmogorov's standard construction scheme there exist probability measures $\mathbb{P}_x$, $x \in H_0$, on $\Omega = H_0^{\mathbb{R}^+}$, equipped with product σ–field $\mathcal{F}^0$, so that $\mathbb{M}^0 := (\Omega, \mathcal{F}^0, (\mathcal{F}_t^0)_{t\geq 0}, (X_t^0)_{t\geq 0}, (\mathbb{P}_x)_{x\in H_0})$ is a normal Markov process on H_0 with transition semigroup $(p_t)_{t>0}$. Here $X_t^0 : H_0^{\mathbb{R}^+} \to H_0$ are the coordinate maps, and $\mathcal{F}_t^0 := \sigma(X_s^0 |\ s \leq t)$.

The following lemma is more or less obvious, but we include a proof for the reader's convenience. Define

$$\mathbb{P}_\nu := \int_{H_0} \mathbb{P}_x \nu(dx). \tag{6.1}$$

Lemma 6.1. $(X_t^0)_{t\geq 0}$ *is stochastically continuous under* $\mathbb{P}_\nu$. *Hence there exists a measurable process* $(X_t)_{t\geq 0}$ *such that*

$$\mathbb{P}_\nu\left[X_t^0 \neq X_t\right] = 0,\ \forall\ t > 0.$$

Proof. For $t > s$, $k \in \mathbb{N}$, we have for f_k as in (5.5)

$$\int_\Omega \left|f_k(X_t^0) - f_k(X_s^0)\right|^2 d\mathbb{P}_\nu$$

$$= \int_H \int_H (f_k(y) - f_k(x))^2 p_{t-s}(x, dy)\nu(dx)$$

$$= 2\int_H f_k^2 d\nu - 2\int_H f_k P_{t-s} f_k d\nu$$

where we used that ν is invariant for (p_t). By the strong continuity of P_t the latter converges to 0 for $|t-s|\to 0$. This implies the stochastic continuity of $(X_t^0)_{t\geq 0}$ under $\mathbb{P}_\nu$, since $f_k,\ k\in\mathbb{N}$, generate the topology of H_0. The second part of the assertion is a well known consequence, see e.g. [13, Proposition 3.2]. □

Remark 6.2. The following proposition is formulated in the situation studied in this paper, but it is of quite general nature. It works for a large class of operators, replacing N_2, which have a nice infinitesimally invariant measure and which are diffusion operators in the sense of [16, Appendix B].

Theorem 6.3. *Let $\lambda>0,\ f\in C_b^2(H)$, and*

$$g:=R_\lambda f.$$

(with R_λ as defined in (5.2)). Then there exists a constant $c(g)>0$ such that for all $t,s>0$

$$\int_\Omega |g(X_t^0)-g(X_s^0)|^4 d\mathbb{P}_\nu \leq c(g)|t-s|^{3/2}. \tag{6.2}$$

Proof. Let $t>s$.

Step 1. Let $\varphi\in\mathcal{E}_A(H)$. Then $\varphi,\varphi^2,\varphi^3,\varphi^4\in\mathcal{E}_A(H)\subset D(N_2)$. Hence setting $\Gamma(\varphi,\varphi):=|C^{1/2}D\varphi|^2$ we obtain

$$\begin{aligned}\int_\Omega |\varphi(X_t^0)-\varphi(X_s^0)|^4 d\mathbb{P}_\nu &= \int_\Omega |\varphi(X_t)-\varphi(X_s)|^4 d\mathbb{P}_\nu\\
&= \int_\Omega \Big[\varphi^4(X_t)-4\varphi^3(X_t)\varphi(X_s)+6\varphi^2(X_t)\varphi^2(X_s)\\
&\quad -4\,\varphi(X_t)\varphi^3(X_s)+\varphi^4(X_s)\Big]d\mathbb{P}_\nu\\
&= 2\int_H \varphi^4 d\nu - 4\int_H P_{t-s}\varphi^3\ \varphi d\nu - 4\int_H P_{t-s}\varphi\ \varphi^3 d\nu + 6\int_H P_{t-s}\varphi^2\ \varphi^2 d\nu.\end{aligned}$$

It follows that

$$\begin{aligned}\int_\Omega |\varphi(X_t)-\varphi(X_s)|^4 d\mathbb{P}_\nu &= 2\int_H \varphi^4 d\nu - 4\int_H [\varphi^3+\int_0^{t-s} N_2(P_r\varphi^3)dr]\varphi d\nu\\
&\quad -4\int_H[\varphi+\int_0^{t-s}N_2(P_r\varphi)dr]\varphi^3 d\nu + 6\int_H[\varphi^2+\int_0^{t-s}N_2(P_r\varphi^2)dr]\varphi^2 d\nu\\
&= 6\int_0^{t-s} dr\int_H P_r(N_0\varphi^2)\varphi^2 d\nu - 4\int_0^{t-s} dr\int_H P_r(N_0\varphi^3)\varphi d\nu\\
&\quad -4\int_0^{t-s} dr\int_H P_r(N_0\varphi)\varphi^3 d\nu.\end{aligned}$$

Since

$$N_0(\varphi^3) = 3\varphi^2 N_0\varphi + 3\varphi\Gamma(\varphi,\varphi),$$

we obtain

$$\begin{aligned}
&\int_\Omega |\varphi(X_t)-\varphi(X_s)|^4 d\mathbb{P}_\nu \\
&= 4\int_0^{t-s}\int_\Omega [3\varphi^2(X_0)\varphi(X_r) - 3\varphi(X_0)\varphi^2(X_r) - \varphi^3(X_0)]N_0\varphi(X_r)d\mathbb{P}_\nu \\
&\quad + 6\int_0^{t-s}\int_\Omega [\varphi^2(X_0) - 2\varphi(X_0)\varphi(X_r)]|C^{1/2}D\varphi(X_r)|^2 d\mathbb{P}_\nu,
\end{aligned}$$

which can be written as

$$\begin{aligned}
&\int_\Omega |\varphi(X_t)-\varphi(X_s)|^4 d\mathbb{P}_\nu \\
&= 4\int_0^{t-s} dr \int_\Omega [\varphi(X_r)-\varphi(X_0)]^3 N_0\varphi(X_r)d\mathbb{P}_\nu \\
&\quad + 6\int_0^{t-s} dr \int_\Omega [\varphi(X_r)-\varphi(X_0)]^2 |C^{1/2}D\varphi(X_r)|^2 d\mathbb{P}_\nu \\
&\quad - 4\int_0^{t-s} dr \int_\Omega \varphi(X_r)^3 N_0\varphi(X_r)d\mathbb{P}_\nu \\
&\quad - 6\int_0^{t-s} dr \int_\Omega \varphi(X_r)^2 |C^{1/2}D\varphi(X_r)|^2 d\mathbb{P}_\nu.
\end{aligned}$$

Since

$$N_0(\varphi^4) = 4\varphi^3 N_0\varphi + 6\varphi^2\Gamma(\varphi,\varphi),$$

we see that the two last terms are equal to

$$-(t-s)\int_H N_0(\varphi^4)d\nu = 0$$

by the invariance of ν. In conclusion we have

$$\begin{aligned}
&\int_\Omega |\varphi(X_t)-\varphi(X_s)|^4 d\mathbb{P}_\nu \\
&= 4\int_0^{t-s} dr \int_\Omega [\varphi(X_r)-\varphi(X_0)]^3 N_0\varphi(X_r)d\mathbb{P}_\nu \\
&\quad + 6\int_0^{t-s} dr \int_\Omega [\varphi(X_r)-\varphi(X_0)]^2 |C^{1/2}D\varphi(X_r)|^2 d\mathbb{P}_\nu := I_1 + I_2.
\end{aligned} \tag{6.3}$$

Step 2. Let $\varphi \in D(L, C^1_{b,2}(H)) \cap C_b(H)$. Let $\varphi_{\overline{n}}$, $\overline{n} \in \mathbb{N}^4$, be as in Lemma 2.2. Applying (6.3) to $\varphi_{\overline{n}}$ replacing φ, by Hypothesis 1.2–(i) and dominated convergence, we can take limits as in Lemma 2.2 to obtain (6.3) for φ. Now we note that

$$\begin{aligned}\int_\Omega |\varphi(X_r) - \varphi(X_0)|^2 d\mathbb{P}_\nu &= 2\int_H \varphi^2 d\nu - \int_H \varphi P_r \varphi d\nu \\ &= -2\int_H \varphi \int_0^r P_{r'}(N_0\varphi) dr' d\nu \\ &\le 2\|\varphi\|_0 \int_0^r P_{r'}(|N_0\varphi|) dr' d\nu \\ &= 2r\|\varphi\|_0 \|N_0\varphi\|_{L^1(H,\nu)} \le 2r\|\varphi\|_0\|N_0\varphi\|_{L^2(H,\nu)}.\end{aligned} \tag{6.4}$$

Consequently

$$\begin{aligned}|I_1| &\le 4\int_0^{t-s} dr \left(\int_\Omega |\varphi(X_r) - \varphi(X_0)|^6 d\mathbb{P}_\nu\right)^{\frac12} \left(\int_\Omega |N_0\varphi(X_r)|^2 d\mathbb{P}_\nu\right)^{\frac12} \\ &\le 4(2\|\varphi\|_0)^2 \int_0^{t-s} dr \left(\int_\Omega |\varphi(X_r) - \varphi(X_0)|^2 d\mathbb{P}_\nu\right)^{\frac12} \|N_0\varphi\|_{L^2(H,\nu)}.\end{aligned}$$

Taking into account (6.4), it follows that

$$\begin{aligned}|I_1| &\le 4(2\|\varphi\|_0)^2 (2\|\varphi\|_0\|N_0\varphi\|_{L^2(H,\nu)})^{1/2} \int_0^{t-s} r^{\frac12} dr \|N_0\varphi\|_{L^2(H,\nu)} \\ &= \frac{2^{11/2}}{3} \|\varphi\|_0^{5/2} \|N_0\varphi\|^{3/2}_{L^2(H,\nu)} (t-s)^{3/2}.\end{aligned} \tag{6.5}$$

Moreover,

$$\begin{aligned}|I_2| &\le 6\int_0^{t-s} dr \left(\int_\Omega |\varphi(X_r) - \varphi(X_0)|^4 d\mathbb{P}_\nu\right)^{1/2} \|\Gamma(\varphi,\varphi)\|_{L^2(H,\nu)} \\ &\le 12\|\varphi\|_0 \int_0^{t-s} dr \left(\int_\Omega |\varphi(X_r) - \varphi(X_0)|^2 d\mathbb{P}_\nu\right)^{1/2} \|\Gamma(\varphi,\varphi)\|_{L^2(H,\nu)} \\ &\le 2^{7/2}\|\varphi\|_0^{3/2} \|N_0\varphi\|^{1/2}_{L^2(H,\nu)} (t-s)^{3/2} \|\Gamma(\varphi,\varphi)\|_{L^2(H,\nu)}.\end{aligned} \tag{6.6}$$

So (6.5) and (6.6) imply

$$\begin{aligned}\int_\Omega |\varphi(X^0_t) - \varphi(X^0_s)|^4 d\mathbb{P}_\nu &\le 2^6 \|\varphi\|_0^{3/2} \|N_0\varphi\|^{1/2}_{L^2(H,\nu)} \\ &\quad \times \left(\|\varphi\|_0 \|N_0\varphi\|_{L^2(H,\nu)} + \|\Gamma(\varphi,\varphi)\|_{L^2(H,\nu)}\right)(t-s)^{3/2}.\end{aligned} \tag{6.7}$$

Step 3. Let g be as in the assertion. Let φ_n be as in Corollary 2.4. Then applying (6.7) with φ_n replacing φ and taking the limit as $n \to \infty$, we obtain (6.2) for g. □

Define a metric on H_0 (which is in general not complete) by

$$d(x, y) := \sum_{n\in\mathbb{N}} \frac{2^{-n}}{c(g_n)} \inf\{|g_n(x) - g_n(y), 1\}, \; x, y \in H_0, \tag{6.8}$$

where the g_n are as in (5.6), (5.7) and $c(g_n)$ is as in (6.2) with g_n replacing g. Then the following is a straightforward consequence of Theorem 6.3 and Lemma 5.5.

Corollary 6.4. *(i) d generates the topology of H_0.*
(ii) For all $t, s > 0$

$$\int_\Omega d(X_t^0, X_s^0)^4 d\mathbb{P}_\nu \le |t - s|^{3/2}.$$

7. Construction of a diffusion weakly solving SDE (0.1)

By the proof of Kolmogorov's continuity criterion Corollary 6.4 implies that $\mathbb{P}_\nu$–a.e. path in $H_0^{\mathbb{R}^+}$ is uniformly continuous on the dyadics with respect to the metric d. Below we are going to apply the technique developed in [15] to show that this is also true $\mathbb{P}_x$–a.s., for all $x \in H_0$.

Unfortunately, the results in [15] do not apply directly, but a modification of the arguments leads to the desired conclusions. We shall give a reasonably self–contained presentation below (but giving credit to [15] at respective points).

We consider the same situation as in the previous section and we also adopt the notation there. In particular, d denotes the metric defined in (6.8), $\Omega = H_0^{\mathbb{R}_+}$, and $H_0 := \operatorname{supp} \nu$.

For $k, l \in \mathbb{N}$ define (as in [15])

$$A_k^{(l)} := \Big\{ \omega \in \Omega | \; \exists\, n_0 \; \forall\, n \ge n_0, \; \forall\, s, t \in S_n \cap [0, l], \; |s - t| \le 2^{-n_0} : \\ d(X_s^0(\omega), X_t^0(\omega)) \le 2^{-k} \Big\} \tag{7.1}$$

where $S_n := \{k2^{-n} | \; k \in \mathbb{N} \cup \{0\}\}$, and

$$\Lambda_0 := \bigcap_{k,l\in\mathbb{N}} A_k^{(l)}. \tag{7.2}$$

Let $\Theta_t : \Omega \to \Omega, \; t > 0$, be the canonical shift, i.e. $\Theta_t(\omega) = \omega(\cdot + t), \omega \in \Omega$. Then it is easy to check that

$$\Theta_t^{-1}(\Lambda_0) \supset \Lambda_0 \;\; \forall\, t \in D, \tag{7.3}$$

where $D := \bigcup_{n\in\mathbb{N}} S_n$.(cf. [15]), and we know by the proof of Kolmogorov's continuity criterion and Corollary 6.4 that

$$\mathbb{P}_\nu(\Lambda_0) = 1. \tag{7.4}$$

The main trick is contained in the following lemma:

Lemma 7.1. *Suppose* $A \in \mathcal{F}^0$, $t > 0$, *such that* $\mathbb{P}_\nu(\Theta_t^{-1}(A)) = 1$. *Then*

$$\mathbb{P}_x(\Theta_t^{-1}(A)) = 1, \ \forall\, x \in H_0. \tag{7.5}$$

Proof. We have for all $x \in H_0$ by the Markov property that

$$\begin{aligned}\mathbb{P}_x(\Theta_t^{-1}(A)) &= \mathbb{E}_x\left[\mathbb{E}_x(1_A \circ \Theta_t |\, \mathcal{F}_t^0)\right]\\ &= \mathbb{E}_x\left[\mathbb{E}_{X_t^0}(1_A)\right]\\ &= p_t(\mathbb{E}_\cdot(1_A))(x),\end{aligned}$$

where $\mathbb{E}_x(\cdot)$, $\mathbb{E}_x(\cdot|\, \mathcal{F}_t^0)$ denotes expectation conditional expectation, with respect to $\mathbb{P}_x$ respectively. By the strong Feller property of p_t this implies that

$$x \to \mathbb{P}_x(\Theta_t^{-1}(A))$$

is continuous on H_0. But since $\mathbb{P}_\nu(\Theta_t^{-1}(A)) = 1$, it follows that

$$\mathbb{P}_x(\Theta_t^{-1}(A)) = 1 \text{ for } \nu - \text{a.e. } x \in H_0.$$

Consequently (7.5) follows by continuity. □

Define as in [15]

$$\Lambda_0' = \bigcap_{t \in D, t > 0} \Theta_t^{-1}(\Lambda_0). \tag{7.6}$$

Then Λ_0' consists of all paths locally uniformly continuous on $(t, \infty) \cap D$ for all $t > 0$. Set (as in [15])

$$\Lambda_1 := \left\{\omega \in \Omega |\ \lim_{s \downarrow 0, s \in D} X_s^0(\omega) \text{ exists in } H_0\right\}, \tag{7.7}$$

and

$$\Lambda := \Lambda_0' \cap \Lambda_1. \tag{7.8}$$

Then it suffices to show that

$$\mathbb{P}_x(\Lambda) = 1 \ \ \forall\, x \in H_0. \tag{7.9}$$

By Lemma 7.1, (7.3) and (7.4) we already know that $\mathbb{P}_x(\Lambda_0') = 1$. So (7.9) follows from the following result (whose proof is slightly different from the corresponding result (i.e. Lemma 2.10) in [15].

Proposition 7.2. *Let* $x \in H_0$. *Then*

$$\lim_{t \downarrow 0} X_t^0 = x \quad \mathbb{P}_x - \text{a.s.}. \tag{7.10}$$

Proof. Let $k, m \in \mathbb{N}$ and let f_k be as defined in (5.5). Then (as is well known and easily follows from the Markov property) $(e^{-mt} m R_m f_k(X_t^0))_{t\geq 0}$ is a positive supermartingale, so by the martingale convergence theorem $\mathbb{P}_x$–a.s.

$$\lim_{t\downarrow 0} e^{-mt} m R_m f_k(X_t^0) \text{ exists in } \mathbb{R},$$

i.e. using the notation introduced in (5.6), (5.7)

$$\lim_{t\downarrow 0} g_n(X_t^0) \text{ exists in } \mathbb{R},\ \forall\, n \in \mathbb{N}. \tag{7.11}$$

But since g_n, g_n^2 are bounded and Lipschitz, it follows by Proposition 5.2 that for all $n \in \mathbb{N}$

$$\mathbb{E}_x\left[(g_n(X_t^0) - g_n(x))^2\right] = p_t g_n^2(x) - 2g_n(x) p_t g_n(x) + g_n^2(x) \to 0,$$

as $t \to 0$, which in turn together with (7.11) implies that $\mathbb{P}_x$–a.s.

$$\lim_{t\downarrow 0} g_n(X_t^0) = g_n(x) \ \ \forall\, n \in \mathbb{N}.$$

Since $g_n, n \in \mathbb{N}$, generate the topology, (7.10) follows. □

Taking e.g. right limits of $(X_t^0)_{t\in D}$, the above considerations imply that we obtain a process having continuous sample paths $\mathbb{P}_x$–a.s. for all $x \in H_0$. But since our metric is not complete in general, the so constructed process will take values only in the d–completion of H_0 and may be not in H_0. To prove that this is, in fact, not the case we have to employ methods based on the capacity determined by $(R_\lambda)_{\lambda\geq 0}$. These have been developed in detail in [23] and in a way, particularly useful for our case, in [22]. In order to apply the corresponding result in [22] (i.e. Theorem 1.9 in Chapter II), in addition to Hypotheses 1.1, 1.2 and $C^{-1} \in L(H)$, we need to assume:

Hypothesis 7.3. *A is self–adjoint.*

Now we can prove the main result of this section.

Theorem 7.4. *(i) There exists a conservative strong Markov process* $\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in H_0})$ *with continuous sample paths having transition semigroup* $(p_t)_{t\geq 0}$ *(as defined in Proposition 5.7 (iii)).*

(ii) For every $x \in H_0$, $\mathbb{P}_x$ *solves the martingale problem for* N_2 *with test function space*

$$D_0 := \left\{\varphi \in D(N_2) \cap C_b(H) \mid N_2\varphi \in L^\infty(H, \nu)\right\}$$

and initial condition x*, i.e.. under* $\mathbb{P}_x$

$$\varphi(X_t) - \int_0^t N_2\varphi(X_s)ds, \quad t \geq 0, \tag{7.12}$$

is an $(\mathcal{F}_t)$*–martingale with* $X_0 = x$ *for all* $\varphi \in D_0$.

Proof. (i). Since $C^{-1} \in L(H)$ and Hypotheses 1.1, 1.2 and 7.3 hold, we can apply [4, Theorem 1.1] to conclude that

$$\nu = \rho \cdot N_Q$$

and $\rho^{1/2} \in W^{1,2}(H, N_Q)$, i.e. the closure of $C_b^1(H)$ with respect to the norm $\|\cdot\|_{1,2}$ given by

$$\|\varphi\|_{1,2}^2 := \int_H \left(|C^{1/2}D\varphi|^2 + \varphi^2\right) dN_Q, \ \varphi \in C_b^1(H).$$

Then we can write for $\varphi \in \mathcal{E}_A(H)$

$$N_0\varphi = L^0\varphi + \langle \beta, D\varphi \rangle$$

where

$$L^0\varphi = L\varphi + 2\left\langle \frac{CD\rho^{1/2}}{\rho^{1/2}}, D\varphi \right\rangle$$

and

$$\beta := F_0 - 2\frac{CD\rho^{1/2}}{\rho^{1/2}}.$$

Note that L^0 is symmetric on $L^2(H, \nu)$ and that β has ν–divergence zero, i.e.

$$\int_H \langle \beta, D\varphi \rangle d\nu = 0, \quad \forall\, \varphi \in \mathcal{E}_A(H).$$

So, we can apply [22, Chapter II, Theorem 1.9] to conclude that

$$\mathbb{P}_\nu(\Lambda_0 \cap \Lambda_2) = 1$$

where Λ_0 is as in (7.2) and

$$\Lambda_2 := \left\{ \omega \in \Omega | \lim_{s \downarrow t, s \in D} X_s^0(\omega) \text{ exists in } \Omega\ \forall\, t \in [0, \infty) \right\}.$$

D denotes the dyadics as in the previous section. Repeating the arguments there with $\Lambda_0 \cap \Lambda_2$ replacing Λ_0 we see that

$$\mathbb{P}_x(\Lambda \cap \Lambda_2) = 1, \ \forall\, x \in H_0,$$

where Λ is as defined in (7.8). Now we define for $\omega \in \Lambda \cap \Lambda_2$

$$X_t(\omega) := \lim_{s \downarrow t, s \in D} X_s^0(\omega)$$

to obtain continuous sample paths $\mathbb{P}_x$–a.s. for all $x \in H_0$. It is standard to check that this gives the desired Markov process, (see [15] for details). Also the strong Markov property is obvious, since we have continuous sample paths and a (strong) Feller transition semigroup.

(ii). First note that for $f \in B_b(H),\ f \geq 0,\ x \in H_0$,

$$\mathbb{E}_x\left[\int_0^t f(X_s)ds\right] \leq e^t\mathbb{E}_x\left[\int_0^\infty e^{-s}f(X_s)ds\right] = e^t R_1 f(x).$$

In particular, this is always finite. If, in addition, $f = 0$ ν–a.e., then by Corollary 5.4 also $R_1 f(x) = 0$ for all $x \in H_0$. Hence the integral in (7.12) is well defined independent of the ν–version taken for $N_2\varphi,\ \varphi \in D_0$. Furthermore, we know that for $\varphi \in D_0$,

$$P_t\varphi - \varphi = \int_0^t P_r N_2\varphi dr \text{ in } L^2(H,\nu).$$

Hence, since $p_r\varphi$, $p_r(N_2\varphi)$ are ν–versions of $P_r\varphi$, $P_r(N_2\varphi)$ respectively, which are continuous on H_0, it follows that

$$p_t\varphi(x) - \varphi(x) = \int_0^t p_r(N_2\varphi)(x)dr \quad \forall\ x \in H_0$$

(by dominated convergence). The rest of the proof of (ii) is then standard by the Markov property (cf. also the proof of Proposition 8.2 below). □

Remark 7.5. (i). Both assumptions $C^{-1} \in L(H)$ and Hypothesis 7.3 were made to avoid technical complications and can be relaxed. E.g. in Hypothesis 7.3 it is enough to assume that A is sectorial, and $C^{-1} \in L(H)$ can be dropped if $(N_0, \mathcal{E}_A(H))$ satisfies the weak sector condition on $L^2(H,\nu)$, which in turn is the case if it is symmetric.

(ii). [22, Chapter II, Proposition 1.9] implies directly the continuity of sample paths $\mathbb{P}_\nu$ a.s.. Using this the above arguments can be shortened, since we can avoid to use Corollary 6.4. We presented the proof above based on the results in Section 6, which are certainly of their own interest, because it is more transparent. In particular, no further assumptions are necessary to get continuity of sample paths on dyadics. If, however, we assume that A is sectorial and $C^{-1} \in L(H)$ and if we use [22, Chapter II, Proposition 1.9] instead of Corollary 6.4, then in Hypothesis 1.2–(i) the assumption $\int_H |x|^{12}\nu(dx) < \infty$ can be weakened again to $\int_H |x|^4\nu(dx) < \infty$.

8. Uniqueness

Consider the situation of the previous section. We shall prove uniqueness in an even larger class of diffusions. First we need to introduce a "ν–version" of our martingale problem. We restrict to the class of diffusion processes which are Feller, i.e. their transition semigroups map $C_b(H)$ into $C_b(H)$.

Definition 8.1. *A Feller diffusion process* $\mathbb{M}' = (\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\geq 0}, (X'_t)_{t\geq 0}, (\mathbb{P}'_x)_{x\in H_0})$ *on* H_0 *with transition semigroup* $(p'_t)_{t\geq 0}$ *is said to satisfy the* $L^2(H,\nu)$*–martingale problem for* $(N_0, \mathcal{E}_A(H))$*, if*

(i) For some $M', \varepsilon' \in (0,\infty)$

$$\int_{H_0} (p'_t f)^2 d\nu \leq M' \int_{H_0} f^2 d\nu,\ \forall\ f \in C_b(H), t \in (0,\varepsilon').$$

(ii) For all $\varphi \in \mathcal{E}_A(H)$

$$\varphi(X'_t) - \int_0^t N_0\varphi(X'_s)ds, \ t \geq 0,$$

is an $(\mathcal{F}'_t)_{t\geq 0}$–martingale under $\mathbb{P}'_\nu := \int_{H_0} \mathbb{P}'_x \nu(dx)$.

Below as usual we denote the expectation, conditional expectation of $\mathbb{P}'_x$ by $\mathbb{E}'_x(\cdot)$, $\mathbb{E}'_x(\cdot|\ \mathcal{F}^0_t)$ respectively.

One should note that for any $\mathbb{M}'$ as in the Definition 8.1 (as is easy to see) $(p'_t)_{t\geq 0}$ gives rise to a C_0–semigroup $(P'_t)_{t\geq 0}$ on $L^2(H,\nu)$ and for its infinitesimal generator N'_2 we have for sufficiently big $\lambda > 0$ that $(\lambda - N'_2)(D(N'_2)) = L^2(H,\nu)$ and

$$R(\lambda, N'_2) = (\lambda - N'_2)^{-1} = \int_0^\infty e^{-\lambda t} P'_t dt. \tag{8.1}$$

(see e.g. A. Pazy [20], Chapter I, Theorem 5.3 and its proof).

For $\mathbb{E}'_\nu(\cdot) := \int_{H_0} \mathbb{E}'_x(\cdot)\nu(dx)$ and $f \in L^1(H,\nu)$ it follows that

$$\begin{aligned}\mathbb{E}'_\nu\left[|\int_0^t f(X'_s)ds| \right] &\leq \int_{H_0}\int_0^t P'_s|f|dsd\nu \\ &\leq e^t \int_{H_0} R(\lambda, N'_2)|f|d\nu \\ &\leq e^t \|R(\lambda, N'_2)|f|\|_{L^2(H,\nu)} < \infty.\end{aligned}$$

Hence, in particular, the expression in Definition 8.1–(ii) is well defined (i.e. independent of the ν–class taken for $N_0\varphi$) and in $L^1(\Omega', \mathbb{P}'_\nu)$.

Proposition 8.2. *The diffusion $\mathbb{M}$ from Theorem 7.4 solves the $L^2(H,\nu)$–martingale problem for $(N_0, \mathcal{E}_A(H))$*

Proof. 8.1–(i) is obvious. To show 8.1–(ii) let $\varphi \in \mathcal{E}_A(H)$. (Note that 8.1–(ii) does not follow directly from Theorem 7.4–(ii), since $N_0\varphi$ is not bounded in general.) Then for $t > s$ and any $\mathcal{F}_s$–measurable, bounded function $F_s : \Omega \to \mathbb{R}$ by the Markov property

$$\begin{aligned}&\mathbb{E}_\nu\left[F_s\left(\varphi(X_t) - \varphi(X_s) - \int_s^t N_0\varphi(X_r)dr\right)\right] \\ &= \int_{H_0} \nu(dx)\mathbb{E}_x\left[F_s\mathbb{E}_x\left(\varphi(X_t) - \varphi(X_s)|\mathcal{F}_s\right)\right] \\ &- \int_{H_0} \nu(dx)\mathbb{E}_x\left[F_s\int_s^t \mathbb{E}_x\left(N_0\varphi(X_r)|\mathcal{F}_s\right)dr\right] \\ &= \int_{H_0} \nu(dx)\mathbb{E}_x\left[F_s\mathbb{E}_{X_s}\left(\varphi(X_{t-s}) - \varphi(X_s)\right)\right]\end{aligned}$$

$$-\int_{H_0} \nu(dx)\mathbb{E}_x \left[F_s \int_s^t \mathbb{E}_{X_s} \left(N_0\varphi(X_{r-s})\right) dr \right]$$

$$= \mathbb{E}_\nu \left[F_s \left(p_{t-s}\varphi(X_s) - \varphi(X_s) - \int_0^{t-s} p_r(N_0\varphi)(X_s)dr \right) \right].$$

But since ν is invariant for (p_t),

$$\mathbb{E}_\nu \left[\left| p_{t-s}\varphi(X_s) - \varphi(X_s) - \int_0^{t-s} p_r(N_0\varphi)(X_s)dr \right| \right]$$

$$= \int_{H_0} \left| P_{t-s}\varphi - \varphi - \int_0^{t-s} P_r N_0 \varphi dr \right| = 0. \qquad \square$$

Theorem 8.3. *Let* $\mathbb{M}' = (\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\geq 0}, (X'_t)_{t\geq 0}, (\mathbb{P}'_x)_{x\in H_0})$ *be a Feller diffusion process on* H_0 *satisfying the* $L^2(H,\nu)$*–martingale problem for* $(N_0, \mathcal{E}_A(H))$. *Then* $\mathbb{M}'$ *has the same finite dimensional distributions as* $\mathbb{M}$ *from Theorem* 7.4.

Proof. Let $(p'_t)_{t\geq 0}$ be the transition semigroup of $\mathbb{M}'$ and $t > 0$. We have to show that

$$p'_t = p_t.$$

To this end, let $\varphi \in \mathcal{E}_A(H)$, $g \in L^2(H,\nu)$. Then

$$\int_{H_0} g \left(P'_t\varphi - \varphi - \int_0^t P'_s N_0 \varphi ds \right) d\nu$$

$$= \mathbb{E}_\nu \left[g(X'_0) \left(\varphi(X'_t) - \varphi(X'_0) - \int_0^t N_0\varphi(X_s)ds \right) \right] = 0.$$

Hence

$$P'_t\varphi - \varphi = \int_0^t P'_s N_0 \varphi ds,$$

so $\varphi \in D(N'_2)$ and $N_0\varphi = N'_2\varphi$. But $\mathcal{E}_A(H)$ is a core for N_2 (cf. Theorem 2.3), consequently,

$$D(N_2) \subset D(N'_2) \text{ and } N_2 = N'_2 \text{ on } D(N_2),$$

hence for all $\lambda > 0$

$$(\lambda - N'_2)(D(N'_2)) \supset (\lambda - N_2)(D(N_2)) = L^2(H,\nu).$$

So,

$$(\lambda - N'_2)(D(N'_2)) = (\lambda - N'_2)(D(N_2))$$

and taking $\lambda > 0$ large enough it follows by (8.1) that

$$D(N'_2) = D(N_2),$$

consequently $N_2' = N_2$. Therefore,

$$P_t' = P_t,$$

hence for all $f \in C_b(H)$

$$p_t' f(x) = p_t f(x) \ \text{ for } \nu - \text{a.e. } x \in H.$$

By continuity it follows that

$$p_t' f(x) = p_t f(x) \ \text{ for all } \ x \in H_0 = \text{ supp } \nu,$$

hence $p_t' = p_t$, by a monotone class argument. □

9. Application

9.1. Gradient systems

Let us first consider a general situation and then concrete examples. We adopt the notation from the previous sections.

Hypothesis 9.1. *(i) A is a self–adjoint linear operator on H such that there exists* $\omega > 0$ *such that*

$$\langle Ax, x\rangle \le -\omega |x|^2, \ \forall\, x \in H,$$

and A^{-1} *is of trace class.*

(ii) $C := I$. *(Hence for Q from Hypothesis 1.1, we have* $Q = -\frac{1}{2}\, A^{-1}$.*)*

(iii) Let $U : H \to (-\infty, +\infty]$ *be convex, lower semicontinuous, such that* $\{U < +\infty\}$ *is open and* $\mu(\{U < +\infty\}) > 0$, *where* $\mu := N_Q$, *and such that*

$$\rho := Z^{-1} e^{-2U(x)} \in L^1(H, \mu)$$

with $Z := \int_H e^{-2U(x)} \mu(dx)$, *so that* $\nu(dx) := \rho(x)\mu(dx)$ *is a probability measure on* $(H, \mathcal{B}(H))$.

(iv) Let ∂U *denote the subdifferential of* U, *i.e.* $D(\partial U) := \{U < +\infty\}$ *and for* $x \in D(\partial U)$

$$\partial U(x) := \{y \in H |\ U(x+h) - U(x) \ge \langle y, h\rangle\ \forall\, h \in H\}\,.$$

Then $F := \partial U$ *is maximal dissipative, so* F_0 *can be defined as in §1. Assume*

$$\int_H (|x|^{12} + |F_0(x)|^2 + |x|^4 |F_0(x)|^2)\nu(dx) < +\infty.$$

Note that Hypothesis 9.1 implies that $\nu(D(\partial U)) = 1$. So Hypotheses 1.1, 1.2 and 7.3 and $C^{-1} \in L(H)$ hold except for 1.2–(ii). But we have the following result.

Proposition 9.2. *Suppose* $\rho^{1/2} \in W^{1,2}(H,\mu)$ *such that*

$$2\frac{D\rho^{1/2}}{\rho^{1/2}} = F_0.$$

Then, if as before,

$$N_0\varphi := \frac{1}{2}\,\mathrm{Tr}\,[D^2\varphi] + \langle \cdot, A^* D\varphi\rangle + \langle F_0, D\varphi\rangle, \quad \varphi \in \mathcal{E}_A(H),$$

we have

$$\int_H N_0\varphi d\nu = 0 \;\forall\; \varphi \in \mathcal{E}_A(H).$$

So, Hypothesis 1.2–(ii) also holds, so all results in Sections 1–8 apply.

Proof. Let $\varphi, \psi \in \mathcal{E}_A(H)$. Then, e.g. by [21, Proposition 2.1] and the proof of Theorem 3.5, in particular formula (3.17) in [5]

$$\int_H \psi N_0\varphi d\nu = \int_H \varphi N_0\psi d\nu.$$

Choosing $\psi = 1$, the result follows. □

Example 9.3. Take $H = \mathbb{R}$, $-A = C = I$, and

$$U(x) := \begin{cases} -\log x, & x > 0, \\ +\infty, & x \le 0. \end{cases}$$

Then

$$\rho(x) = \begin{cases} x^2, & x > 0, \\ 0, & x \le 0. \end{cases}$$

and

$$F_0(x) = \frac{2}{x}, \; x \in D(F) = (0, +\infty).$$

So, Hypothesis 9.1 and the assumptions in Proposition 9.2 are satisfied. Hence by Theorem 7.4 there exists a strong Feller diffusion process on supp $\nu = [0, +\infty)$ solving the martingale problem corresponding to

$$\begin{cases} dX(t) = \left(-X(t) + \dfrac{2}{X(t)}\right) dt + dW(t), \\ \\ X(0) = x, \end{cases}$$

which is unique in the sense of Theorem 8.2.

Example 9.4. Let H be a separable Hilbert space, and take A as in Hypothesis 9.1–(i), $C = I$. Let $B_1(0)$ denote the open unit ball in H. Set

$$U(x) := \begin{cases} -\log(1-|x|^2), & \text{if } x \in B_1(0), \\ +\infty, & \text{otherwise.} \end{cases}$$

Then

$$\rho(x) = \begin{cases} (1-|x|^2)^2, & \text{if } x \in B_1(0), \\ 0, & \text{otherwise,} \end{cases}$$

and

$$F_0(x) = \frac{2x}{1-|x|^2},\ x \in D(\partial U) = B_1(0).$$

So, Hypothesis 9.1 and the assumptions in Proposition 9.2 are satisfied. Hence by Theorem 7.4 there exists a strong Feller diffusion process on supp $\nu = \overline{B_1(0)}$ solving the martingale problem corresponding to

$$\begin{cases} dX(t) = \left(-X(t) + \dfrac{2X(t)}{1-|X(t)|^2}\right) dt + dW(t), \\ \\ X(0) = x, \end{cases}$$

which is unique in the sense of Theorem 8.2. We note that both in this and in the previous example the relation of the martingale problem to the stochastic differential equation is somewhat informal since supp $\nu = H_0 \neq H$.

9.2. Applications to Reaction–Diffusion equations

Let D be an open bounded subset of $\mathbb{R}^d$ with regular boundary ∂D. Set $H = L^2(D)$ and let A be the linear operator in H defined as

$$\begin{cases} Ax = \Delta_\xi x,\ x \in D(A), \\ \\ D(A) = H^2(D) \cap H_0^1(D). \end{cases} \tag{9.1}$$

It is well known that A is self–adjoint. Moreover there exist an orthonormal basis $\{e_k\}$ in H and a nondecreasing sequence of positive numbers $\{\alpha_k\}$ such that

$$Ae_k = -\alpha_k e_k,\ k \in \mathbb{N}^d.$$

Finally $\alpha_k \uparrow \infty$ and α_k behaves as $|k|^2$ at infinity, see e.g. [1, Theorem 14.6].

Therefore Hypothesis 1.1–(i) is fulfilled with $\omega = \inf_{k\in\mathbb{N}} \alpha_k$.

Set now $C := (-A)^{-\delta}$ with $\delta \geq 0$, and $Q = \int_0^\infty Ce^{2tA}dt = \frac{1}{2}\ (-A)^{-1-\delta}$. Since

$$\mathrm{Tr}\ Q \simeq \sum_{k\in\mathbb{N}^d} \frac{1}{|k|^{2(1+\delta)}},$$

Hypothesis 1.1–(ii) is fulfilled provided $2(1+\delta) > d$, i.e.

$$\delta > \frac{d}{2} - 1, \tag{9.2}$$

that we shall assume from now on.

Let us now consider a continuous decreasing function

$$f : \mathbb{R} \to \mathbb{R}, \quad \rho \to f(\rho).$$

We shall denote by f_α its Yosida approximations.

We shall assume that

Hypothesis 9.5. *There exist $m, N \in \mathbb{N}$, $a, b > 0$ such that*

$$|f_\alpha(\rho)| \le a(1 + |\rho|^m), \ \forall \rho \ \in \mathbb{R}, \ \alpha > 0,$$

and

$$|f_\alpha(\rho) - f(\rho)| \le b\alpha(1 + |\rho|^N), \ \forall \rho \ \in \mathbb{R}, \ \alpha > 0.$$

Finally for $\alpha \ge 0$, we set $F_\alpha(x) := f_\alpha \circ x, \ x \in H$, and

$$F(x) = f \circ x, \ \forall x \in D(F) = \{x \in H | \ f \circ x \in H\}.$$

Obviously $F_0(x) = F(x)$.

Let us give an example. Define the non locally Lipschitz function

$$f(\rho) := \begin{cases} \sqrt{-\rho}, \text{ if } \ \rho < 0, \\ -\sqrt{-\rho}, \text{ if } \ \rho \ge 0. \end{cases}$$

Then an easy calculation shows that Hypothesis 9.5 holds.

We are going to show that, under Hypothesis 9.5, F fulfills Hypothesis 1.2. For this it is enough to show, by Remark 3.4, that for any $m \in \mathbb{N}$ there exists $c_m > 0$ such that

$$\int_H \left[\int_D |x(\xi)|^{2m} d\xi \right] \nu_\alpha(dx) = \int_H |x|^{2m}_{L^{2m}(D)} \nu_\alpha(dx) \le c_m, \tag{9.3}$$

where ν_α is the invariant measure of the operator N_α defined by (3.1). This is a consequence of the following lemma, which is a generalization of Lemma 3.1 and Corollary 3.2.

Note that in comparison with Remark 3.4 we only have that for $h \in D(A^*) = D(A)$

$$x \to \int_D h(\xi) f \circ x(\xi) d\xi$$

is continuous on $L^{2m}(D)$ rather than on $H = L^2(D)$ where m is as in Hypothesis 9.5. But because of (9.3) this is enough to get (3.13).

Lemma 9.6. *For any $m \in \mathbb{N}$ there exists $c_m > 0$ such that* (9.3) *holds.*

Proof. We shall denote by X_α the solution of (3.2) and by W_A the stochastic convolution defined by (3.3). Then we proceed in several steps.

Step 1. There exists $c_{1,m} > 0$ such that

$$\mathbb{E}|F_\alpha(W_A(t))|^{2m}_{L^{2m}(D)} \le c_{1,m} t^m, \ t \ge 0. \tag{9.4}$$

The proof of Step 1 is straightforward.

Step 2. There is $c_{2,m} > 0$ such that

$$\mathbb{E}|X_\alpha(t,x)|^{2m} \le c_{2,m} t^m (1 + e^{-m\omega t}|x|^{2m}). \tag{9.5}$$

Setting $Y(t,\xi) = X_\alpha(t,x)(\xi) - W_A(t,\xi)$, $Y(t,\xi)$ is the solution to

$$\begin{cases} Y'(t,\xi) = \Delta_\xi Y(t,\xi) + f_\alpha(Y(t,\xi) + W_A(t,\xi)) \\ \\ Y(0) = x. \end{cases} \tag{9.6}$$

Multiplying the first equation by $Y(t,\xi)^{2m-2}Y(t,\xi)$, and taking into account the dissipativity of F_α, we obtain, for a suitable constant $c_{3,m}$

$$\begin{aligned} \frac{1}{2m}\frac{d}{dt}\, Y(t,\xi)^{2m} &= Y(t,\xi)^{2m-1}\Delta_\xi Y(t,\xi) \\ &+ (f_\alpha(Y(t,\xi)+W_A(t,\xi)) - f_\alpha(W_A(t,\xi)))Y(t,\xi)^{2m-1} \\ &+ f_\alpha(W_A(t,\xi))Y(t,\xi)^{2m-1} \\ &\le Y(t,\xi)^{2m-1}\Delta_\xi Y(t,\xi) + f_\alpha(W_A(t,\xi))Y(t,\xi)^{2m-1}. \end{aligned} \tag{9.7}$$

Now notice that

$$\int_D Y(t,\xi)^{2m-1}\Delta_\xi Y(t,\xi)d\xi = -(2m-1)\int_D |\nabla_\xi Y(t,\xi)|^2 Y(t,\xi)^{2m-2}d\xi. \tag{9.8}$$

Then, integrating (9.7) with respect to ξ, and taking into account (9.8), yields

$$\begin{aligned} \frac{1}{2m}\frac{d}{dt}\int_D Y(t,\xi)^{2m}d\xi + (2m-1)\int_D |\nabla_\xi Y(t,\xi)|^2 Y(t,\xi)^{2m-2}d\xi \\ \le \int_D f_\alpha(W_A(t,\xi))Y(t,\xi)^{2m-1}d\xi. \end{aligned} \tag{9.9}$$

But, recalling the Poincaré inequality, there is $c_{4,m} > 0$ such that

$$\begin{aligned} (2m-1)\int_D |\nabla_\xi|^2 Y^{2m-2}d\xi &= \frac{2m-1}{m^2}\int_D |\nabla_\xi Y(t,\xi)|^2 d\xi \\ &\ge c_{4,m}\int_D Y(t,\xi)^{2m}d\xi. \end{aligned} \tag{9.10}$$

Moreover there exists $c_{5,m} > 0$ such that

$$\begin{aligned}&\int_D f_\alpha(W_A(t,\xi))Y(t,\xi)^{2m-1}d\xi\\&\le \frac{1}{2}\,c_{4,m}\int_D Y(t,\xi)^{2m}d\xi + c_{5,m}\int_D f_\alpha(Y(t,\xi))^{2m}d\xi.\end{aligned} \tag{9.11}$$

Substituting (9.10) and (9.11) into (9.9) yields

$$\begin{aligned}\frac{d}{dt}\int_D Y(t,\xi)^{2m}d\xi &\le -mc_{4,m}\int_D Y(t,\xi)^{2m}d\xi\\&\quad + 2mc_{5,m}\int_D f_\alpha(Y(t,\xi))^{2m}d\xi.\end{aligned} \tag{9.12}$$

By a classical comparison result we find

$$\begin{aligned}\int_D Y(t,\xi)^{2m}d\xi &\le e^{-mc_{4,m}t}\int_D x(\xi)^{2m}d\xi\\&\quad + 2mc_{5,m}\int_0^t e^{-mc_{4,m}(t-s)}\int_D f_\alpha(Y(s,\xi))^{2m}d\xi ds,\end{aligned} \tag{9.13}$$

and Step 2 follows from Step 1.

Step 3. Conclusion.

Arguing as in the proof of Corollary 3.2 we obtain (9.3). □

Remark 9.7. One can study the stochastic differential equation

$$dX = (\Delta_\xi X + F(X))dt + \sqrt{C}dW(t),\quad X(0) = x,$$

and the corresponding transition semigroup, see [14, Theorem 11.4.1] and [7, Proposition 6.2.2]. But in this way, in contrast to the "double approximation " performed in our paper, one cannot prove that the corresponding generator N_2 is the closure of N_0 with respect to $L^2(H,\nu)$. But, under the assumptions of [7] in [11] it was proved that N_2 is the closure of N_0, defined on a different core.

Remark 9.8. The semigroup P_t is strong Feller provided $\delta \le 1$. Since by (9.2) $d/2 - 1 < \delta$, this is possible for $d \le 3$. In this case all results in Sections 5–8, apart from Theorem 7.4, apply.

Remark 9.9. We would like to emphasize that, as pointed out in the previous remark, for the very particular examples studied in this section our general results are more suitable to prove the strong Feller property of the transition semigroup rather than existence and uniqueness of solutions to the underlying stochastic equation. The latter could be proved by more direct techniques (under even weaker assumptions). We would like to thank one of the referees for pointing this out to us.

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